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COMPACTIFICATIONS AND QUOTIENT LATTICES OF WALLMAN BASES

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We improve the intuition and some ideas of G. D. Faulkner and M. C. Vipera presented in the article *"Remainders of compactifications and their relation to a quotient lattice of the topology"* [*Proc. Amer. Math. Soc.* **122** (1994)] in connection with the question about internal conditions for locally compact spaces X and Y under which $\beta X \setminus X \cong \beta Y \setminus Y$ or, more generally, under which the remainders of compactifications of X belong to the collection of the remainders of compactifications of Y. We point out the reason why the quotient lattice of the topology considered by Faulkner and Vipera cannot lead to a satisfactory answer to the above question. We replace their lattice by the qoutient lattice of a new equivalence relation on a Wallman base in order to describe a method of constructing a Wallman-type compactification which allows us to deduce more complete solutions to the problems investigated by Faulkner and Vipera.

0. INTRODUCTION

All topological spaces considered below will be locally compact and Hausdorff. The word "space" will be used to refer to such a topological space, unless otherwise specified.

For a space X, the symbol $\mathcal{K}(X)$ will denote the lattice of all compactifications of X, while $\mathcal{R}(X)$ will stand for the collection of all remainders of X, that is, $\mathcal{R}(X) = \{\alpha X \setminus X : \alpha X \in \mathcal{K}(X)\}$ if we do not distinguish between homeomorphic spaces.

Although it is generally known that non-homeomorphic spaces can have homeomorphic remainders of their Čech-Stone compactifications, the nature of such spaces is not well understood. For instance, how to compare such a trivial space as the space $\mathbb{N}$ of positive integers with a pseudocompact non-normal space $\Lambda = \beta\mathbb{R} \setminus (\beta\mathbb{N} \setminus \mathbb{N})$ (see [**10**, 6P])? But the reason for which $\beta\mathbb{N} \setminus \mathbb{N} \cong \beta\Lambda \setminus \Lambda$ must surely be hidden somewhere in the internal structures of the topologies of $\mathbb{N}$ and Λ.

The question of when a space K can be homeomorphic to $\beta X \setminus X$ is also non-trivial. Let us mention that, for instance, the statement "every Parovičenko space of weight 2^ω is homeomorphic to $\beta\mathbb{N} \setminus \mathbb{N}$" is equivalent to the continuum hypothesis (see [**8**]). Furthermore, by Magill's theorem (see [**6**, 7.2]), the inclusion $\mathcal{R}(X) \subseteq \mathcal{R}(Y)$

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holds if and only if $\beta X \setminus X \cong K$ for some $K \in \mathcal{R}(Y)$. This turns our attention to the question about internal conditions for X and Y under which $\mathcal{R}(X) \subseteq \mathcal{R}(Y)$. Since one of the most familiar methods of constructing a compactification of a space is to add to the space as new points some ultrafilters in certain lattices of sets associated with the space, the theory of lattices and Boolean algebras very often offers an efficient tool for the study of compactifications (see for example [**11**]). Such an approach to the above problems on remainders is presented by Faulkner and Vipera in [**9**]. Namely, the authors of [**9**] consider the lattice of equivalence classes of the relation on the topology of X which identifies all open sets A and B such that the symmetric difference $A \triangle B$ is relatively compact. In terms of this lattice, they obtain some sufficient conditions for X and Y to have $\beta X \setminus X \cong \beta Y \setminus Y$ or $\mathcal{R}(X) \subseteq \mathcal{R}(Y)$. However, to get internal conditions under which a given space K is homeomorphic to $\beta X \setminus X$, Faulkner and Vipera must keep their considerations within the limits of the class of normal spaces X having the property that every non-relatively compact subset of X contains a closed non-compact set (see [**9**, condition (C), p.936]). Some other results of [**9**] also require the assumption of normality.

In the present paper we shall observe that, according to the theory of Wallman-type compactifications, one should not be surprised that the relation introduced by Faulkner and Vipera frequently restricts its applications only to normal spaces. We shall give a number of examples illustrating the reason for which the quotient lattice considered in [**9**] cannot lead to more satisfactory solutions of the basic problems on remainders posed in [**9**]. Since βX is always a Wallman-type compactification with respect to the collection of all zero-sets of X, it is necessary to have a much deeper look at the structure of a Wallman-type compactification in order to find answers to the problems investigated by Faulkner and Vipera. Therefore, we shall propose a new equivalence relation on a Wallman base which has a wider range of applicability than the relation of Faulkner and Vipera. The quotient lattice of our relation will allow us to give some answers to the following questions:

(1) Given a compact space K and a Wallman base $\mathcal{C}$ for a space X, what are necessary and sufficient internal conditions for K and X to have K homeomorphic to the remainder of the Wallman compactification of X which arises from $\mathcal{C}$?

(2) Given Wallman bases $\mathcal{C}$ and $\mathcal{D}$ for spaces X and Y, respectively, under what internal conditions for X and Y do the Wallman compactifications arising from $\mathcal{C}$ and $\mathcal{D}$ have homeomorphic remainders?

(3) What internal conditions for spaces X and Y guarantee that $\mathcal{R}(X) \subseteq \mathcal{R}(Y)$?

Moreover, we shall apply our lattices to the ESH-compactifications introduced in [**5**].

The algebra of continuous real functions defined on X will be denoted by $C(X)$.

