

ON THE SOLUTION OF A CLASS OF SECOND-ORDER QUASI-LINEAR PDES AND THE GAUSS EQUATION

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(Received 17 March 1997; revised 14 January 2000)

Abstract

Gauss' *Theorema Egregium* produces a partial differential equation which relates the Gaussian curvature K to components of the metric tensor and its derivatives. Well-known partial differential equations (PDEs) such as the Schrödinger equation and the sine-Gordon equation can be derived from Gauss' equation for specific choices of K and coördinate systems. In this paper we consider a class of Bäcklund Transformations which corresponds to coördinate transformations on surfaces with a given Gaussian curvature. These Bäcklund Transformations lead to the construction of solutions to certain classes of non-linear second order PDEs of hyperbolic type by identifying these PDEs as the Gauss equation in some coördinate system. The possibility of solving the *Cauchy Problem* has also been explored for these classes of equations.

1. Introduction

In this paper we develop a method for solving certain types of PDEs based on finding *Bäcklund Transformations* [10]. The PDE from Gauss' *Theorema Egregium* [1], the Gauss equation, plays a central rôle in this method. The basic idea is that, if a PDE can be interpreted as the Gauss equation in a particular coördinate system for a surface of known Gaussian curvature, then a solution may be obtainable by the use of Bäcklund Transformations which, in essence, can be interpreted as coördinate transformations on surfaces having the same Gaussian curvature.

Many nonlinear and some linear PDEs of interest correspond to the Gauss equation. Konopelchenko [6] showed that the Schrödinger equation

$$\psi_{xx} + K\psi = 0,$$

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where ψ and K are both functions of x and y , corresponds to the Gauss equation for surfaces of Gaussian curvature K in geodesic coördinates. The sine-Gordon equation corresponds to Gauss' equation for a pseudosphere in a *Tchebychef net* [11, 12] coördinate system.

Given a nonlinear PDE, it may be possible to interpret it as a statement of Gauss' *Theorema Egregium* for some K , which is known in terms of the given coördinates and possibly the unknown function. If this PDE is the Gauss equation in disguise, then a coördinate transformation to geodesic coördinates yields the Schrödinger equation (which may be easier to solve than the given PDE).

We use this technique to solve a class of second-order quasi-linear partial differential equations of the hyperbolic type. As an example, we obtain a family of solutions for the sine-Gordon equation.

Finally, solutions to the Cauchy problem are discussed and the sine-Gordon equation is used as an example.

2. Solving a class of second-order quasi-linear PDEs

The Gauss equation, in terms of E , F and G , the coefficients of the first fundamental form on a surface with local coördinates u and v , is

$$K(u, v) = \frac{1}{2H} \left\{ \left(\frac{F}{HE} E_v - \frac{1}{H} G_u \right)_u + \left(\frac{2}{H} F_u - \frac{1}{H} E_v - \frac{F}{HE} E_u \right)_v \right\}, \quad (1)$$

where K is the Gaussian curvature and $H^2 = EG - F^2$. Suppose E , F and G are of the form $E(\phi)$, $F(\phi)$ and $G(\phi)$. Here ϕ is some function of u and v . Under this assumption (1) becomes

$$G_\phi \phi_{uu} - 2F_\phi \phi_{uv} + E_\phi \phi_{vv} + 2KH^2 + \Theta(u, v, \phi, \phi_u, \phi_v) = 0, \quad (2)$$

where

$$\Theta(u, v, \phi, \phi_u, \phi_v) = \Theta_1(u, v, \phi)(\phi_u)^2 + \Theta_2(u, v, \phi)(\phi_v)^2 + \Theta_3(u, v, \phi)\phi_u\phi_v$$

and

$$\begin{aligned} \Theta_1(u, v, \phi) &= -\frac{\{G_\phi E_\phi G + E(G_\phi)^2 - 2G_\phi F_\phi F - 2H^2 G_{\phi\phi}\}}{2H^2}, \\ \Theta_2(u, v, \phi) &= -\frac{\{G_\phi E_\phi E + G(E_\phi)^2 - 2E_\phi F_\phi F - 2H^2 E_{\phi\phi}\}}{2H^2}, \\ \Theta_3(u, v, \phi) &= \frac{\{F_\phi E_\phi G + F_\phi G_\phi E - 2F(F_\phi)^2 - 2H^2 F_{\phi\phi}\}}{H^2}. \end{aligned}$$

Equation (2) is of the form

$$A\phi_{uu} + 2B\phi_{uv} + C\phi_{vv} + D = 0, \quad (3)$$

which is the general form for a second-order quasi-linear PDE with coefficients A , B and C depending on u , v and ϕ and D depending on u , v , ϕ , ϕ_u and ϕ_v .

