12

DLCQ and the spectra of QCD with fundamental and adjoint fermions

12.1 Discretized light-cone quantization

So far we have analyzed the mesonic spectra of QCD in low dimensions using the methods of the large N_c limit ('t Hooft model), and bosonization or currentization. In both these methods we have chosen a light-cone gauge and implemented a correlated light-front quantization. To get a broader perspective of the spectra in this framework, and in particular to extract the mesonic spectra using fermionic degrees of freedom with finite N_c, we now invoke another tool, the discrete light-cone quantization. We first describe the method and then apply it to both QCD with fundamental quarks and with adjoint quarks.

The discretized light-cone quantization (DLCQ) is a method devised to compute spectra and wave functions of physical states of quantum field theories.[1] It is based on the following ingredients:

- A Hamiltonian formulation of the theory.
- Calculations in momentum representation.
- Periodic boundary conditions and hence discretized momenta.
- Light-front quantization.

The Hamiltonian approach is used since it is more convenient for analyzing the structure of bound states. The periodic boundary conditions assure that charges associated with symmetries are strictly conserved. For a conserved current $\partial^+ J_+ + \partial^- J_- = 0$ the light-front charge is conserved,

$$Q(x^+) \equiv \int_{-L}^{L} \mathrm{d}x^- J^+(x^-, x^+) \quad \frac{\mathrm{d}Q(x^+)}{\mathrm{d}x^+} = 0, \tag{12.1}$$

provided that,

$$J^+(x^+, +L) - J^+(x^+, -L) = 0, \tag{12.2}$$

which is guaranteed by the periodic boundary conditions.

Note that, the light-front plane of constant x^+, serving as "time", is generally called "light-cone quantization", although the plane is only tangential to the light cone. In general d-dimensional space-time one may view the DLCQ

[1] For reviews see [48] and [47].

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approach as a projection into non-relativistic dynamics since for a fixed light-cone momentum P^+ the Hamiltonian $H = P^-$ is quadratic in the transverse momenta $H = \frac{P_i^2}{2P^+} + \frac{M^2}{2P^+}$ where P^i are the transverse momenta.

In spite of a lot of progress in handling problems faced by the DLCQ there are still several unresolved issues such as:

- There is no proof that the light-front dynamics is fully equivalent to that of ordinary time evolution, in particular for massless chiral fermions.
- There are subtleties with the renormalization of a Hamiltonian matrix with a cutoff such that all physical results are independent of the cutoff.
- The implementation of a proper quantization for the system, with constraints which emerge from gauge fixing.

12.2 Application of DLCQ to QCD_2 with fundamental fermions

Instead of describing the method in general, we demonstrate the application of the DLCQ method to the case of QCD_2 with fundamental fermions. The light-front action of two-dimensional $SU(N)$ YM gauge fields coupled to Dirac fermions in the fundamental representation of the gauge group in the light-cone gauge $A^+ = 0$ reads,[2]

$$S_{QCD_2} = \int \mathrm{d}x^+ \mathrm{d}x^- \frac{1}{2} \mathrm{Tr}\left[(\partial_- A_+)^2 + \bar{\Psi}\left(i\,\not{\partial} - m - \frac{g}{\sqrt{N}}\gamma_- A_+ \right)\Psi \right)\Big], \quad (12.3)$$

where $\Psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \bar{\psi}_{\mathrm{R}} \end{pmatrix}$, with ψ_L and $\bar{\psi}_R$ are Weyl fermions, the trace is over the color indices which are not written explicitly. To simplify the analysis we restrict ourselves to the case of a single flavor.

The corresponding equations of motion take the form,

$$i\partial_- \psi_{\mathrm{L}} = m\psi_{\mathrm{R}}, \quad (i\partial_+ + gA_+)\psi_{\mathrm{R}} = m\psi_{\mathrm{L}}, \quad \partial_-^2 A_+^a = g\psi_{\mathrm{R}}^\dagger T^a \psi_R. \quad (12.4)$$

One can then express ψ_{L} and A_+ in terms of ψ_{R} only,

$$\begin{aligned} \psi_{\mathrm{L}}(x^-, x^+) &= \frac{-im}{2}\int_{-L}^{+L} \mathrm{d}y^- \epsilon(x^- - y^-)\psi_{\mathrm{R}}(x^+, y^-) \\ A_+^a(x^-, x^+) &= \frac{g}{2}\int_{-L}^{+L} \mathrm{d}y^- |x^- - y^-| \psi_{\mathrm{R}}^\dagger T^a \psi_{\mathrm{R}}(x^+, y^-), \end{aligned} \quad (12.5)$$

where $\epsilon(x^- - y^-)$ is $+1$ for positive argument and -1 for negative.

