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SECOND ORDER OPTIMALITY CONDITIONS FOR MATHEMATICAL PROGRAMMING WITH SET FUNCTIONS

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Abstract

Second order necessary and sufficient conditions are given for a class of optimization problems involving optimal selection of a measurable subset from a given measure subspace subject to set function inequalities. Relations between twice-differentiability at Ω and local convexity at Ω are also discussed.

1. Introduction

Let $(X, \mathscr{A}, \mu)$ be a finite measure space and $F, G_i, \ldots, G_m$ be real-valued set functions on $\mathscr{A}$. The problem considered in this paper is to find a measurable set $\Omega^* \in \mathscr{A}$ which minimizes $F(\Omega)$ subject to constraints $G_i(\Omega) \leqslant 0$, $i = 1, \ldots, m$. This type of optimization problem has received attention lately due to its diverse applications and theoretical interest. These include applications in fluid flow [1], electrical insulator design [3], optimal plasma confinement [12], first order necessary and sufficient optimal conditions [11], and duality theories set functions [6] and [7].

The difficulty of the above problem, as pointed out by Morris in [11], lies in the poorly structured feasible domain which is not convex, not open, and actually nowhere dense. Morris [11] overcame these difficulties and derived several necessary and sufficient optimality conditions with properly defined notions of first-order differentiability and convexity of set functions. By continuing to work in the setting of Morris [11] and Luenberger [8], Lai, Yang and Hwang [7] proved the Fenchel duality theorem for set functions.

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The purpose of this paper is to obtain second-order necessary and sufficient optimality conditions for the above problem. In Section 2, we begin with a definition of second differentiability of a set function on a measure space. This is followed by several theorems concerning properties of a set function with second differentiability. Section 3 contains the main results of this paper, in which sufficient conditions are presented in a way close to that in [4], [10].

Numerous results on necessary and sufficient optimality conditions in optimization problems for point functions under second order differentiability assumptions have been given by many researchers; some recent ones include [2] and [5].

2. Second differentiability of set functions

Throughout this paper, we assume that the measure space $(X, \mathscr{A}, \mu)$ is finite and atomless with $L_1(X, \mathscr{A}, \mu) = L_1(\mu)$ separable. Let ρ be a pseudometric on $\mathscr{A}$ defined by $\rho(\Omega_1, \Omega_2) = \mu(\Omega_1 \Delta \Omega_2)$ for $\Omega_1, \Omega_2 \in \mathscr{A}$, and we identify any set $\Omega \in \mathscr{A}$ with its characteristic function $\chi_\Omega \in L_1(\mu)$. Thus $\mathscr{A}$ can be regarded as a subset $\chi_{\mathscr{A}} = \{\chi_\Omega | \Omega \in \mathscr{A}\}$ of $L_1(\mu)$. Note that $\rho(\Omega_1, \Omega_2) = \|\chi_{\Omega_1} - \chi_{\Omega_2}\|_{L_1}$. For $f \in L_1(\mu)$ and $w \in L_1(\mu_1 \times \mu_1)$ we denote the integral $\int_\Omega f$ by the functional notation $\langle f, \chi_\Omega \rangle$, $\int_{\Omega_1 \times \Omega_2} w$ by $\langle w, \chi_{\Omega_1} \times \chi_{\Omega_2} \rangle$. The diagonal of w, denoted by diag w, is defined as a function on $\mathscr{A}$ in the following way:

$$\operatorname{diag} w(\Omega) = \langle w, \chi_\Omega \times \chi_\Omega \rangle, \qquad \Omega \in \mathscr{A}.$$

Moreover, diag w is said to be w^*-continuous if $\chi_{\Omega_n} \to^{w^*} \chi_\Omega$ implies that diag $w(\Omega_n) \to$ diag $w(\Omega)$ where $\chi_{\Omega_n} \to^{w^*} \chi_\Omega$ means $\langle f, \chi_{\Omega_n} \rangle \to \langle f, \chi_\Omega \rangle$ for all $f \in L_1(\mu)$.