If $B^2 - AC$ does not change sign, (3) can be classified into one of three types: hyperbolic, parabolic or elliptic, corresponding to $B^2 - AC > 0$, $B^2 - AC = 0$, or $B^2 - AC < 0$ respectively. We shall focus on the hyperbolic case.

3. Equations of hyperbolic type

Consider a second-order quasi-linear hyperbolic PDE of the type

$$\phi_{uv} = f(\phi, \phi_u, \phi_v). \quad (4)$$

Equation (2) will be of this form if $F_\phi \neq 0$ and either

- (a) $E_\phi = G_\phi = 0$ or
- (b) $G_\phi\phi_{uu} + E_\phi\phi_{vv} = 0$.

In this paper we shall consider only the first case and find the conditions under which (4) can be solved by our method. In this case $E = \text{constant}$ and $G = \text{constant}$.

By an appropriate scaling of the coördinates u and v we can put E and G equal and without loss of generality we thus take $E = 1$ and $G = 1$. This choice of E and G corresponds to a Tchebychef net coördinate system and (2) reduces to

$$\phi_{uv} = \frac{KH^2}{F_\phi} - \frac{1}{F_\phi} \left\{ F_{\phi\phi} + \frac{F(F_\phi)^2}{H^2} \right\} \phi_u\phi_v. \quad (5)$$

Equation (5) is of the form

$$\phi_{uv} = M(\phi) + A(\phi)\phi_u\phi_v, \quad (6)$$

where

$$M(\phi) = \frac{K(1 - F^2)}{F_\phi} \quad (7)$$

and

$$A(\phi) = \frac{-1}{F_\phi} \left\{ F_{\phi\phi} + \frac{F(F_\phi)^2}{(1 - F^2)} \right\}. \quad (8)$$

The above arguments indicate that a given second-order quasi-linear hyperbolic PDE of the form (6) can be identified as the Gauss equation, where the (u, v) coördinate

system corresponds to a Tchebychef net ($E = 1$, $G = 1$ and $F = F(\phi)$), on a surface of curvature

$$K = \frac{M(\phi)F_\phi}{1 - F^2},$$

where the function F can be determined from (8).

The angle χ between the coördinate lines on a surface described by $\mathbf{r}(u, v)$ is given by

$$\frac{\mathbf{r}_u \cdot \mathbf{r}_v}{\|\mathbf{r}_u\| \|\mathbf{r}_v\|} = \cos \chi,$$

where χ depends on ϕ ; thus

$$F = \cos \chi(\phi). \quad (9)$$

Substituting (9) into (8) and solving for χ yields

$$\chi(\phi) = \lambda \int \exp \left\{ - \int_0^\phi A(\zeta) d\zeta \right\} d\phi + \lambda_1, \quad (10)$$

where λ and λ_1 are arbitrary constants. We can thus identify the PDE given by (6) with a surface defined intrinsically by the quantities

$$E = 1, \quad F = \cos \chi(\phi), \quad G = 1 \quad \text{and} \quad K = -\frac{M(\phi)}{\sin \chi(\phi)} \chi'(\phi). \quad (11)$$

Note that if $A(\phi) = 0$, then

$$F = \cos(\lambda\phi + \lambda_1). \quad (12)$$

From (6) and (10) we obtain

$$\chi_{uv} = M_1(\chi), \quad (13)$$

where

$$M_1(\chi) = \lambda \exp \left\{ - \int_0^\phi A(\zeta) d\zeta \right\} M(\phi).$$

We can thus reduce a PDE of the form (6) to an equation of the form (13) by using a transformation defined by (10). It is therefore sufficient to investigate the case when $A(\phi) = 0$. For this case (6) reduces to

$$\phi_{uv} = M(\phi) \quad (14)$$

and from (10) and (11) we have

$$E = 1, \quad F = \cos(\lambda\phi + \lambda_1), \quad G = 1 \quad \text{and} \quad K = -\frac{\lambda M(\phi)}{\sin(\lambda\phi + \lambda_1)}. \quad (15)$$