[2] The application of the discrete light-front quantization to two-dimensional QCD was done in [128] and in [127].

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The light-cone momentum and energy are given by,

$$\begin{aligned}
P^+ &= \int_{-\mathrm{L}}^{+\mathrm{L}} \mathrm{d}x^- \psi_{\mathrm{R}}^\dagger i\partial_- \psi_{\mathrm{R}}(x^+, x^-) \\
P^- &= \frac{-im^2}{2} \int_{-\mathrm{L}}^{+\mathrm{L}} \mathrm{d}x^- \int_{-\mathrm{L}}^{+\mathrm{L}} \mathrm{d}y^- \psi_{\mathrm{R}}^\dagger(x^-)\epsilon(x^- - y^-)\psi_{\mathrm{R}}(y^-) \\
&\quad - \frac{g^2}{2} \int_{-\mathrm{L}}^{+\mathrm{L}} \mathrm{d}x^- \int_{-\mathrm{L}}^{+\mathrm{L}} \mathrm{d}y^- \psi_{\mathrm{R}}^\dagger(x^-) T^a \psi_{\mathrm{R}}(x^-)|x^- - y^-|\psi_{\mathrm{R}}^\dagger(y^-) T^a \psi_{\mathrm{R}}(y^-).
\end{aligned} \tag{12.6}$$

When one substitutes into these expressions the expansion of the fields in anti-commuting modes, subjected to anti-periodic boundary conditions, one gets,

$$P^+ = \frac{2\pi}{L} \sum_{n=\frac{1}{2},\frac{3}{2},\ldots} n(b_n^\dagger b_n + d_n^\dagger d_n), \tag{12.7}$$

where,

$$\psi_{\mathrm{R}}(x^-) = \frac{1}{\sqrt{2L}} \sum_{n=\frac{1}{2},\frac{3}{2},\ldots} \left[b_n \mathrm{e}^{-i\frac{n\pi x^-}{L}} + d_n^\dagger \mathrm{e}^{i\frac{n\pi x^-}{L}}\right]. \tag{12.8}$$

The creation and annihilation operators $b_n^\dagger, d_n^\dagger, b_n, d_n$ are all taken to be in the fundamental representation of $SU(N)$ and obey the usual algebra,

$$\{b_n^\dagger, b_m\} = \delta_{nm} \quad \{d_n^\dagger, d_m\} = \delta_{nm}. \tag{12.9}$$

Since the eigenvalues of the momentum are proportional to $\frac{2\pi}{L}$ it is natural to define a dimensionless momentum,

$$K = \frac{L}{2\pi} P^+. \tag{12.10}$$

Similarly one defines a dimensionless Hamiltonian,

$$H \equiv \frac{2\pi}{L}\frac{1-\hat{\lambda}^2}{m^2} P^- \quad \hat{\lambda} \equiv \sqrt{\frac{1}{1+\frac{\pi m^2}{g^2}}}, \tag{12.11}$$

where we introduce the dimensionless coupling $\hat{\lambda}$. The rationale behind this parameterization is that the spectrum and wave function depend, apart from an overall mass scale, only on the ratio of $\frac{g}{m}$. The Hamiltonian H is decomposed into a free kinetic term H_0 and the potential V,

$$H = (1-\hat{\lambda}^2)H_0 + \hat{\lambda}^2 V, \tag{12.12}$$

where,

$$H_0 = \sum_{n=\frac{1}{2},\frac{3}{2},\ldots} \frac{1}{n}(b_n^\dagger b_n + d_n^\dagger d_n), \tag{12.13}$$

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and,

$$V = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} J^a(k) \frac{1}{k^2} J^a(-k), \quad k \neq 0 \tag{12.14}$$

with the currents given by,

$$J^a(k) = \sum_{k=-\infty}^{\infty} [\theta(n) b_n^\dagger + \theta(-n) d_n] T^a [\theta(n-k) b_{n-k}^\dagger + \theta(k-n) d_{k-n}]. \tag{12.15}$$

We already have the expression for the potential (11.16).