DEFINITION 1 [11]. *A set function $F: \mathscr{A} \to R$ is said to be differentiable at $\Omega_0 \in \mathscr{A}$ if there exists $DF_{\Omega_0} \in L_1(\mu)$, called the first derivative of F at Ω_0, such that*

$$F(\Omega) = F(\Omega_0) + \langle DF_{\Omega_0}, \chi_\Omega - \chi_{\Omega_0} \rangle + E(\Omega_0, \Omega).$$

where $E(\Omega_0, \Omega) = o[\rho(\Omega_0, \Omega)]$, i.e., $\lim_{\rho(\Omega_0, \Omega) \to 0} [E(\Omega_0, \Omega)/\rho(\Omega_0, \Omega)] = 0$.

REMARKS. Definition 1 differs from that of the usual Fréchet derivative [8] in that $\Omega \to \Omega_0$ passing through only points of the subset $\chi_{\mathscr{A}}$ which is not even a linear subspace. However, if $\tilde{F}$ is a Fréchet differentiable functional on $L_1(\mu)$ and if we define a set function F on $\mathscr{A}$ by $F(\Omega) = \tilde{F}(\chi_\Omega)$, then F is a differentiable set function. In this case, note that the Fréchet derivative of $\tilde{F}$ at Ω coincides with DF_Ω due to the uniqueness of the derivative of a set function [11].

LEMMA 1 [11]. *For any $\Omega \in \mathscr{A}$ and $\alpha \in [0, 1]$ there exists a sequence $\{\Omega_n\}$ with $\Omega_n \subset \Omega$ for all n and $\chi_{\Omega_n} \to^{w^*} \alpha \chi_\Omega$.*

LEMMA 2. *For $\Omega, \Omega_0 \in \mathscr{A}$ and $\alpha \in [0,1]$ there exists a sequence $\{\Omega_n(\alpha)\}$ in $\mathscr{A}$ such that $\chi_{\Omega_n(\alpha)} - \chi_{\Omega_0} \to^{w^*} \alpha(\chi_\Omega - \chi_{\Omega_0})$.*

PROOF. Let $\Omega^+ = \Omega \sim \Omega_0$ and $\Omega^- = \Omega_0 \sim \Omega$. Then

$$\chi_\Omega - \chi_{\Omega_0} = \chi_{\Omega^+} - \chi_{\Omega^-}.$$

By Lemma 1 there exist sequences $\{\Omega_n^\pm(\alpha)\}$ satisfying

$$\chi_{\Omega_n^\pm(\alpha)} \overset{w^*}{\to} \alpha\chi_{\Omega^\pm}.$$

Let $\Omega_n(\alpha)$ denote $(\Omega_n^+(\alpha) \cup \Omega_0) \sim \Omega_n^-(\alpha)$, then

$$\chi_{\Omega_n(\alpha)} - \chi_{\Omega_0} = \chi_{\Omega_n^+(\alpha)} - \chi_{\Omega_n^-(\alpha)}$$

and

$$\chi_{\Omega_n(\alpha)} - \chi s_{\Omega_0} \overset{w^*}{\to} \alpha(\chi_{\Omega^+} - \chi_{\Omega^-}) = \alpha(\chi_\Omega - \chi_{\Omega_0}). \qquad Q.E.D.$$

DEFINITION 2. *A set function $F\colon \mathscr{A} \to R$ is said to be twice differentiable at $\Omega_0 \in \mathscr{A}$ if it has a first derivative DF_{Ω_0} at Ω_0, and there exists $D^2F_{\Omega_0} \in L_1(\mu \times \mu)$ such that the function defined by $q_{\Omega_0}(\Omega) = \operatorname{diag} D^2F_{\Omega_0}(\chi_\Omega - \chi_{\Omega_0})$, called the second derivative of F at Ω_0, is w^*-continuous, is $o[\rho(\Omega, \Omega_0)]$, and satisfies*

$$F(\Omega) = F(\Omega_0) + \langle DF_{\Omega_0}, \chi_\Omega - \chi_{\Omega_0}\rangle + \langle D^2F_{\Omega_0}, (\chi_\Omega - \chi_{\Omega_0})^2\rangle + E(\Omega, \Omega_0)$$

where $E(\Omega, \Omega_0) = o[\rho^2(\Omega, \Omega_0)]$, i.e. $\lim_{\rho(\Omega,\Omega_0)\to 0}[E(\Omega, \Omega_0)/\rho^2(\Omega, \Omega_0)] = 0$.

