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A THEOREM OF MATSUSHIMA

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In [7], Matsushima studied the vector bundles over a complex torus. One of his main theorems is: A vector bundle over a complex torus has a connection if and only if it is homogeneous (Theorem (2.3)). The aim of this paper is to prove the characteristic $p > 0$ version of this theorem. However in the characteristic $p > 0$ case, for any vector bundle E over a scheme defined over a field k with char. $k = p$, the pull back F^*E of E by the Frobenius endomorphism F has a connection. Hence we have to replace the connection by the stratification (cf. (2.1.1)). Our theorem states: Let A be an abelian variety whose p-rank is equal to the dimension of A. Then a vector bundle over A has a stratification if and only if it is homogeneous (Theorem (2.5)).

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§ 1. Preliminaries

(1.1) All schemes are of finite type over an algebraically closed field k, unless the contrary is stated as in (1.4.1). Let char. $k = p \geq 0$. Let P, X be schemes. Let $\pi: P \to X$ be a morphism. Let G be a group scheme. We denote by $P(X, G, \pi)$ a principal G-bundle over X: G operates on P from the right satisfying the following conditions:

(i) The diagram

$$\begin{tikzcd} P \times_k G \arrow[r] \arrow[dr] & P \arrow[d] \\ & X \end{tikzcd}$$

is commutative.

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(ii) The morphism

$$\begin{aligned} G \times_k P &\longrightarrow P \times_X P \\ (g, p) &\longmapsto (p \cdot g, p) \end{aligned}$$

is an isomorphism.

(iii) There exists an (fpqc) extension $X' \to X$ such that $P' = P \times_X X'$ is isomorphic to $G \times_k X'$.

(1.1.1) Let K be a group scheme. Let H be a subgroup scheme. Then the quotient $K \xrightarrow{\pi} K/H$ exists (S.G.A.D. Exposé VI_A) and $K(K/H, H, \pi)$ is a principal H-bundle.

We need some results of Grothendieck [4].

(1.2) THEOREM A. *An (fpqc) (faithfully flat and quasi-compact) morphism is a strict descent morphism for the category of affine schemes and for the category of quasi-coherent sheaves.*

In particular this means two important things:

(1.2.1) Let S', S be schemes. Let $\alpha: S' \to S$ be an (fpqc) morphism. Let X, Y be schemes affine over S. Let X' (resp. Y') be the pull-back of X (resp. Y). Then to define a morphism $X \to Y$ over S, it is sufficient to define a morphism $X' \to Y'$ over S' commuting with the descent data.

(1.2.2) In the category of affine schemes (or in the category of quasi-coherent sheaves) descent data descends through an (fpqc) morphism.

(1.3) Let $P(X, G, \pi)$ be a principal G-bundle over X with group G. Let G' be a group scheme. Let $\varphi: G \to G'$ be a morphism of group schemes.

LEMMA (1.3.1) *Assume that G, G' are affine over k. Then there exists a unique principal G'-bundle $P'(X, G', \pi')$ such that there exists a morphism f of P to P' over X with $f(x^g) = f(x)^{\varphi(g)}$, $x \in P$, $g \in G$.*

Proof. Let $X' \to X$ be an (fpqc) morphism trivializing P. Let P^* denote the pull-back of P on X'. The pull-back $P^* \simeq G \times_k X'$ carries naturally a descent data, so that it induces a descent data on $G' \times_k X'$. We put $P^{*\prime} \simeq G' \times_k X'$. Since $P^{*\prime}$ is affine over X', by (1.2.2), $P^{*\prime}$ descends and defines a principal G'-bundle over X. This shows the existence.

We shall show the uniqueness. Let P_1' and P_2' be two principal G'-bundles over X having the property described in the lemma. Let $X' \to X$ be an (fpqc) morphism trivializing P, P_1 and P_2. We denote the pull-

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back by $*$. We have $f_i^*: P^* \simeq X' \times G \to X' \times G'$, $i = 1, 2$.

$$(x, g) \to (x, f_i^*(x, g))$$

$$\begin{tikzcd} P^* \arrow[r, "f_1^*"] \arrow[dr, "f_2^*"'] & P_1'^* \arrow[d, "\ell"] \\ & P_2'^* \end{tikzcd}$$

$f_i^*(x, g) = f_i^*((x, 1)^g) = f_i^*(x, 1)^{\varphi(g)}$, $x \in X'$, $g \in G$. If we put $\ell(x, g') = (x, f_2^*(x, 1) f_1^*(x, 1)^{-1} g')$ for $(x, g') \in X' \times G'$, then the diagram above commutes and gives an isomorphism between $P_1'^*$ and $P_2'^*$. It is easy to check that this isomorphism commutes with the descent data for $P_1'^*$ and for $P_2'^*$. Hence this gives an isomorphism of P_1' and P_2' over X.