Note that there is no contribution from $k = 0$, since we apply the Hamiltonian P^- only on singlet states, and $j^a(0)$ on these vanishes. Normal ordering of the potential gives,

$$V =: V : + \frac{\hat{\lambda}^2 C_{\mathrm{F}}}{\pi} \sum_{n=\frac{1}{2},\frac{3}{2},\ldots} \frac{I_n}{n} (b_n^\dagger b_n + d_n^\dagger d_n), \tag{12.16}$$

where $C_{\mathrm{F}} = \frac{N^2-1}{2N}$ is the second Casimir operator in the fundamental representation, and I_n is the self-induced inertia $I_n = -\frac{1}{2n} + \sum_{m=1}^{n+1/2} \frac{1}{m^2}$. The self-induced inertia terms cancel the infrared singularity in the interaction term in the continuum limit. $: V :$ involves a sum of eight quartic terms in the fermionic creation and annihilation operators. For instance one such term is,

$$-\frac{1}{4N} \left(N \delta_{c_2}^{c_1} \delta_{c_4}^{c_3} - \delta_{c_4}^{c_1} \delta_{c_2}^{c_3} \right) \frac{1}{(n_4 - n_2)^2} \delta_{n_4 - n_2 + n_3 - n_1, 0} b^{\dagger}{}_{n_4}^{c_4} b^{\dagger}{}_{n_3}^{c_3} b_{n_2, c_2} b_{n_1, c_1}, \tag{12.17}$$

where c_i are the color indices that have been suppressed before and there is an implicit summation over the half integers n_i such that the momentum is conserved. Now since P^- and P^+ (or H and K) commute they can be diagonalized simultaneously. One fixes the value of $K = 1, 2, 3 \ldots$ and the corresponding Fock space is finite dimensional. One then diagonalizes H in the restricted subspace of gauge singlets such that the masses are given by,

$$M^2 = 2P^+ P^- = \frac{2m^2}{1 - \hat{\lambda}^2} K H(K). \tag{12.18}$$

Notice that the dependence of the invariant masses on L the size of the space drops out.

In Fig. 12.1 the DLCQ spectrum of low-lying mesons is drawn as a function of m/g for N = 2,3,4 and compared with the t' Hooft large N calculation. A comparison with lattice calculation is presented in Fig. 12.2.

In performing these calculations it was found that, except for very small quark masses, there is a quick convergence of the numerics. This is a manifestation of the fact that the lowest Fock states dominate the hadronic state. It was found out that typically the momentum carried by sea quarks is less than one percent.

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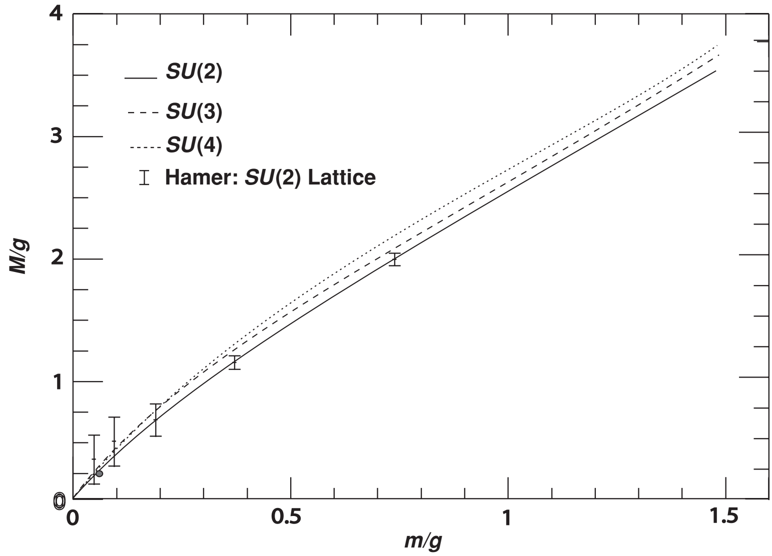

Fig. 12.1. Comparison of the DLCQ meson spectra for N = 2,3,4 and the spectrum derived from lattice calculations [127].