LEMMA 3. *If F is twice differentiable at Ω_0, then for any $\Omega \in \mathscr{A}$ and $\alpha \in [0,1]$ there exists a sequence $\{\Omega_n(\alpha)\}$ in $\mathscr{A}$ such that*

$$\chi_{\Omega_n(\alpha)} - \chi_{\Omega_0} \overset{w^*}{\to} \alpha(\chi_\Omega - \chi_{\Omega_0})$$

and

$$\lim_{n\to\infty} F(\Omega_n(\alpha)) = F(\Omega_0) + \alpha\langle DF_{\Omega_0}, \chi_\Omega - \chi_{\Omega_0}\rangle + \alpha^2\langle D^2F_{\Omega_0}, (\chi_\Omega - \chi_{\Omega_0})^2\rangle + o(\alpha^2).$$

PROOF. Fix Ω. For $\alpha \in [0,1]$, let $\{\Omega_n(\alpha)\}$ be the sequence in Lemma 2 satisfying

$$\chi_{\Omega_n(\alpha)} - \chi_{\Omega_0} \overset{w^*}{\to} \alpha(\chi_\Omega - \chi_{\Omega_0}).$$

Then

$$\lim_{n\to\infty} F(\Omega_n(\alpha)) = F(\Omega_0) + \alpha\langle DF_{\Omega_0}, \chi_\Omega - \chi_{\Omega_0}\rangle + \alpha^2\langle D^2F_{\Omega_0}, (\chi_\Omega - \chi_{\Omega_0})^2\rangle + \lim_{n\to\infty} E(\Omega_n(\alpha), \Omega_0).$$

We need only to show that $\lim_{n\to\infty} E(\Omega_n(\alpha), \Omega_0) = o(\alpha^2)$.

It suffices to show that, given $\varepsilon > 0$, there exists $\delta > 0$ such that $0 \leqslant \alpha < \delta$ implies $\lim_{n\to\infty}|E(\Omega_n(\alpha), \Omega_0)| \leqslant \varepsilon\alpha^2$. Since $E(\Omega', \Omega_0) = o[\rho^2(\Omega', \Omega_0)]$ there exists $\gamma > 0$ such that $|E(\Omega', \Omega_0)| \leqslant \varepsilon\rho^2(\Omega', \Omega_0)$ for $\Omega' \in \mathscr{A}$ satisfying $\rho(\Omega', \Omega_0) < \gamma$. Now let $\delta = \gamma/\rho(\Omega, \Omega_0)$. Then

$$\rho(\Omega_n(\alpha), \Omega_0) = \|\chi_{\Omega_n(\alpha)} - \chi_{\Omega_0}\|_{L_1} \to \alpha\|\chi_\Omega - \chi_{\Omega_0}\|_{L_1} = \alpha\rho(\Omega, \Omega_0)$$

implies that for $\alpha < \delta$ and for all large n's we have $\rho(\Omega_n(\alpha), \Omega_0) < \gamma$. Hence $|E(\Omega_n(\alpha), \Omega_0)| \leqslant \varepsilon\rho^2(\Omega_n(\alpha), \Omega_0)$ and therefore $\lim_{n\to\infty} E(\Omega_n(\alpha), \Omega_0) \leqslant \varepsilon\alpha^2$. *Q.E.D.*

Theorem 1. *If F is twice differentiable at Ω_0, then both the first and second derivative are unique.*