The constants λ and λ_1 correspond to the magnification and the shift of the angle between the characteristics respectively. If we choose $\lambda = 1$ and $\lambda_1 = 0$ then we can identify a PDE of the form (14) as Gauss' equation on a surface Σ with

$$E = 1, \quad F = \cos \phi, \quad G = 1 \quad \text{and} \quad K = -\frac{M(\phi)}{\sin \phi}. \tag{16}$$

Next we derive a transformation on Σ from a geodesic coordinate system (x, y) to the (u, v) coordinate system (Tchebychef net). Let \hat{E}, \hat{F} and \hat{G} be the coefficients of the first fundamental form in the geodesic coordinate system; thus, $\hat{E} = 1$ and $\hat{F} = 0$. Moreover, the Gauss equation reduces to the Schrödinger equation

$$\hat{H}_{xx} + K(x, y)\hat{H} = 0, \tag{17}$$

where $\hat{H}^2 = \hat{E}\hat{G} - \hat{F}^2 = \hat{G}$.

A solution to (14) can be obtained provided:

- (i) a coordinate transformation between geodesic coordinates and Tchebychef net coordinates can be determined;
- (ii) the Schrödinger equation can be solved.

The metric tensor transformation properties yield the following relations between the coordinates:

$$x_u^2 + \hat{H}^2 y_u^2 = 1, \tag{18}$$

$$x_u x_v + \hat{H}^2 y_u y_v = F = \cos \phi, \tag{19}$$

$$x_v^2 + \hat{H}^2 y_v^2 = 1. \tag{20}$$

One solution to the above system of equations can be determined under the assumption that

$$x_u = x_v = \left\{ 1 - \hat{H}^2(x, y) \right\}^{1/2} \quad \text{and} \quad y_u^2 = y_v^2 = 1, \tag{21}$$

provided that $y_u \neq y_v$. Using (21) we can define a transformation implicitly by

$$x = \int \left\{ 1 - \hat{H}^2(x, y) \right\}^{1/2} du + f(v) \tag{22}$$

and

$$y = u - v, \tag{23}$$

where $f(v)$ satisfies

$$f'(v) + \frac{\partial}{\partial v} \left\{ \int \left(1 - \hat{H}^2(x, y) \right)^{1/2} du \right\} = \left\{ 1 - \hat{H}^2(x, y) \right\}^{1/2}.$$

Clearly this transformation is non-singular since the Jacobian is non-zero. We also note that $\hat{H} < 1$ unless $K = 0$, that is, $M(\phi) = 0$.

Equations (19) and (21) imply that

$$\hat{H} = \sin\left(\frac{\phi}{2}\right). \quad (24)$$

Substituting (24) into the Schrödinger equation (17) yields

$$\phi_{xx} - \frac{1}{2} \tan \frac{\phi}{2} (\phi_x)^2 = M(\phi) \sec^2 \frac{\phi}{2}. \quad (25)$$

The above equation is essentially a second-order ordinary differential equation which can be reduced to a first-order equation (using a Prüfer substitution) which in turn can be solved. Equation (25) thus yields the relation

$$x = \frac{1}{2} \int \frac{\cos(\phi/2)}{\{\int M(\phi) d\phi + c_1(y)\}^{1/2}} d\phi + c_2(y), \quad (26)$$

where $c_1(y)$ and $c_2(y)$ are arbitrary functions of y .

Now (26) defines an implicit relationship

$$\Lambda(x, y, \phi) = 0 \quad (27)$$

between x , y and ϕ . Under the assumption that one set of values x_0 , y_0 , ϕ_0 can be found to satisfy (27) and that, near (x_0, y_0, ϕ_0) , Λ and its first partial derivatives are continuous and $\partial\Lambda/\partial\phi \neq 0$, the *implicit function theorem* [8] implies that in a region of the xy plane containing (x_0, y_0) , there is precisely one differentiable function

$$\phi = \alpha(x, y) \quad (28)$$

which reduces (27) to an identity and is such that $\phi_0 = \alpha(x_0, y_0)$. Under these assumptions (21) gives, using (23), (24) and (28),

$$x_u = x_v = \cos(\alpha(x, u - v)), \quad (29)$$

where, by (26), $\alpha(x, y)$ contains the two arbitrary functions $c_1(y)$ and $c_2(y)$. Hence we have determined a coordinate transformation

$$\begin{aligned} x &= \beta(u, v), \\ y &= u - v. \end{aligned} \quad (30)$$

Note that the Jacobian $\partial(x, y)/\partial(u, v) \neq 0$, so that this transformation is invertible. The general solution to the given PDE (6) is

$$\phi = \alpha(x, y) = \alpha(\beta(u, v), u - v) = \Phi(u, v). \quad (31)$$

Example: The sine-Gordon equation The sine-Gordon equation

$$\phi_{uv} = \sin \phi, \quad (32)$$

is of the form (14) with $M(\phi) = \sin \phi$ and $K = -1$. For this choice of M (26) becomes

$$x = \frac{1}{2} \int \frac{\cos(\phi/2)}{\{-\cos \phi + c_1(y)\}^{1/2}} d\phi + c_2(y), \quad (33)$$

where $c_1(y)$ and $c_2(y)$ are arbitrary functions of y .