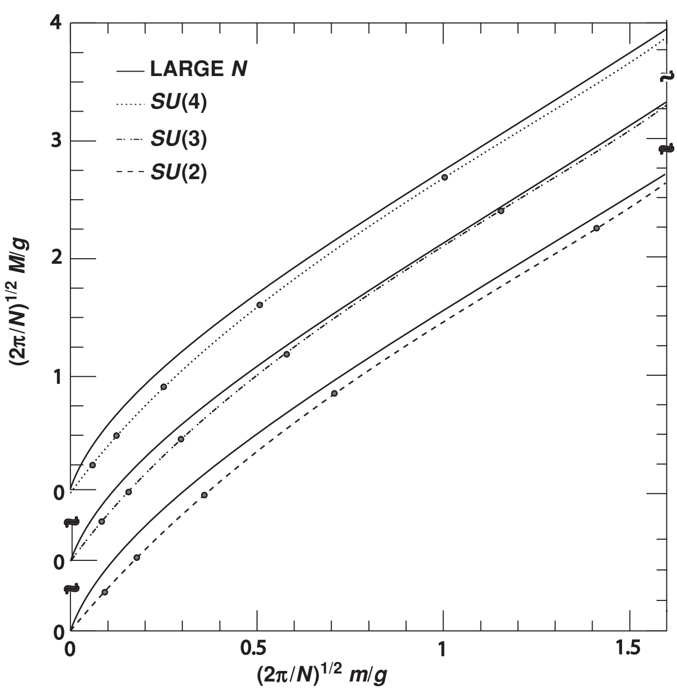

Fig. 12.2. Comparison of the DLCQ meson spectra for N = 2,3,4 and the 't Hooft large N spectrum [127].

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Several further properties were extracted from the DLCQ spectrum and wave functions:

- The scaling of the lightest mesonic and baryonic masses with N. It was found that there is fair agreement with the result deduced from bosonization for small $\frac{m}{g}$, namely, that
$$\frac{M_{\text{meson}}}{M_{\text{baryon}}} = 2\sin\left[\frac{\pi}{2(2N-1)}\right]. \tag{12.19}$$
The results "measured" were found to be $1, .62(5), .46(4)$ for $N = 2, 3, 4$ comparing with bosonization result $1, .618, .445$. In the large N limit this result implies that the baryon mass is proportional to N times the mass of the meson.
- The mesonic form factors were shown to be in accordance with analytical work.
- The "deuteron", a loosely bound state of two nucleons, was shown to be stable, in QCD_2 with two colors and two flavors.
- The "anti Pauli-blocking" effect, for which the sea quarks with the same flavor as that of the majority of the valence ones, are not suppressed in spite of their fermionic nature.

12.3 The spectrum of QCD_2 with adjoint fermions

Our starting point is the action of two-dimensional $SU(N)$ YM theory coupled to Majorana fermions in the adjoint representation.[3] The latter is expressed in terms of a traceless Hermitian matrix ψ_{ij}. The action reads,

$$\begin{aligned} S_{\text{adj}} &= \int \mathrm{d}^2x \mathrm{Tr}\left[i\psi^T\gamma^0\gamma^\mu D_\mu\psi - m\psi^T\gamma^0\psi - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}\right] \\ &= \int \mathrm{d}x^+\mathrm{d}x^-\mathrm{Tr}\left[i(\psi\partial_+\psi + \bar\psi\partial_-\bar\psi) - i\sqrt{2}m\bar\psi\psi + \frac{1}{2g^2}(\partial_-A_+)^2 + A_+J^+\right], \end{aligned} \tag{12.20}$$

where we have parameterized the Majorana fermions as follows,

$$\psi_{ij} = \frac{1}{\sqrt{\sqrt{2}}}\begin{pmatrix}\psi_{ij}\\ \bar\psi_{ij}\end{pmatrix} \qquad J^+_{ij} = 2\psi_{ik}\psi_{kj}, \tag{12.21}$$

where ψ and $\bar\psi$ are Weyl Majorana spinors written as $N\times N$ traceless Hermitian matrices. In the second line we have imposed the light-cone gauge $A_- = A^+ = 0$ and used $\gamma^0 = \sigma_2$ and $\gamma^1 = i\sigma_1$. Note that the action does not include time (x^+) derivatives of A_+ and of $\bar\psi$ and hence both of them are non-dynamical. The equal time (x^+) anti-commutation relation for the dynamical Majorana fermions is given by,

$$\{\psi_{ij}(x^-), \psi_{kl}(y^-)\} = \frac{1}{2}\delta(x^- - y^-)\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right). \tag{12.22}$$

[3] Two-dimensional QCD with adjoint fermions was analyzed in several papers. Here we follow [38], [72], [147].