(1.4) Let X be a projective scheme over k. Let $P(X, GL(r, k), \pi)$ be a principal $GL(r, k)$-bundle over X. $\mathscr{G}(P)$ is a functor from the category of k-schemes to the category of groups defined by the following formula:

$\mathscr{G}(p)(T) = \{f \in \mathrm{Hom}_T(P \times_k T, P \times_k T) \mid f$ is an automorphism of $P \times_k T$. $f(x^g) = f(x)^g$ for any $g \in GL(r, k).\}$ for a scheme T over k. Since X is the quotient of P by the action of $GL(r, k)$, f induces the following commutative diagram:

$$\begin{tikzcd} P \times_k T \arrow[r, "f"] \arrow[d, "\pi"] & P \times_k T \arrow[d, "\pi"] \\ X \times_k T \arrow[r, "\bar{f}"] & X \times_k T . \end{tikzcd}$$

PROPOSITION (1.4.1) *The functor $\mathscr{G}(P)$ is represented by a scheme locally of finite type over k.*

Proof. $GL(r, k)$ is an open subscheme of the scheme of $r \times r$ matrices $M(r \times r, k)$. $GL(r, k)$ operates on $M(r \times r, k) \simeq A_k^{r^2}$ from the left and the right as linear automorphisms of the affine space $A_k^{r^2}$. Since a linear automorphism of $A_k^{r^2}$ can be prolonged to an automorphism of $\boldsymbol{P}_k^{r^2}$, the actions from the left and the right of $GL(r, k)$ on $A_k^{r^2}$ can be extended equivariantly to the actions on $\boldsymbol{P}_k^{r^2}$. Hence $GL(r, k) \subset \boldsymbol{P}_k^{r^2}$ is an equivariant completion of $GL(r, k)$ with respect to the both actions. Since a principal $GL(r, k)$-bundle is locally trivial for the Zariski topology, we can associate $P_k^{r^2}$-bundle $P' = P \times^{GL(r,k)} P^{r^2} \xrightarrow{\pi'} X$ with $P(X, GL(r,k), \pi)$. Then P is an open subscheme of P' and $GL(r,k)$ operates on P' from the right. From the existence of the Hilbert scheme (See Grothendieck [5]), we deduce that the functor $\mathrm{Aut}_k P'$ is represented by a group scheme G locally of finite type over k since P' is projective over X and since X is projective over k by hypothesis. Now let $Y = P' - P$ and we regard Y as a reduced closed subscheme of P'. Consider the subfunctor F of $\mathrm{Aut}_k P'$:

$F = \{$Automorphisms of P' leaving the closed subscheme Y fixed$\}$. Then F is representable. Now we show $\tilde{\mathscr{G}}(P)$ is a subgroup functor of F. It is sufficient to show that any element of $\tilde{\mathscr{G}}(P)$ can be extended to an automorphism of P'. First we assume that P is trivial. In this case, letting f be an element of $\tilde{\mathscr{G}}(P)$, $f(g) = f(I_r g) = f(I_r^g) = f(I_r)^g = f(I_r)g$. Hence f is nothing but the multiplication by $f(I_r)$ from the left. Since P' is an equivariant completion, the multiplication by $f(I_r)$ can be extended to P'. In the case P is not trivial, take an open covering $\{U_\alpha\}$ of X so that $\pi^{-1}(U_\alpha) = P_\alpha$ is trivial. $\tilde{\mathscr{G}}(P)$ acts on P_α. By what we have seen above, the action of $\tilde{\mathscr{G}}(P)$ on P_α can be extended to an automorphism of the restriction of P' on U_α. Since the extension is unique, the extensions over U_α and U_β coincide if $U_\alpha \cap U_\beta \neq \emptyset$. Hence the operation of $\tilde{\mathscr{G}}(P)$ on P can be extended to the operation on P'. This is what we had to show. $GL(r,k)$ operates on P' from the right and leaves Y fixed. Hence $GL(r,k)$ is a closed subgroup scheme of F (See S.G.A.D. Exposé VI_B Cor. 1.4.2.). $\tilde{\mathscr{G}}(P)$ is the centralizer of the closed subgroup scheme $GL(r,k)$ of the group scheme F. Hence $\tilde{\mathscr{G}}(P)$ is representable. q.e.d.