Proof. Let f and $\bar{f}$ both be the first derivatives of F at Ω_0; and h and $\bar{h}$ the second derivatives. Set $g = f - \bar{f}$ and $\gamma = h - \bar{h}$. Then $\langle g, \chi_\Omega - \chi_{\Omega_0}\rangle = o[\rho(\Omega, \Omega_0)]$ and $\gamma(\Omega) = o[\rho^2(\Omega, \Omega_0)]$ where $\gamma(\Omega) = \langle w, (\chi_\Omega - \chi_{\Omega_0})^2\rangle$ for some $w \in L_1(\mu \times \mu)$. Given $\Omega \in \mathscr{A}$, by Lemma 2, for any $\alpha \in [0, 1]$ there exists a sequence $\{\Omega_n(\alpha)\}$ with

$$\chi_{\Omega_n(\alpha)} - \chi_{\Omega_0} \overset{w^*}{\to} \alpha(\chi_\Omega - \chi_{\Omega_0}).$$

Then

$$\langle g, \chi_{\Omega_n(\alpha)} - \chi_{\Omega_0}\rangle \to \alpha\langle g, \chi_\Omega - \chi_{\Omega_0}\rangle$$

and

$$\langle w, (\chi_{\Omega_n(\alpha)} - \chi_{\Omega_0})^2\rangle \to \alpha^2\langle w, (\chi_\Omega - \chi_{\Omega_0})^2\rangle.$$

Since $\rho(\Omega_n(\alpha), \Omega_0) \to \alpha\rho(\Omega, \Omega_0)$, by a similar argument as used in the proof of Lemma 3 we have

$$\lim_{n\to\infty} \langle g, \chi_{\Omega_n(\alpha)} - \chi_{\Omega_0}\rangle = \alpha\langle g, \chi_\Omega - \chi_{\Omega_0}\rangle = o(\alpha)$$

and

$$\alpha^2\langle w, (\chi_\Omega - \chi_{\Omega_0})^2\rangle = o(\alpha^2).$$

This implies that $\langle g, \chi_\Omega - \chi_{\Omega_0}\rangle = 0$ and $\langle w, (\chi_\Omega - \chi_{\Omega_0})^2\rangle = 0$ for any $\Omega \in \mathscr{A}$. Let $\Omega_+ = g^{-1}([0, \infty)]$ and $\Omega_- = g^{-1}((-\infty, 0])$. Then $\langle g, \chi_{\Omega_0}\rangle = \langle g, \chi_{\Omega_+}\rangle \geqslant 0$ and $\langle g, \chi_{\Omega_0}\rangle = \langle g, \chi_{\Omega_-}\rangle \leqslant 0$ which implies $\langle g, \chi_{\Omega_0}\rangle = 0$. Therefore, $\langle g, \chi_\Omega\rangle = 0$ for all $\Omega \in \mathscr{A}$. Hence, $g = 0$, a.e. on X. Similarly, $\gamma = 0$ a.e. on X. *Q.E.D.*

Remarks. (i) If F is twice differentiable at Ω_0, then F is differentiable at Ω_0. Since $q_{\Omega_0}(\Omega) \in o[\rho(\Omega, \Omega_0)]$ by assumption, hence DF_{Ω_0} is unique by Proposition

2.2 [10]. The first derivative of F is a linear functional on $\mathscr{A}$ defined by $\Omega \to \langle DF_{\Omega_0}, \chi_\Omega - \chi_{\Omega_0} \rangle$ rather than just DF_{Ω_0}, an L_1-function. However, we may identify the first derivative with DF_{Ω_0} [10]. For the second derivative $q_{\Omega_0}(\Omega) = \langle D^2F_{\Omega_0}, (\chi_\Omega - \chi_{\Omega_0})^2 \rangle$, it is the quadratic form defined by $D^2F_{\Omega_0}$. We may always assume $D^2F_{\Omega_0}$ is symmetric in Definition 2, i.e., $D^2F_{\Omega_0}(x, y) = D^2F_{\Omega_0}(y, x), \forall x, y \in \mathscr{A}$, since $\frac{1}{2}(D^2F_{\Omega_0}(x, y) + D^2F_{\Omega_0}(y, x))$ is symmetric and defines the same quadratic form.

(ii) If F is countably additive and absolutely continuous with respect to μ, then DF_Ω is simply the Radon-Nikodym derivative $Df/d\mu$, and the second derivative $q_\Omega \equiv 0$ for all $\Omega \in \mathscr{A}$.