Integrating (33) and solving for $\phi(x, y)$ yields

$$\phi(x, y) = 2 \sin^{-1} \left\{ \sqrt{(c_1(y) - 1)/2} \sinh\{x - c_2(y)\} \right\}; \quad (34)$$

hence (24) implies that

$$\hat{H} = \left\{ \sqrt{(c_1(y) - 1)/2} \sinh\{x - c_2(y)\} \right\}, \quad (35)$$

which is a solution to the Schrödinger equation $\hat{H}_{xx} - \hat{H} = 0$. Equations (21) and (35) imply that

$$x_u = x_v = \left\{ 1 - ((c_1(y) - 1)/2) \sinh^2(x - c_2(y)) \right\}^{1/2}. \quad (36)$$

From (36) it follows that

$$u + \rho(v) = (k/2)F(\alpha, k) = (k/2) \operatorname{sn}^{-1}(\sin \alpha, k), \quad (37)$$

where $\rho(v)$ depends upon the arbitrary functions c_1 and c_2 ,

$$\begin{aligned} \rho'(v) &= \frac{\partial}{\partial v} \left\{ \frac{k}{2} \operatorname{sn}^{-1}(\sin \alpha, k) \right\}, \\ \alpha &= \sin^{-1} \left\{ (1/k) \tanh(x - c_2(y)) \right\} \end{aligned} \quad (38)$$

and

$$k = \sqrt{2/(c_1(y) + 1)}. \quad (39)$$

Here, F denotes an *elliptic function of the first kind* [3, 7]. Equations (37) and (38) indicate that

$$x = \tanh^{-1} \left\{ k \operatorname{sn} \left((2/k)(u + \rho(v)), k \right) \right\} + c_2(y), \quad (40)$$

where $y = u - v$. Using (34), (39) and (40) (after some algebraic manipulations) we thus obtain the family of solutions to the *sine-Gordon equation*:

$$\phi(u, v) = 2 \sin^{-1} \left\{ \frac{\operatorname{sn} \left((2/k)(u + \rho(v)), k \right)}{\left\{ 1 + (k^2/(1 - k^2)) \operatorname{cn}^2 \left((2/k)(u + \rho(v)), k \right) \right\}^{1/2}} \right\}, \quad (41)$$

where k is an arbitrary function of $u - v$ and $\rho(v)$ is an arbitrary function of v .

4. The Cauchy Problem for the sine-Gordon equation

The *Cauchy Problem* [5] for (6) consists of solving this PDE for given initial data along a non-characteristic curve. Let Ω be a smooth non-intersecting curve defined in the plane in parametric form by $u = U(t)$ and $v = V(t)$. Let

$$\phi_u = p \quad \text{and} \quad \phi_v = q$$

and suppose that, on Ω , the *Cauchy data* are

$$\phi = \Phi(t), \quad p = P(t) \quad \text{and} \quad q = Q(t). \quad (42)$$

For compatibility it is required that

$$\dot{\Phi}(t) = P(t)\dot{U}(t) + Q(t)\dot{V}(t),$$

where $\dot{}$ denotes differentiation with respect to t . The following system of equations must therefore be satisfied along Ω :

$$\phi_{uv} = f(\Phi(t), P(t), Q(t)),$$

$$\dot{U}(t)\phi_{uu} + \dot{V}(t)\phi_{vv} = \dot{P}(t),$$

and

$$\dot{U}(t)\phi_{uv} + \dot{V}(t)\phi_{vv} = \dot{Q}(t).$$

The Cauchy data are non-characteristic if

$$\dot{U} \neq 0 \quad \text{and} \quad \dot{V} \neq 0.$$

To solve the Cauchy problem for the sine-Gordon equation (32), we shall take ϕ as in (34) (using (x, y) geodesic coordinates) rather than as in (41) which uses the Tchebychef net (u, v) , as the calculations for ϕ_u and ϕ_v are easier this way. Equation (34) implies that

$$\phi_u + \phi_v = 4\sqrt{(c_1(y) - 1)/2} \cosh(x - c_2(y))$$

and

$$\phi_u - \phi_v = \frac{\{-4((c_1(y) - 1)/2) \cosh(x - c_2(y))c_2'(y) + \sinh(x - c_2(y))c_1'(y)\}}{2\sqrt{(c_1(y) - 1)/2} \{1 - ((c_1(y) - 1)/2) \sinh^2(x - c_2(y))\}^{1/2}}.$$