(1.4.2) We have the natural homomorphism of group schemes $q\colon \tilde{\mathscr{G}}(P) \to \mathrm{Aut}\, X$ as we remarked above. The kernel of this homomorphism is the group $\mathrm{Aut}_X(P)$ which is connected and affine over k (See M. Maruyama: On a family of algebraic vector bundles, Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki Kinokuniya, Tokyo, 1973, 95–146). We apply (1.4.1) to an abelian variety A.

COROLLARY (1.4.3) *Let A be an abelian variety. Let $P = P(A, GL(r,k), \pi)$. Consider a subgroup functor $\mathscr{G}(P)$ of $\tilde{\mathscr{G}}(P)$ whose value is defined for any k-scheme T by the following formula:*

$\mathscr{G}(P)(T) = \{(x_T, f) \mid x_T$ *is a T-valued point of A.* $f \in \tilde{\mathscr{G}}(P)(T)$ *such that the diagram*

$$\begin{array}{ccc} P \times_k T & \xrightarrow{\quad f \quad} & P \times_k T \\ {\scriptstyle \pi \times \mathrm{id}_T}\downarrow & & \downarrow{\scriptstyle \pi \times \mathrm{id}_T} \\ A \times_k T & \xrightarrow[\text{translation by } x_T]{} & A \times_k T \end{array}$$

commutes$\}$.

Then the group functor $\mathscr{G}(P)$ is represented by a scheme of finite type over k.

Proof. The abelian variety A can be regarded in a natural way as a subgroup scheme of $\mathrm{Aut}_k A$. Let $i: A \hookrightarrow \mathrm{Aut}_k A$. From the definition $\mathscr{G}(P)$ is isomorphic to the fibre product $\tilde{\mathscr{G}}(P) \times_{\mathrm{Aut}_k A} A$. Since A and $\mathrm{Aut}_A P$ are connected, $\mathscr{G}(P)$ is isomorphic to the fibre product $\tilde{\mathscr{G}}^0(P) \times_{\mathrm{Aut}_k A} A$ where $\tilde{\mathscr{G}}^0(P)$ is the connected component containing the identity of the group scheme $\mathscr{G}(P)$.

$$\begin{array}{ccc} \mathscr{G}(P) & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle i} \\ \tilde{\mathscr{G}}^0(P) & \xrightarrow{q} & \mathrm{Aut}_k A \end{array}$$

$\tilde{\mathscr{G}}^0(P)$ is of finite over k. Hence $\mathscr{G}(P)$ is of finite type over k. q.e.d.

DEFINITION (1.4.4) Using the notation of (1.4.3),

$$1 \to \mathrm{Aut}_A P \to \mathscr{G}(P) \to A \ .$$

The quotient group scheme $\mathscr{G}(P)/\mathrm{Aut}_A P$ is denoted by $H(P)$. $H(P)$ is a closed subgroup scheme of A (See S.G.A.D. Exposé VI_A).

A principal $GL(r,k)$-bundle $P(A, GL(r,k), \pi)$ over an abelian variety A is said to be homogeneous if $H(P) = A$.

If P is homogeneous,

$$I \to \mathrm{Aut}_A P \to \mathscr{G}(P) \to A \to 0 \ .$$

Remark (1.4.5) The set of all the k-valued points of $H(P)$ is $\{k$-valued point x of $A \mid P \simeq T_x^* P$ where T_x is the translation by $x\}$.

