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REGIONS CUT BY ARRANGEMENTS OF TOPOLOGICAL SPHERES

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ABSTRACT. We define an arrangement of pseudohyperplanes as an image of a topological sphere arrangement with appropriate intersections, and prove that the complement components are then homologically trivial. We apply this to extend a formula of Winder and Zaslavsky.

1. **Introduction and main results.** The Winder-Zaslavsky formula (equation (1.1) below) gives a concise expression for the number of regions induced by an arrangement of hyperplanes in terms of intersection degeneracies. We place this formula in a more general topological setting by considering a version which applies to certain families of topological spheres. We then define an arrangement of pseudohyperplanes in terms of topological spheres, and show that equation (1.1) continues to hold by showing that the components of the complement of the arrangement are homologically trivial.

Suppose $A_1, \ldots, A_k$ are subsets of a topological n-sphere. We will say that $A_1, \ldots, A_k$ have *spherical intersections* if, for any nonempty $I \subseteq \{1, \ldots, k\}$, the set $A_I := \bigcap_{i \in I} A_i$ is either a single point or homeomorphic to a sphere of some dimension. (The empty set is a sphere of dimension -1.) This condition obviously holds for Euclidean spheres. If instead each A_I is a closed set having the same Čech cohomology as a sphere of some dimension, or the same cohomology as a point, we will say that the collection has *sphere-like intersections.* If A_I has the cohomology of a sphere of dimension $r \geq -1$ we will define the corresponding *degeneracy index* d_I to be $r - (n - |I|)$ and if A_I has the cohomology of a point we set $d_I = \infty$. We set $d_\emptyset = 0$.

A collection of hyperplanes in $\mathbf{R}^n$ can be regarded, via stereographic projection, as a collection of $(n-1)$-spheres $A_1, \ldots, A_k$ in $\mathbf{S}^n$, having spherical intersections, and passing through the north pole N of $\mathbf{S}^n$. The hyperplanes cut $\mathbf{R}^n$ into P regions; in our notation we have

$$P = \sum_{d_I < \infty} (-1)^{d_I} = \#\{I : d_I \text{ is even}\} - \#\{I : d_I \text{ is odd}\}. \tag{1.1}$$

This is shown in [6] when the hyperplanes intersect in a point; [7] gives the general case.

Let $\chi(A)$ denote the Euler characteristic of A.

THEOREM 1. *Let $A_1, \ldots, A_k$ be closed nonempty subsets of $\mathbf{S}^n$ having sphere-like intersections and suppose $\mathbf{S}^n \setminus \cup A_i$ has path components $C_1, \ldots, C_P$. Then*

$$\chi\Big(\mathbf{S}^n \setminus \bigcup_1^k A_i\Big) = \sum_1^P \chi(C_i) = \sum_{d_I < \infty} (-1)^{d_I}. \tag{1.2}$$

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This theorem, in a bit less topological generality, follows from Theorem 6.1 of [8], where dissection problems are treated comprehensively from a lattice theoretic viewpoint. We include a short direct proof in the next section. If *each C_i has the homology of a point, i.e.* $\tilde{H}_q(C_i) = 0$ for all q, where $\tilde{H}_*$ is reduced singular homology, then $\chi(C_i) = 1$ for each i, and (1.2) reduces to (1.1).

DEFINITION 1. An *arrangement of pseudohyperplanes* in $\mathbf{R}^n$ is the image under stereographic projection of a collection of proper subsets $A_1, \ldots, A_k$ of $\mathbf{S}^n$ having spherical intersections and such that

i) each A_i is a topological $(n-1)$-sphere which contains the pole N,
ii) for $I \subseteq \{1, \ldots, k\}$, $I \neq \emptyset$, either $A_I = \{N\}$ or A_I is a topological sphere of dimension $\geq n - |I| \geq 0$.

Clearly, k Euclidean hyperplanes form an arrangement of pseudohyperplanes.

Our definition is more general than usual since nontransverse intersections are allowed. Note that not every arrangement of pseudohyperplanes is the image under a homeomorphism of $\mathbf{R}^n$ of an arrangement of Euclidean hyperplanes, even if all intersections are transverse ([2]).

The regions of $\mathbf{R}^n$ cut by a set of Euclidean hyperplanes are convex sets and hence have the homology of a point. Our main result gives a purely topological version of this fact for arrangements of pseudohyperplanes.

THEOREM 2. *Let $A_1, \ldots, A_k$ be closed subsets of $\mathbf{S}^n$ such that for some point $Q \in \mathbf{S}^n$, $Q \in A_i$ for all i and such that for $I \subseteq \{1, \ldots, k\}$, $I \neq \emptyset$, either A_I is a topological sphere of dimension at least $n - |I| \geq 0$ or $A_I = \{Q\}$. If $\mathbf{S}^n \setminus \cup A_i \neq \emptyset$ then each component of $\mathbf{S}^n \setminus \cup A_i$ has the homology of a point.*

The hypothesis of this theorem corresponds to the *geometric intersection property* of [8]. Theorem 2 suggests that the assumption of cellular regions in [8, Theorem 3; 10, p. 70] is superfluous.

Classical enumeration formulas for regions cut by lines and hyperplanes can be derived from (1.1).

If $A_1, \ldots, A_k$ have spherical intersections in $\mathbf{S}^n$, we say they are in *general position* if any $\ell \leq n$ of them intersect in a topological sphere of dimension $n - \ell$, and the intersection of any $\ell > n$ of them is empty. In this case, $d_I = 0$ for $|I| \leq n$ and $d_I = -1 - n + |I|$ for $|I| > n$. Using elementary binomial coefficient identities we get

$$\sum_{d_I < \infty} (-1)^{d_I} = 2 \sum_{\ell=0}^{m} \binom{k-1}{\ell} =: M_{n,k}, \quad m = \min(k-1, n).$$

(Note that $M_{n,k} = 2^k$ for $k \leq n+1$.)