Suppose now that the initial conditions in the (u, v) coordinate system (Tchebychef net) $\phi_u = P(t)$ and $\phi_v = Q(t)$ are transformed to $P_1(t)$ and $Q_1(t)$ in the (x, y) (geodesic) coordinate system. The last two expressions indicate that

$$P_1 + Q_1 = 4\sqrt{(c_1(y_0) - 1)/2} \cosh(x_0 - c_2(y_0)) \quad (43)$$

and

$$P_1 - Q_1 = \frac{\{-4((c_1(y_0) - 1)/2) \cosh(x_0 - c_2(y_0))c'_2(y_0) + \sinh(x_0 - c_2(y_0))c'_1(y_0)\}}{\sqrt{(c_1(y_0) - 1)/2} \{1 - ((c_1(y_0) - 1)/2) \sinh^2(x_0 - c_2(y_0))\}^{1/2}}, \tag{44}$$

where $x_0(t)$ and $y_0(t)$ correspond to the initial curve in the (x, y) geodesic coordinate system. Solving the above expression for $c'_1(y_0)$ yields the first order differential equation

$$c'_1(y_0) = A \{1 - (B - c_1(y_0))^2\}^{1/2}, \tag{45}$$

where

$$A = -\frac{(P_1 - Q_1)}{2} \quad \text{and} \quad B = \left(\frac{P_1 + Q_1}{2\sqrt{2}}\right)^2. \tag{46}$$

In principle, the differential equation (45) can be solved for $c'_1(y_0)$ in terms of $c_2(y_0)$. Equation (43) can then be used to determine $c_2(y_0)$. The Cauchy Problem for the sine-Gordon equation can thus be reduced to a problem involving the solution of a nonlinear first order ODE and inversions.

For a concrete example, suppose that Ω is parametrized in the Tchebychef net coordinate system by

$$u + v = c_0 \quad \text{constant}, \tag{47}$$

and on Ω ,

$$\phi = \Phi_0, \quad p = P_0 \quad \text{and} \quad q = Q_0, \tag{48}$$

where Φ_0, P_0 and Q_0 are constants. It can be shown using (31) that the initial conditions given in (48), when transformed to the geodesic (x, y) coordinates, are of the same form, that is,

$$\phi = \Phi_1, \quad p = P_1 \quad \text{and} \quad q = Q_1, \tag{49}$$

where Φ_1, P_1 and Q_1 are constants. Now the quantities A and B (given in (46)) are also constants and (45) thus has the solution

$$c_1(y_0) = B - \sin(Ay_0 + C),$$

where the constant of integration is given by

$$C = \sin^{-1} \{2 \sin^2(\Phi_1/2) - 1\} - Ay_0.$$

Generally,

$$c_1(y) = B - \sin(Ay + C) \quad (50)$$

and, by (43),

$$c_2(y) = x_0 - \cosh^{-1} \sqrt{\frac{B}{(B-1) - \sin(Ay + C)}}. \quad (51)$$

Hence a solution for the Cauchy Problem of the sine-Gordon equation in geodesic coordinates is given by (34), where $c_1(y)$ and $c_2(y)$ are defined by (50) and (51) respectively. To get a solution in terms of the original coordinates we can use (23) and (26) to express x and y in terms of u and v .