LEMMA (1.4.6). *Let A,B be abelian varieties. Let $\varphi: A \to B$ be a*

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homomorphism. We set $N = \operatorname{Ker} \varphi \xhookrightarrow{i} A$ *(scheme theoretic kernel, of course). Let* $P'(B, GL(r, k), \pi')$ *be a principal* $GL(r, k)$*-bundle over* B*. Let* $P(A, GL(r, k), \pi) = \varphi^* P'(B, GL(r, k), \pi')$*. Then* N *is a subgroup scheme of* $H(P)$*.*

Proof. Let $f: T \to N$ be a T-valued point of N. Then

$$
\begin{array}{ccc}
(B \times_k T) \times_T (B \times_k T) & \xrightarrow{\mu_B} & B \times_k T \\
\uparrow {\scriptstyle \varphi_T \times \varphi_T} & & \uparrow {\scriptstyle \varphi_T} \\
(A \times_k T) \times_T (A \times_k T) & \xrightarrow{\mu_A} & A \times_k T \\
\uparrow {\scriptstyle i_T \times \mathrm{Id}} & & \\
(N \times_k T) \times_T (A \times_k T) & & \\
\uparrow {\scriptstyle f_T \times \mathrm{Id}} & & \\
A \times_k T \xrightarrow[\Psi]{\sim} T \times_T (A \times_k T) \,, & &
\end{array}
$$

where μ_A (resp. μ_B) is the law of composition of the abelian scheme A (resp. B) and Ψ is the natural isomorphism. Hence we get:

$$
\begin{aligned}
f_T^* P_T &= \Psi^* \circ (f_T \times \mathrm{Id})^* \circ (i_T \times \mathrm{Id})^* \circ \mu_A^* P_T \\
&= \Psi^* \circ (f_T \times \mathrm{Id})^* \circ (i_T \times \mathrm{Id})^* \circ \mu_A^* \circ \varphi_T^* P'_T \\
&\simeq \Psi^* \circ (f_T \times \mathrm{Id})^* \circ (i_T \times \mathrm{Id})^* \circ (\varphi_T \times \varphi_T)^* \circ \mu_B^* P'_T \\
&= \varphi_T^* P'_T \\
&= P_T \,.
\end{aligned}
$$

This shows we have a section s on N of $\mathscr{G}(P) \to H(P)$:

$$
\begin{array}{ccc}
\mathscr{G}(P) & \longrightarrow & H(P) \\
& \nwarrow {\scriptstyle s} & \cup \\
& & N \,.
\end{array}
$$

q.e.d.

Remark (1.4.7) Let X be a scheme. Consider a principal $GL(r, k)$-bundle $P(X, GL(r, k), \pi)$. Let E be the associated vector bundle to P. We define the functor $\tilde{\mathscr{G}}(E)$ by the following formula:

$\tilde{\mathscr{G}}(E)(T) = \{(\varphi, \varphi') \mid \varphi$ is a T-automorphism of $X \times_k T$ and φ' is an isomorphism of E_T to $\varphi^* E_T\}$ for a k-scheme T. Then it is easy to see that

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$\tilde{\mathscr{G}}(P) = \tilde{\mathscr{G}}(E)$ as group functors. Moreover $\mathrm{Aut}_X P = \mathrm{Aut}_X E$.

If $X = A$ is an abelian variety, we define a subgroup functor $\mathscr{G}(E)$ of $\tilde{\mathscr{G}}(E)$ by the following formula; for any k-scheme T,

$$\mathscr{G}(E)(T) = \{(\varphi, \varphi') \in \tilde{\mathscr{G}}(E)(T) \mid \varphi \text{ is the translation by a } T\text{-valued point } x_T\}.$$

Then $\mathscr{G}(P) = \mathscr{G}(E)$. Let $H(E)$ be the quotient group scheme of $\mathscr{G}(E)$ by $\mathrm{Aut}_A E$. The group schemes $H(P)$ and $H(E)$ are isomorphic. We say that E is homogeneous when P is so (cf. Miyanishi [8], Umemura [10] and Remark (1.4.5)).

§ 2. Main theorem

(2.1) We recall the definition of stratification (Grothendieck [6]).