In analogy to the expression of P^+ and P^- given for the fundamental fermions in Section 12.2 and for the bosonized case in Section 11.3, we have now,

$$P^+ = \int \mathrm{d}x^- \mathrm{Tr}[i\psi\partial_-\psi]$$
$$P^- = \int \mathrm{d}x^- \mathrm{Tr}\left[\frac{-im^2}{2}\psi\frac{1}{\partial_-}\psi - \frac{1}{2}g^2 J^+ \frac{1}{\partial_-^2}J^+\right]. \qquad (12.23)$$

Denoting by Φ the physical states of the system, that obey the zero charge condition,

$$\int \mathrm{d}x^- J^+ |\Phi> = 0, \qquad (12.24)$$

which is simultaneously an eigenstate of both P^+ and P^- since $[P^+, P^-] = 0$, the spectrum is then determined as usual in the light-cone quantization via,

$$2P^+P^-|\Phi> = M^2|\Phi> . \qquad (12.25)$$

Next we introduce the mode expansion and transform the expressions from the configuration space to the space of momenta. In Section 12.2 we have done this directly in the discretized formalism. Here for completeness we first consider a continuous momentum and then perform the discretization.

The mode expansion reads,

$$\psi_{ij}(x^-) = \frac{1}{2\sqrt{\pi}}\int_0^\infty \mathrm{d}k^+[b_{ij}(k^+)\mathrm{e}^{-ik^+x^-} + b_{ij}^\dagger(k^+)\mathrm{e}^{ik^+x^-}], \qquad (12.26)$$

and the non-trivial part of the algebra of the creation and annihilation operators is given by,

$$\{b_{ij}(k^+), b_{kl}(q^+)\} = \frac{1}{2}\delta(k^+ - q^+)\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right). \qquad (12.27)$$

From here on we will omit the + of k^+ and denote it as k. Plugging the mode expansion into (12.23) we get,

$$P^+ = \int_0^\infty \mathrm{d}k\ k b_{ij}^\dagger(k)b_{ij}(k), \qquad (12.28)$$

and

$$P^- = \frac{m^2}{2}\int_0^\infty \frac{\mathrm{d}k}{k}b_{ij}^\dagger(k)b_{ij}(k) + \frac{g^2N}{\pi}\int_0^\infty \mathrm{d}kC(k)b_{ij}^\dagger(k)b_{ij}(k)$$
$$+ \frac{g^2N}{2\pi}\int_0^\infty \mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4[A(k_i)\delta(k_1+k_2-k_3-k_4)b_{kj}^\dagger(k_3)b_{ji}^\dagger(k_4)b_{kl}(k_1)b_{li}(k_2)$$
$$+ B(k_i)\delta(k_1+k_2+k_3-k_4)(b_{kj}^\dagger(k_4)b_{kl}(k_1)b_{li}(k_2)b_{ij}(k_3)$$
$$- b_{kj}^\dagger(k_1)b_{jl}^\dagger(k_2)b_{li}^\dagger(k_3)b_{ki}(k_4))], \qquad (12.29)$$

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where

$$A(k_i) = \frac{1}{(k_4 - k_2)^2} - \frac{1}{(k_1 + k_2)^2},$$
$$B(k_i) = \frac{1}{(k_2 - k_3)^2} - \frac{1}{(k_1 + k_2)^2},$$
$$C(k) = \int_0^k \mathrm{d}p \frac{k}{(p-k)^2}. \tag{12.30}$$

From these expressions of P^+ and P^- it is obvious that the vacuum is annihilated by both P^+ and P^-,

$$P^+|0\rangle = 0 \quad P^-|0\rangle = 0. \tag{12.31}$$

The bosonic and fermionic states of the system take the following form,

$$\begin{aligned} |\Phi_b(p^+)\rangle &= \sum_{j=1}^{\infty} \int_o^{P^+} \mathrm{d}k_1 \ldots \mathrm{d}k_{2j} \delta\left(\sum_{i=1}^{2j} k_i - P^+\right) \\ &\quad f_{2j}(k_1, k_2, \ldots k_{2j}) N^{-j} \mathrm{Tr}[b^\dagger(k_1) \ldots b^\dagger(k_{2j})]|0\rangle, \\ |\Phi_f(p^+)\rangle &= \sum_{j=1}^{\infty} \int_o^{P^+} \mathrm{d}k_1 \ldots \mathrm{d}k_{2j+1} \delta\left(\sum_{i=1}^{2j} k_i - P^+\right) \\ &\quad f_{2j}(k_1, k_2, \ldots k_{2j}) N^{-j} \mathrm{Tr}[b^\dagger(k_1) \ldots b^\dagger(k_{2j+1})]|0\rangle, \end{aligned} \tag{12.32}$$

where the wave functions obey the cyclicity relation due to the fermionic nature of the creation and annihilation operators,

$$f_i(k_2, k_3, \ldots, k_i, k_1) = (-1)^{i-1} f_i(k_1, k_2, \ldots k_i). \tag{12.33}$$