(iii) Another example of a twice differentiable set function is $F(\Omega) = h(\int_\Omega v_1\,d\mu, \ldots, \int_\Omega v_n\,d\mu)$ where h: $R^n \to R$ is differentiable and $v_1, \ldots, v_n$ are in $L_1(\mu)$. Then its first derivative

$$DF_\Omega = \sum_{i=1}^{n} h_i\left(\int_\Omega v_1\,d_\mu, \ldots, \int_\Omega v_n\,d_\mu\right) v_i$$

and its second derivative

$$D^2F_\Omega = \sum_{j=1}^{n} \sum_{i=1}^{n} h_{ij}\left(\int_\Omega v_1\,d_\mu, \ldots, \int_\Omega v_n\,d_\mu\right) v_i v_j$$

where h_i denotes the ith first partial derivative, and h_{ij} is the ijth second partial derivative of h.

(iv) If F and G are differentiable (twice differentiable) at Ω_0, then for $c \in R$, $c \cdot F$, and $F \pm G$ are differentiable (twice differentiable) at Ω_0.

In order to obtain sufficient conditions for a constrained local minimum, Morris [11] introduced the concept of local convexity of a set function as follows.

DEFINITION 3 [11]. *A differentiable set function F: $\mathscr{A} \to R$ is locally convex at Ω_0 if there exists $\varepsilon > 0$ such that $\rho(\Omega_0, \Omega) < \varepsilon$ implies*

$$F(\Omega) \geqslant F(\Omega_0) + \langle DF_{\Omega_0}, \chi_\Omega - \chi_{\Omega_0} \rangle.$$

The following lemmas give relationships between local convexity of a set function and its second derivative.

LEMMA 4. *Let F: $\mathscr{A} \to R$ be a set function which is twice differentiable at Ω_0. If F is locally convex at Ω_0 then there exists $\varepsilon > 0$ such that $\rho(\Omega_0, \Omega) < \varepsilon$ implies $\langle D^2F_{\Omega_0}, (\chi_\Omega - \chi_{\Omega_0})^2 \rangle \geqslant 0$, i.e., $D^2F_{\Omega_0}$ is locally positive semidefinite.*

PROOF. Using the sequence $\{\Omega_n(\alpha)\}$ given in Lemma 3, the proof of this lemma is similar to that of Theorem 1 in [9, page 89]. *Q.E.D.*

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LEMMA 5. *Let $F: \mathscr{A} \to R$ be a set function which is twice differentiable at Ω_0. If there exists $\gamma > 0$ such that*

$$\langle D^2F_{\Omega_0}, (\chi_\Omega - \chi_{\Omega_0})^2 \rangle \geqslant \gamma\rho^2(\Omega, \Omega_0)$$

for all Ω with $\rho(\Omega, \Omega_0) < \varepsilon$ for some $\varepsilon > 0$, then F is locally convex at Ω_0.

PROOF. The result follows directly from Lemma 3 and the definition of "o". *Q.E.D.*

3. Optimality conditions of second order

In this section we consider the problem mentioned at the beginning of Section 1:

$$\text{Min } F(\Omega) \quad \text{subject to } G_i(\Omega) \leqslant 0,\ i = 1,\dots,m. \tag{1}$$

$\Omega_0 \in \mathscr{A}$ is a local minimum for problem (1) if there exists $\varepsilon > 0$ such that for Ω satisfying $\rho(\Omega_0, \Omega) < \varepsilon$, $G_i(\Omega) \leqslant 0$, $i = 1,\dots,m$, it follows that $F(\Omega) \geqslant F(\Omega_0)$.

The first-order necessary condition to this problem was given by Morris in [11].