DEFINITION (2.1.1) Let X be a smooth scheme defined over k. Let E be a vector bundle on X. For each positive integer n, we denote by $\varDelta^1(n)$ the n-th infinitesimal neighbourhood of the diagonal of $X \times_k X$. $\varDelta^2(n)$ denotes the n-th infinitesimal neighbourhood of the diagonal of $X \times_k X \times_k X$. Then we have the usual diagram of projections:

$$\begin{tikzcd} X & \varDelta^1(n) \arrow[l, shift right, "p_1(n)"'] \arrow[l, shift left, "p_2(n)"] & \varDelta^2(n) \arrow[l, shift right=2, "p_{31}(n)"'] \arrow[l, "p_{32}(n)"] \arrow[l, shift left=2, "p_{21}(n)"] \end{tikzcd} .$$

An n-connection is an isomorphism

$$\varphi : P_1(n)^*(E) \xrightarrow{\sim} P_2(n)^*E$$

satisfying the cocycle condition

$$P_{31}^*(\varphi) = P_{32}^*(\varphi)P_{21}^*(\varphi) .$$

A stratification on E is a system of an n-connection for each positive integer n so that if $n \leq n'$, the n'-connection induces the given n-connection.

We need two facts:

(2.1.2) In the case k is the field of complex numbers $\boldsymbol{C}$, E has a stratification if and only if E has an integrable connection.

(2.1.3) In the case the characteristic of k is positive, then E has a stratification if and only if E descends through any power of Frobenius

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endomorphism F^m of X. Sketch of the proof. Consider the fibre product $X^{(m)}$:

$$\begin{tikzcd} X^{(m)} \arrow[r, "q_1"] \arrow[d, "q_2"'] & X \arrow[d, "F^m"] \\ X \arrow[r, "F^m"'] & X . \end{tikzcd}$$

Then $X^{(m)}$ is a closed subscheme of $X \times_k X$ defined by the ideal $I^{(m)}$ where I is the ideal defining the diagonal $\Delta^1(1)$ and $I^{(m)}$ is the m-th Frobenius power, the ideal generated by the p^m-th power of all the elements of I.

Let E be a vector bundle descending through F^m for any m. Then it induces an isomorphism $q_1^*E \xrightarrow{\sim} q_2^*E$ satisfying the cocycle condition. For any integer n, if we take m sufficiently large, we have $\Delta^1(n) \subsetneq X^{(m)}$ since $\Delta^1(n)$ defined by I^n. The isomorphism $q_1^*E \xrightarrow{\sim} q_2^*E$ induces an isomorphism

$$p_1(n)^*E = q_1^*E_{|\Delta^1(n)} \xrightarrow{\sim} q_2^*E_{|\Delta^1(n)} = p_2(n)^*E$$

satisfying the cocycle condition hence it defines an n-connection. Hence E has a stratification.

Conversely, suppose E has a stratification, for any n, we have $I^n \supset I^{(m)}$ provided m is sufficiently large. Hence we have $X^{(m)} \subsetneq \Delta^1(n)$. This immersion and the isomorphism $p_1(n)^*E \xrightarrow{\sim} p_2(n)^*E$ give a descent data on E. By the descent theory (1.2), E descends through F^m for any m.

DEFINITION (2.2). Let G be a complex Lie group. Let B be a closed normal subgroup of G such that the quotient G/B is a complex torus T. Hence G is a principal B-bundle over T. Let $\rho: B \to GL(r, \boldsymbol{C})$ be a representation of B. Let $P = G \times {}^B GL(r, \boldsymbol{C})$ be the principal $GL(r, \boldsymbol{C})$-bundle over T associated with this representation. Let E be the vector bundle over T associated with the principal $GL(r, \boldsymbol{C})$-bundle P. We say that the vector bundle E is associated with the Lie group G.

The following theorem was proven by Matsushima [7].

THEOREM (Matsushima) (2.3). *Let T be a complex torus (not necessarily an abelian variety). Let E be a vector bundle over T.*

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Then the following are equivalent.

(1) *E has a holomorphic connection.*

(2) *E has an integrable connection.*

(3) *E is associated with a Lie group.*

(4) *E is homogeneous.*

DEFINITION (2.4). Let G be a group scheme. Let B be a normal subgroup scheme such that B is affine and G/B is an abelian scheme A. By (1.1.1), G is a principal B-bundle over A. Let $\rho: B \to GL(r, k)$ be a representation. Since B and $GL(r, k)$ are affine, we can associate the principal $GL(r, k)$-bundle P over A by Lemma (1.3.1). Let E be the vector bundle associated to the principal $GL(r, k)$-bundle P. We say that the vector bundle E is associated to the group scheme G.