Unlike the case of fundamental fermions, pairs of adjoint fermions are not suppressed by additional factor of $\frac{1}{N}$ and hence the eigenstates are generated by applying operators on the vacuum with a mixture of different numbers of creation operators. This renders the extraction of the spectrum for adjoint fermions much harder to determine than that of the fundamental ones. These states are obviously eigenstates of P^+. We will have to ensure that they are also eigenstates of P^-. Following the same procedure as for 't Hooft's model of Chapter 10 and of Chapter 11 one derives a set of equations for the wavefunctions f_i by applying (12.25) on the bosonic and fermionic eigenstates which take the form,

$$\begin{aligned} &M^2 f_i(x_1, x_2, \ldots x_i) = \frac{m^2}{x_1} f_i(x_1, x_2, \ldots x_i) \\ &+ \frac{g^2 N}{\pi (x_1 + x_2)^2} \int_0^{x_1 + x_2} \mathrm{d}y f_i(y, x_1 + x_2 - y, x_3, \ldots x_i) \\ &\frac{g^2 N}{\pi} \int_0^{x_1 + x_2} \frac{\mathrm{d}y}{(x_i - y)^2} [f_i(x_1, x_2, \ldots x_i) - f_i(y, x_1 + x_2 - y, x_3, \ldots x_i)] \end{aligned}$$

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$$\frac{g^2N}{\pi}\int_0^{x_1}\mathrm{d}y\int_0^{x_1-y}\mathrm{d}z f_{i+2}(y,z,x_1-y-z,x_2,\dots x_i)\left[\frac{1}{(y+z)^2}-\frac{1}{(x_1-y)^2}\right]$$
$$\left[\frac{1}{(x_1+x_2)^2}-\frac{1}{(x_2+x_3)^2}\right]\pm\text{cyclic}, \tag{12.34}$$

where $x_i=\frac{k_i^+}{P^+}$ and the last term of the equation stands for cyclic permutations of $(x_1,x_2,\dots,x_i)$ which for odd i comes with a $+$ sign and for even i with alternating signs. Similar to what happens in the 't Hooft model, the equation does not have an ambiguity once we incorporate a principal value prescription to the Coulomb double pole since at $x_1=y$ the numerator also vanishes.

At this point we implement the idea of discretizing the light-cone momenta in the following way,

$$x\rightarrow\frac{n}{K}\qquad\int_0^1\mathrm{d}x\rightarrow\frac{2}{K}\sum_{\text{odd }n>0}^{K}, \tag{12.35}$$

where n is an odd positive integer and $K\rightarrow\infty$ is the continuum limit. The constraint $\sum_{j=1}^{i}x_j=1$ eliminates all states with over K partons, where a parton is a state created from the vacuum by a single creation operator. In this way the discretized eigenvalue problem becomes finite dimensional. With this discretization the Fourier transform (12.26) translates into a sum,

$$\psi_{ij}(x^-)=\frac{1}{2\sqrt{\pi}}\sum_{\text{odd }n>0}[b_{ij}(n)\mathrm{e}^{-ik^+x^-}+b_{ij}^\dagger(n)\mathrm{e}^{ik^+x^-}]. \tag{12.36}$$

Similar to (12.8), the creation and annihilation operators of (12.27) also take discretized values, and obviously the Dirac delta function in (12.27) is replaced by a Kronecker delta function. The eigenvalue problem now reads,