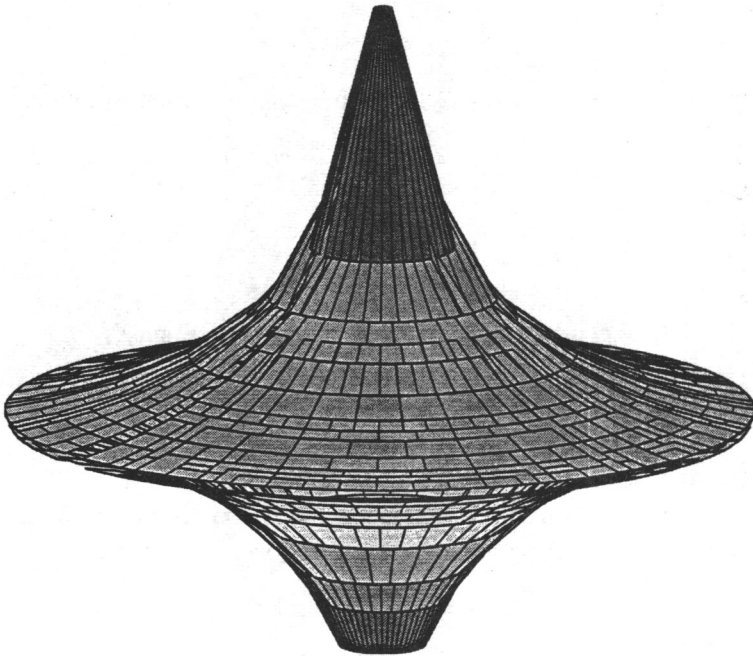


FIGURE 1. Beltrami surface in the (X, Y) coordinate system

Though perhaps not obvious, this solution corresponds to a *Beltrami surface* [9] (see Figure 1). Recall that the Beltrami surface is a surface of revolution described by

$$\mathbf{r}(X, Y) = (\sin X \cos Y, \sin X \sin Y, \cos X + \ln \tan(X/2)),$$

where the (X, Y) coordinates correspond to the lines of curvature.

The coefficients E , F and G of the first fundamental form and the Gaussian curvature K in this coordinate system are

$$E = \cot^2 X, \quad F = 0, \quad G = \sin^2 X \quad \text{and} \quad K = -1,$$

and thus $H = \sqrt{EG - F^2} = \cos X$.

A transformation between the (X, Y) coördinate system and the Tchebychef net (u, v) coördinate system is defined by the system

$$X_u^2 \cot^2 X + Y_u^2 \sin^2 X = 1, \quad (52)$$

$$X_u X_v \cot^2 X + Y_u Y_v \sin^2 X = \hat{F} = \cos \phi, \quad (53)$$

$$X_v^2 \cot^2 X + Y_v^2 \sin^2 X = 1, \quad (54)$$

and a solution to this system can be determined under the assumption that

$$Y_u^2 = Y_v^2 = 1.$$

Choosing

$$Y = u - v, \quad (55)$$

(52) and (54) give $X_u = X_v = \sin X$. A suitable transformation is defined by (55) and

$$u + v = \ln |\csc X - \cot X|. \quad (56)$$

Equation (53) implies

$$\phi = 2X; \quad (57)$$

further,

$$\phi_u = \phi_X X_u + \phi_Y Y_u = 2 \sin X, \quad \phi_v = 2 \sin X \quad (58)$$

and consequently

$$\phi_{uv} = 2 \sin X \cos X = \sin 2X = \sin \phi,$$

which is the sine-Gordon equation.

Any curve corresponding to $X = \text{constant}$ is a non-characteristic curve on the pseudosphere. Using the transformation equation (56), we have that

$$u + v = c_0 \quad (\text{constant}). \quad (59)$$

This non-characteristic curve in (u, v) coördinates (Tchebychef net) can be taken as the initial curve for the Cauchy Problem of the sine-Gordon equation. Also using (57) and (58) we see that

$$\phi = c_1, \quad \phi_u = c_2 \quad \text{and} \quad \phi_v = c_3, \quad (60)$$

where c_1, c_2 and c_3 are constants.

The last two equations are the same initial conditions as in (47) and (48). Since the Cauchy data are analytic the *Cauchy-Kowalewski theorem* [2] guarantees a unique solution. Thus the solution which we had for the Cauchy problem of the sine-Gordon equation corresponds to the *Beltrami surface*; the surface representation of this Cauchy Problem is now evident.

5. Conclusions

The purpose of this paper is to use classical differential geometry in order to find Bäcklund Transformations and hence solve certain classes of non-linear partial differential equations. If a PDE can be interpreted as a statement of Gauss' theorem for some known coördinate system, then it may be possible to use some coördinate transformations to convert the PDE into one which is easier to solve. In this paper we have focused on a simple, specific case and used the sine-Gordon equation to illustrate the ideas. It is clear, however, that more complicated nonlinear PDEs can be attacked in this manner and that initial value problems can be reduced to problems involving inversions, quadratures and the solution of a first-order ODE.

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