THEOREM (2.5). *Assume that k is of characteristic $p > 0$. Let A be an abelian variety defined over k. Let P be a principal $GL(r, k)$-bundle over A. Let E be the vector bundle associated to P. Consider the following conditions.*

(1) *E has a stratification.*

(2) *E descends through any power of the Frobenius endomorphism F^m of A.*

(3) *E is associated with a group scheme.*

(4) *E is homogeneous.*

Then (3) *and* (4) *are equivalent one another.* (1) *and* (2) *are equivalent and they imply* (3) *and* (4).

If the p-rank of A is equal to the dimension of A i.e. if the p-linear map $H^1(A, O_A) \to H^1(A, O_A)$ induced by the Frobenius endomorphism of A is bijective, then all the conditions are equivalent.

Proof. By (1.2), (1) and (2) are equivalent. We shall show the equivalence of (3) and (4). Assume that E is associated to a group scheme G. We use the notation of the Definition (2.4). Let x be a point of G. Then the following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow[]{\text{multiplication by } x \text{ from the left}} & G \\ \downarrow & & \downarrow \\ A & \xrightarrow[T_x]{} & A \end{array}$$

This shows that G is homogeneous. Hence the associated vector bundle to G is also homogeneous.

Now suppose that P is homogeneous. Then we have the exact sequence

$$1 \to \mathrm{Aut}_A P \to \mathscr{G}(P) \to A \to 0 .$$

We shall show that P is associated to the group scheme $\mathscr{G}(P)$. Let r be the rank of E. We define a representation $\tilde{\alpha}: \mathrm{Aut}_A P \to GL(r, k)$ as follows:

Let x_0 be a k-valued point of P lying over 0 in A. Let T be a scheme. Consider the map

$$\begin{array}{ccc} \mathrm{Aut}_A P(T) \subset \mathrm{Hom}_k (P \times_k T, P \times_k T) & \longrightarrow & P(T) \\ \cup\!\!\!| & & \cup\!\!\!| \\ g & \longrightarrow & g(x_{0_T}) \end{array}$$

where $x_{0_T}: k \times_k T \xrightarrow{x_0 \times \mathrm{id}} P \times_k T$. Then there exists the unique element $\tilde{\alpha}(g)$ of $GL(r, k)(T)$ such that $g(x_{0_T}) = x_{0_T}^{\tilde{\alpha}(g)}$. We put

$$\begin{aligned} \tilde{\alpha}(T): \mathrm{Aut}_A P(T) &\longrightarrow GL(r, k)(T) . \\ g &\longmapsto \tilde{\alpha}(g) \end{aligned}$$

Since $\tilde{\alpha}$ is functorial we get a representation $\tilde{\alpha}: \mathrm{Aut}_A P \to GL(r, k)$. We define a morphism of fibre bundles

$$\begin{aligned} \alpha: \mathscr{G}(P) &\longrightarrow P . \\ g &\longmapsto gx_0 \end{aligned}$$

Since this is functorial and commutes with the representation $\tilde{\alpha}: \mathrm{Aut}_A P \to GL(r, k)$, we get a homomorphism of fibre bundles over A. By Lemma (1.3.1), P is the principal $GL(r, k)$-bundle associated to the group scheme $\mathscr{G}(P)$.

We prove (2) $\Rightarrow$ (4). Suppose that E descends through any power of the Frobenius endomorphism F^m of A. We denote by N_m the kernel of F^m. Then by Lemma (1.4.6), $N_m \subset H(E) \subset A$. Hence the formal groups $\hat{H}(E)$ and $\hat{A}$ are isomorphic. This implies $H(E) \simeq A$. Hence E is homogeneous.

Assume that the p-rank of A is equal to the dimension of A. We shall show (4) $\Rightarrow$ (2). Suppose that E is homogeneous. If E is decomposable, say $E = E_1 \oplus E_2 \oplus \cdots \oplus E_s$ such that E_i is indecomposable for