$$2P^+P^-=K\left[\frac{g^2N}{\pi}T+m^2V\right], \tag{12.37}$$

where the mass term is given by,

$$V=\sum_n\frac{1}{n}b_{ij}^\dagger(n)b_{ij}(n), \tag{12.38}$$

and,

$$T=4\textstyle\sum_n b_{ij}^\dagger(n)b_{ij}(n)\sum_m^{n-2}\frac{1}{(n-m)^2}+$$
$$\tfrac{2}{N}\textstyle\sum_m\{\delta_{n_1+n_2,n_3+n_4}\left[\frac{1}{(n_4-n_2)^2}-\frac{1}{(n_1+n_2)^2}\right]b_{kj}^\dagger(n)b_{ji}^\dagger(n)b_{kl}(n)b_{li}(n)$$
$$+\ \delta_{n_1+n_2+n_3,n_4}\left[\frac{1}{(n_3+n_2)^2}-\frac{1}{(n_1+n_2)^2}\right]$$
$$b_{kj}^\dagger(n_4)b_{kl}(n_1)b_{li}(n_2)b_{ij}(n_3)-b_{kj}^\dagger(n_1)b_{jl}^\dagger(n_2)b_{li}^\dagger(n_3)b_{ki}(n_4)\}, \tag{12.39}$$

where all the summations are over positive odd integers.

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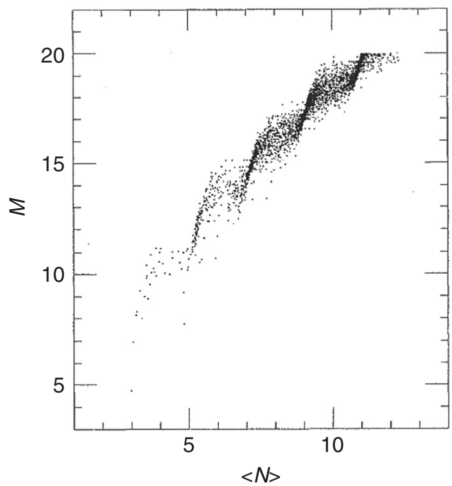

Fig. 12.3. The spectrum of fermionic states for K = 25, $m = 0$ [38].

One chooses a basis of states normalized to 1 in the large N limit,

$$\frac{1}{N^{i/2}\sqrt{s}}\mathrm{Tr}[b^\dagger(n_1)\ldots b^\dagger(n_i)]|0> \qquad \sum_{j=1}^{i} n_j = K. \tag{12.40}$$

The states are defined by ordered partitions of K into i positive odd integers, modulo cyclic permutations. If $(n_1, n_2, \ldots, n_i)$ is taken into itself by s out of i possible cyclic permutations, then the corresponding state receives a normalization factor $\frac{1}{\sqrt{s}}$. Otherwise $s = 1$. For even i, however, all partitions of K where i/s is odd do not give rise to states.

Using the discretized Hamiltonian and the basis of states (12.40) one can diagonalize the Hamiltonian and compute the spectrum for a range of values of K and then extrapolate the results to infinite K, the continuum limit. One can extract certain properties of the spectrum also from the results at a fixed large K. In particular the dependence of the spectrum on the mass of the adjoint quark m is also of interest and the special cases of $m = 0$ and $m^2 = g^2N/\pi$ where the model is supersymmetric.

The fermionic spectrum found by diagonalizing the system with $K = 25$ for the massless case and for $m^2 = \frac{g^2N}{\pi}$ is described in Figs. 12.3 and 12.4 in the form of the mass of the bound state as a function of the expectation value of the parton number. The bosonic spectrum using $K = 24$ for the two masses is drawn in Figs. 12.5 and 12.6.

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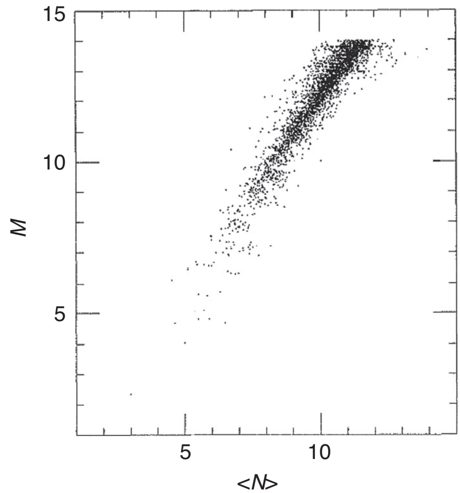

Fig. 12.4. The spectrum of fermionic states for K = 25, $m^2 = g^2/\pi$ [38].