Moreover, when the A_i are in general position, all components of $\mathbf{S}^n \setminus \bigcup_1^k A_i$ have the homology of a point (see [4]) so $M_{n,k}$ gives the number of components. Actually, by [4], $M_{n,k}$ is an upper bound for the number of components of $\mathbf{S}^n \setminus \bigcup_1^k A_i$ for any A_i's with spherical intersections, and this upper bound is achieved if and only if the A_i's are in general position. To show that the upper bound is actually achieved it suffices to show that we can find k Euclidean spheres in general position. Take k large spheres whose centers are close together and in affine general position. By [1, Theorem 7] each $\ell \le n+1$ of them cut $\mathbf{S}^n$ into 2^ℓ regions. By [4] the ℓ spheres are therefore in general position and hence all k spheres are in general position.

2. **Proofs.** H^* (resp. $\tilde{H}^*$) will denote the (resp. *reduced*) Čech cohomology with rational coefficients. If $K \subseteq \mathbf{S}^n$ is closed, we compute the Euler characteristic $\chi(K)$ from cohomology, *i.e.* $\chi(K) = \sum_q (-1)^q \operatorname{rank} H^q(K)$, and we compute $\chi(\mathbf{S}^n \setminus K)$ from singular homology. It is convenient for our purposes that the Čech cohomology Mayer-Vietoris sequence for pairs of closed sets in $\mathbf{S}^n$ is exact without excisiveness assumptions (see [5, p. 291]). From this follows additivity of the Euler characteristic: If A, B are closed in $\mathbf{S}^n$, with all cohomology groups of A, B and $A \cap B$ finitely generated, then $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$. We will need the Alexander-Pontryagin duality theorem (see [3]): For K a closed proper subset of $\mathbf{S}^n$, $\tilde{H}_q(\mathbf{S}^n \setminus K) \approx \tilde{H}^{n-q-1}(K)$.

PROOF OF THEOREM 1. Additivity of χ implies

$$\chi\Big(\bigcup_{i=1}^{k} A_i\Big) = \sum_{I \neq \emptyset} (-1)^{|I|+1} \chi(A_I). \tag{2.1}$$

If A_I is a (cohomology) sphere then $\chi(A_I) = (-1)^{d_I + n - |I|} + 1$, and $\chi(\text{point}) = 1$. Alexander duality gives $\chi(\mathbf{S}^n \setminus K) = (-1)^{n-1}\chi(K) + 1 + (-1)^n$. Let $K = \bigcup_1^k A_i$ and substitute in (2.1) to get (1.1) after some routine manipulations. ∎

PROOF OF THEOREM 2. By Alexander duality, it suffices to show that $\tilde{H}^q(\bigcup_1^k A_i) = 0$ for $0 \le q < n-1$. We proceed by a double induction. The result is clear when $n = 1$, and for (k, n) such that $k = 1, n \ge 2$. For $n = 2$ and any k we need only show $\tilde{H}^0(\bigcup_1^k A_i) = 0$, for which we need only that $\bigcup_1^k A_i$ be connected. But if $\mathbf{S}^2 \setminus \bigcup_1^k A_i \neq \emptyset$, each A_i is a sphere of dimension 1, or is $\{Q\}$, so connectedness follows. Now suppose we have the result for all (k', n') such that either $2 \le n' < n$, $1 \le k'$ or $2 < n' = n$, $k' < k$. Let $X = A_1$ and $Y = A_2 \cup \cdots \cup A_k$. If $\mathbf{S}^n \setminus \bigcup_1^k A_i \neq \emptyset$, we can assume that $\dim A_i = n-1$ for each $i = 1, \ldots, k$ since otherwise $A_i = \{Q\}$ and we could delete it. Let $A_i' = A_1 \cap A_{i+1}$ for $i = 1, \ldots, k-1$. Then $X \cap Y = \bigcup_1^{k-1} A_i'$. None of the sets A_i or A_i' are homeomorphic to the two-point set $\mathbf{S}^0$, hence X, Y, $X \cap Y$ and $X \cup Y$ are connected; in particular $\tilde{H}^0(X \cup Y) = 0$. Also $H^0(X) = H^0(Y) = 1$ and by induction, $H^1(Y) = H^1(X) = 0$. Consider the exact Mayer-Vietoris sequence

$$0 \to H^0(X \cup Y) \to H^0(X) \oplus H^0(Y) \to H^0(X \cap Y) \to H^1(X \cup Y) \to H^1(X) \oplus H^1(Y) \to \cdots.$$

Taking ranks and using exactness we get $1 - 2 + 1 - \operatorname{rank} H^1(X \cup Y) = 0$, so $H^1(X \cup Y) = 0$. Now check that the theorem's hypotheses hold for $A_1', \ldots, A_{k-1}'$ as subsets of $A_1 \approx \mathbf{S}^{n-1}$. Using the induction hypotheses we get $\tilde{H}^q(X \cap Y) = 0$ for $0 \le q < n-2$.

Now suppose we have shown $H^q(X \cup Y) = 0$ for $1 \leq q < n - 2$. By induction we have $H^{q+1}(X) = H^{q+1}(Y) = 0$ and we have seen that $H^q(X \cap Y) = 0$. By exactness of

$$\cdots \longrightarrow H^q(X \cap Y) \longrightarrow H^{q+1}(X \cup Y) \longrightarrow H^{q+1}(X) \oplus H^{q+1}(Y) \longrightarrow \cdots$$

we get $H^{q+1}(X \cup Y) = 0$. ■

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