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$1 \leq i \leq s$, then E_i is homogeneous for $1 \leq i \leq s$. In fact if E_1 were not homogeneous, we would have a closed point x of A such that $T_x^* E_1$ is isomorphic to none of $E_i, 1 \leq i \leq s$ since $\{x \in A \mid T_x^* E_1 \simeq E_i \text{ for some } i\}$ would be a closed subset of dimension strictly less than the dimension of A (cf. (1.4.5)). Then we would have $E_1 \oplus E_2 \oplus \cdots \oplus E_s \simeq E \simeq T_x^* E \simeq T_x^* E_1 \oplus T_x^* E_2 \oplus \cdots \oplus T_x^* E_s$. This is a contradiction to the Krull-Schmidt theorem (Atiyah [1]). Hence we may assume E to be indecomposable. Then by Miyanishi [8], E is isomorphic to $F_r \otimes L$ where L is a line bundle algebraically equivalent to 0 and F_r is a succesive extension of O_A by O_A:

$$
\begin{array}{ccccccccc}
 & & & & F_1 & \xrightarrow{\sim} & O_A & & \\
0 & \longrightarrow & O_A & \longrightarrow & F_2 & \longrightarrow & O_A & \longrightarrow & 0 \\
0 & \longrightarrow & F_2 & \longrightarrow & F_3 & \longrightarrow & O_A & \longrightarrow & 0 \\
 & & & & \vdots & & & & \\
0 & \longrightarrow & F_{r-1} & \longrightarrow & F_r & \longrightarrow & O_A & \longrightarrow & 0 .
\end{array}
$$

Since L descends through any power of the Frobenius endomorphism F^m of A, it is sufficient to show the

LEMMA (2.6). *If the p-linear map $H^1(A, O_A) \to H^1(A, O_A)$ induced by F is bijective, then F_r descends to $F_r^{(m)}$, $r \geq 1$, through any power of the Frobenius endomorphism F^m of A and the p^m-linear map $H^i(A, F_r^{(m)}) \to H^i(A, F_r)$ induced by F^m is bijective for any $m, i \geq 0$.*

Proof. Induction on r. If $r = 1$, then $F_1 \simeq O_A$ descends and by hypothesis the map $H^1(A, O_A) \to H^1(A, O_A)$ is bijective. Since $H^\cdot(A, O_A) = \Lambda^\cdot H^1(A, O_A)$, we conclude that the map $H^i(A, O_A) \to H^i(A, O_A)$ is bijective for any $i \geq 0$. Suppose that the assertion is proven for r. We have exact sequence

$$
(*) \qquad 0 \longrightarrow F_{r-1} \xrightarrow{\alpha} F_r \xrightarrow{\beta} O_A \longrightarrow 0 .
$$

Since the extension is determined by an element of $H^1(A, F_{r-1})$, F_{r-1} descends to $F_{r-1}^{(m)}$ and since F^m induces a bijective morphism $H^1(A, F_{r-1}^{(m)}) \to H^1(A, F_{r-1})$ by induction hypothesis, F_r descends to an extension of O_A by $F_{r-1}^{(m)}$,

$$
(**) \qquad 0 \longrightarrow F_{r-1}^{(m)} \xrightarrow{\alpha^{(m)}} F_r^{(m)} \xrightarrow{\beta^{(m)}} O_A \longrightarrow 0 .
$$

By (*), (**) and induction hypothesis we have

$$\begin{array}{ccccccccccc}
\longrightarrow H^{i-1}(O) & \xrightarrow{\delta} & H^i(F^{(m)}_{r-1}) & \xrightarrow{\alpha^{(m)}} & H^i(F^{(m)}_r) & \xrightarrow{\beta^{(m)}} & H^i(O) & \xrightarrow{\delta} & H^{i+1}(F^{(m)}_{r-1}) \longrightarrow \\
\text{bijective}\downarrow & & \text{bijective}\downarrow & & \downarrow & & \text{bijective}\downarrow & & \text{bijective}\downarrow \\
\longrightarrow H^{i-1}(O) & \xrightarrow{\delta} & H^i(F_{r-1}) & \xrightarrow{\alpha} & H^i(F_r) & \xrightarrow{\beta} & H^i(O) & \xrightarrow{\delta} & H^{i+2}(F_{r-1}) \longrightarrow
\end{array}$$

Then by the five lemma, we conclude that F^m induces a bijective p^m-linear map $H^i(A, F_j^{(m)}) \to H^i(A, F_r^{(m)})$ for any i. This is what we had to show.

Remark (2.7). If the p-rank of A is not equal to the dimension of A, the conditions (2) and (4) in the Theorem (2.5) are not always equivalent. For example, let A be an elliptic curve whose p-rank is 0. Let F_2 be the unique extension of O_A by O_A with $H^0(A, F_2) \neq 0$ (cf. Atiyah [2]). Then F_2 is homogeneous. But F_2 does not descent through the Frobenius endomorphism F of A.

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