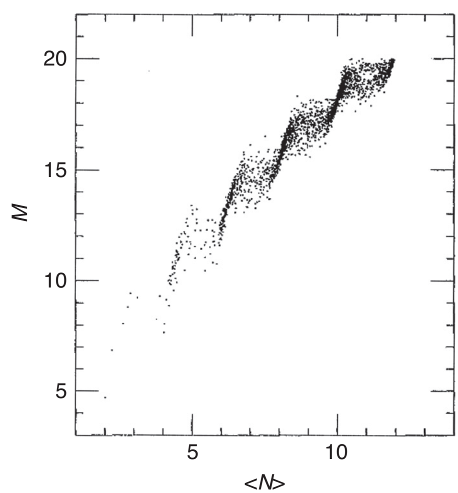

Fig. 12.5. The spectrum of bosonic states for K = 25, $m = 0$ [38].

The characteristic features of the spectra are the following:

- The density of states increases rapidly with the mass, and almost all the states lie within a band bounded by two $< N >\sim M$ lines. The system admits a Hagedorn behavior,

$$\rho(m) \sim m^{\alpha} e^{\beta m}, \tag{12.41}$$

where $\rho(m)$ is the density and from the data it follows that $\beta \sim 0.7\sqrt{\frac{\pi}{g^2 N}}$.

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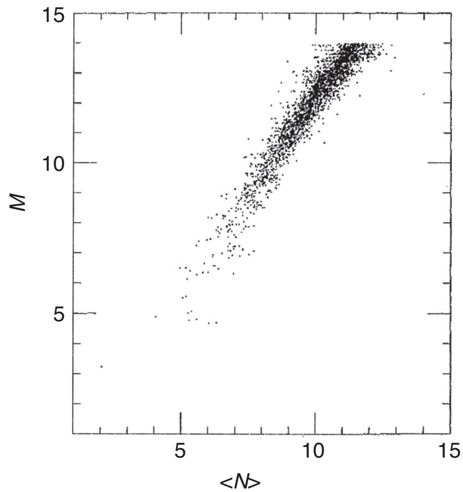

Fig. 12.6. The spectrum of bosonic states for K = 25, $m^2 = g^2/\pi$ [38].

- The mass increases roughly linearly with the average number of partons. Such a behavior characterizes a system of large N non-relativistic particles connected into a closed string by harmonic springs.
- For the low-lying states the wave function strongly peaks on states with a definite number of partons. For instance, for K = 25 the ground state has a probability of 0.9993 of consisting of 3 partons, and the first excited state has a probability of 0.99443 of consisting of 5 partons.
- Thus the low-lying states can be well approximated by truncating the diagonalization to a single parton number sector. For instance the bosonic ground state can be derived from a truncation of (12.34) to a two-parton sector which yields the following equation,

$$M^2\phi(x) = m^2\phi(x)\left(\frac{1}{x} + \frac{1}{1-x}\right) + \frac{2g^2N}{\pi}\int_0^1 \mathrm{d}y\frac{\phi(x)-\phi(y)}{(y-x)^2}, \tag{12.42}$$

  with $\phi(x) = f_2(x, 1-x)$. Note that this equation is the 't Hooft equation discussed in Chapter 10 with the replacement of $g^2 \to 2g^2$. This difference stems from the fact that unlike for mesons built from fundamental quarks, here there are two color flux tubes connecting two partons.
- Due to the fermionic statistics $\phi(x) = -\phi(1-x)$ half of the states of the 't Hooft model including the ground state are now excluded. In particular for $m = 0$ the state $\phi(x) = 1$ which associates with a massless bound state is missing. The absence of a massless ground state even in the limit of $m \to 0$ can be explained huristically as follows. For $m = 0$ the mass of the states is measured in units of the coupling constant g and hence the massless limit can

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be achieved in the strong coupling limit $g \to \infty$ for which the action takes the form,

$$S = \int \mathrm{d}^2x \mathrm{Tr}[i\Psi^T \gamma^0 \gamma^\mu \partial_\mu \Psi + A_\mu J^\mu]. \tag{12.43}$$

Now the left and right currents $J^{\pm}_{ij}$ constitute two independent level N affine Lie algebras for which we have seen in Chapter 3 the corresponding Virasoro anomaly is,

$$c = c_0 - (N^2 - 1)\frac{k}{k+N} = \frac{N^2-1}{2} - (N^2 - 1)\frac{k}{k+N}, \tag{12.44}$$

where c_0 is the central charge before gauging and k is the ALA level. Since $k = N$ it is obvious that $c = 0$ and hence there is no massless bound state. For fundamental quarks in the same limit we get, by taking $k = 1$ *and* $c_0 = N$, that $c = 1$, which means that for this case there is a massless bound state.

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