THEOREM 2 [11]. *Suppose*

(i) *$F, G_1,\dots,G_m$ are differentiable at Ω^* with first derivatives $DF_{\Omega^*}, DG^1_{\Omega^*},\dots,DG^m_{\Omega^*}$, respectively.*

(ii) *Ω^* is a local minimum of problem* (1), *and*

(iii) *Ω^* is regular, i.e., there exists a set $\Omega_1 \in \mathscr{A}$ with $G_i(\Omega^*) + \langle DG^i_{\Omega^*}, \chi_{\Omega_1} - \chi_{\Omega^*}\rangle < 0$, $i = 1,\dots,m$. Then there exists nonnegative reals $\lambda_1,\dots,\lambda_m$ such that*

$$\begin{cases} \left\langle DF_{\Omega^*} + \sum_{i=1}^{m} \lambda_i DG^i_{\Omega^*}, \chi_\Omega - \chi_{\Omega^*} \right\rangle \geqslant 0 & \textit{for all } \Omega \in \mathscr{A}, \textit{ and} \\ \lambda_i = 0 & \textit{if } G_i(\Omega^*) < 0. \end{cases} \tag{2}$$

A set of nonnegative reals $\lambda_1,\dots,\lambda_m$ for which (2) holds is called a Lagrangian multiplier for problem (1) at Ω^* and the associated Lagrangian function is defined as $L(\Omega) = F(\Omega) + \sum_{i=1}^m \lambda_i G_i(\Omega)$. We denote the feasible region of problem (1) by $S = \{\Omega \in \mathscr{A} \mid G_i(\Omega) \leqslant 0,\ i = 1,\dots,m\}$, the index set of active constraints at Ω^* by $I(\Omega^*) = \{i \mid G_i(\Omega^*) = 0\}$, and the first derivative of L at Ω by $DL_\Omega = DF_\Omega + \sum_{i=1}^m \lambda_i DG^i_\Omega$.

THEOREM 3 (Second-Order Necessary Condition). *Let $F, G_1,\dots,G_m$ be twice differentiable at Ω^*. Suppose Ω^* is a local minimum of problem* (1) *and suppose*

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$L(\Omega) = F(\Omega) + \sum_{i=1}^{m} \lambda_i G_i(\Omega)$ is a Lagrangian function associated with a set of Lagrangian multipliers $\lambda_1, \ldots, \lambda_m$ for problem (1) *at Ω^*. Then*

$$\langle D^2 L_{\Omega^*}, (\chi_\Omega - \chi_{\Omega^*})^2 \rangle \geqslant 0$$

for all $\Omega \in S$ satisfying

$$\begin{aligned} &\langle DL_{\Omega^*}, \chi_\Omega - \chi_{\Omega^*} \rangle = 0, \\ &\langle DG^i_{\Omega^*}, \chi_\Omega - \chi_{\Omega^*} \rangle < 0, \qquad i \in I(\Omega^*), \end{aligned} \tag{3}$$

and $\lambda_i G_i(\Omega) = 0$, $i = 1, \ldots, m$.

Proof. For any $\Omega \in S$ satisfying (3) we have $F(\Omega) = L(\Omega)$ and it follows that

$$\begin{aligned} F(\Omega) - F(\Omega^*) &= L(\Omega) - L(\Omega^*) \\ &= \langle D^2 L_{\Omega^*}, (\chi_\Omega - \chi_{\Omega^*})^2 \rangle + E(\Omega, \Omega^*) \end{aligned} \tag{4}$$

where $E(\Omega, \Omega^*) = o[\rho^2(\Omega, \Omega^*)]$.

A sequence $\{\Omega_n(\alpha)\}$ can be constructed as in Lemma 3 so that

$$\lim_{n \to \infty} G_i(\Omega_n(\alpha)) = G_i(\Omega^*) + \alpha \langle DG^i_{\Omega^*}, \chi_\Omega - \chi_{\Omega^*} \rangle + o(\alpha), \qquad i = 1, \ldots, m. \tag{5}$$

If $i \in I(\Omega^*)$ then $G_i(\Omega^*) = 0$. By the definition of $o(\alpha)$, there exists $\delta' > 0$ such that $|o(\alpha)| < \frac{1}{2} |\langle DG^i_{\Omega^*}, \chi_\Omega - \chi_{\Omega^*} \rangle| \alpha$ for $\alpha < \delta'$. Therefore from (3), (5) becomes

$$\lim_{n \to \infty} G_i(\Omega_n(\alpha)) < \frac{\alpha}{2} \langle DG^i_{\Omega^*}, \chi_\Omega - \chi_{\Omega^*} \rangle < 0 \quad \text{for } \alpha < \delta'$$

and hence, for any $\alpha < \delta'$ there exists $M_\alpha > 0$ such that $G_i(\Omega_n(\alpha)) < 0$ for all $n > M_\alpha$.

If $i \notin I(\Omega^*)$ then $G_i(\Omega^*) < 0$, and (5) becomes $\lim_{n \to \infty} (\Omega_n(\alpha)) \to G_i(\Omega^*) < 0$ as $\alpha \to 0$. Therefore, there exists $\delta'' > 0$ so that for any $\alpha < \delta''$ there exists $M_\alpha > 0$ such that $G_i(\Omega_n(\alpha)) < 0$ for all $n > M_\alpha$.

We have shown that there exists $\delta = \min(\delta', \delta'') > 0$, such that for any $\alpha < \delta$ there is $N_\alpha > 0$ so that $G_i(\Omega_n(\alpha)) < 0$, for all $n > N_\alpha$, $i = 1, \ldots, m$. Since Ω^* is a local minimum we have, for any $\alpha < \delta$,

$$F(\Omega_n(\alpha)) \geqslant F(\Omega^*) \quad \text{for all } n > N_\alpha.$$

Therefore

$$\lim_{n \to \infty} F(\Omega_n(\alpha)) \geqslant F(\Omega^*) \quad \text{for } \alpha < \delta.$$

Applying the sequence $\{\Omega_n(\alpha)\}$ to (4), we obtain

$$\lim_{n \to \infty} F(\Omega_n(\alpha)) = F(\Omega^*) + \alpha^2 \langle D^2 L_{\Omega^*}, (\chi_\Omega - \chi_{\Omega^*})^2 \rangle + o(\alpha^2).$$

Dividing both sides by α^2 and letting $\alpha \to 0$ we have

$$\langle D^2 L_{\Omega^*}, (\chi_\Omega - \chi_{\Omega^*})^2 \rangle \geqslant 0$$

for all $\Omega \in S$ satisfying (3). *Q.E.D.*

The following theorem gives first-order sufficient conditions for optimality. The theorem follows the spirit of Theorems 5.3 and 5.6 in [10] and can be proved by a similar argument.

THEOREM 4 (First-Order Sufficient Condition). *Suppose* $\Omega^* \in S$ *and suppose* $L(\Omega) = F(\Omega) + \sum_{i=1}^{m} \lambda_i(\Omega)$ *is a Lagrangian function for problem* (1) *at* Ω^*. *If there is* $\gamma > 0$ *such that*

$$\langle DL_{\Omega^*}, \chi_\Omega - \chi_{\Omega^*} \rangle \geqslant \gamma \cdot \rho(\Omega, \Omega^*) \quad \textit{for all } \Omega \in S$$

then there exist $\alpha > 0$ *and* $\beta > 0$ *such that*

$$F(\Omega) \geqslant F(\Omega^*) + \alpha \cdot \rho(\Omega, \Omega^*) \quad \textit{for all } \Omega \in S$$

with $\rho(\Omega, \Omega^*) \leqslant \beta$.

If we relax the first-order sufficient condition in the above theorem then we need to impose a second-order condition on the set Ω for which the first-order condition is violated, that is,

$$\langle DL_{\Omega^*}, \chi_\Omega - \chi_{\Omega^*} \rangle < \gamma \cdot \rho(\Omega, \Omega^*).$$

THEOREM 5 (Second-Order Sufficient Condition). *Suppose*
(i) $\Omega^* \in S$,
(ii) $L(\Omega) = F(\Omega) + \sum_{i=1}^{m} \lambda_i G_i(\Omega)$ *is a Lagrangian function for problem* (1),
(iii) L *is twice differentiable at* Ω_0, *and*
(iv) *there exists* $\gamma > 0$ *such that* $\langle D^2 L_{\Omega^*}, (\chi_\Omega - \chi_{\Omega^*})^2 \rangle \geqslant \gamma \rho^2(\Omega, \Omega^*)$ *in a neighborhood of* Ω^* *in* S. *Then* Ω^* *is a local minimum of* F *in* S.

PROOF. The proof is straightforward by using Lemma 5. *Q.E.D.*

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