For $\alpha X \in \mathcal{K}(X)$, let $C_\alpha(X) = \{f \in C(X) : f$ is continuously extendable over $\alpha X\}$, $Z_\alpha(X) = \{f^{-1}(0) : f \in C_\alpha(X)\}$, $\mathcal{K}_\alpha(X) = \{\gamma X \in \mathcal{K}(X) : \gamma X \leqslant \alpha X\}$ and $\mathcal{R}_\alpha(X) = \{\gamma X \setminus X : \gamma X \in \mathcal{K}_\alpha(X)\}$. As usual, we denote $Z_\beta(X)$ by $Z(X)$. The collection of all clopen subsets of X will be denoted by $CO(X)$.

Basic facts concerning lattices can be found, for example in [**1**] or [**13,** Chapter 2].

1. Preliminary remarks

Recall that a Wallman base $\mathcal{D}$ for a space X is a base for the closed sets of X which is stable under finite unions and finite intersections and has the following properties:

(i) $\emptyset, X \in \mathcal{D}$;

(ii) if $A \in \mathcal{D}$ and $x \in X \backslash A$, then there exists $B \in \mathcal{D}$ such that $x \in B \subseteq X \backslash A$;

(iii) if $A, B \in \mathcal{D}$ and $A \cap B = \emptyset$, then there exist $C, D \in \mathcal{D}$ such that $A \subseteq X \setminus C \subseteq D \subseteq X \setminus B$.

The Wallman compactification arising from $\mathcal{D}$ is the space $w_{\mathcal{D}}X$ of $\mathcal{D}$-ultrafilters on X (see [**13,** Section 4.4] or [**6,** Section 8]).

In what follows, we assume that $\mathcal{D}$ is a closed base for a space X such that $\emptyset, X \in \mathcal{D}$ and the collection $\mathcal{D}$ is stable under finite unions and finite intersections. For $A, B \in \mathcal{D}$, write $A \sim B$ if and only if the symmetric difference $A \triangle B$ does not contain non-compact members of $\mathcal{D}$, and write $A \sim_k B$ if and only if the set $A \triangle B$ is relatively compact in X. The relation $\sim_k$ was considered in [**9**], but on the collection of all open sets of X. The relation $\sim$ does not seem to have appeared in the literature.

Proposition 1.1. *Both $\sim$ and $\sim_k$ are equivalence relations on $\mathcal{D}$.*

Proof: It is evident that $\sim_k$ is an equivalence relation, but the transitivity of $\sim$ is not clear. We shall give below a direct proof of this fact, although one can also deduce it from properties of Wallman extensions (see Proposition 1.5).

Take any $A, B, C, D \in \mathcal{D}$ such that $A \sim B$, $B \sim C$ and $D \subseteq A \triangle C$. Then $D \cap A \cap B \in \mathcal{D}$ and $D \cap A \cap B \subseteq B \setminus C$, so $D \cap A \cap B$ is compact. Let U be a relatively compact open subset of X such that $D \cap A \cap B \subseteq U$. Since $\mathcal{D}$ is stable under finite intersections, it follows from the compactness of $D \cap A \cap B$ that there exists a set $E \in \mathcal{D}$ such that $(D \cap A) \setminus U \subseteq E$ and $E \cap D \cap A \cap B = \emptyset$. Then $D \cap A \cap E \subseteq A \triangle B$; therefore $D \cap A \cap E$ is compact. This implies that $D \cap A$ is compact because $D \cap A = (D \cap A \cap \operatorname{cl}_X U) \cup (D \cap A \cap E)$. Similarly, $D \cap C$ is compact, too. Hence D is compact, which shows that $A \sim C$. $\square$

Using similar arguments to those above, one can check that finite unions and finite intersections are compatible with both the relations $\sim$ and $\sim_k$. We shall denote by $[D]$ (respectively, by $[D]_k$) the equivalence class of $\sim$ (respectively, of $\sim_k$) which contains

$D \in \mathcal{D}$. The quotient set $L(\mathcal{D})$ of $\sim$ will become a lattice if we put

$$[A] \wedge [B] = [A \cap B] \quad \text{and} \quad [A] \vee [B] = [A \cup B].$$

Let $S(\mathcal{D})$ be the set of all ultrafilters in $L(\mathcal{D})$. For $[A] \in L(\mathcal{D})$, put

$$H[A] = \{a \in S(\mathcal{D}) : [A] \in a\}.$$

The collection $\{H[A] : [A] \in L(\mathcal{D})\}$ forms a base for the closed sets of a topology on $S(\mathcal{D})$. Under that topology $S(\mathcal{D})$ is a T_1-compact space. In the same way, we can convert the quotient set $L_k(\mathcal{D})$ of $\sim_k$ into a lattice, and the set $S_k(\mathcal{D})$ of ultrafilters in $L_k(\mathcal{D})$ into a T_1-compact space.

Let us observe that the mapping $[A]_k \to [A]$ establishes a lattice homomorphism of $L_k(\mathcal{D})$ onto $L(\mathcal{D})$; however, in general, the lattices $L_k(\mathcal{D})$ and $L(\mathcal{D})$ need not be isomorphic. Indeed, if $\mathcal{D}$ is the collection of all closed sets of $[0, \omega_1)$, then $L(\mathcal{D})$ consists of exactly two elements, while $L_k(\mathcal{D})$ has at least three distinct elements (see [**9**, p.936]). More examples will be given in Sections 2 and 3.

Even if the lattices $L(\mathcal{D})$ and $L_k(\mathcal{D})$ are not isomorphic, we do have the following

Proposition 1.2. *The spaces $S(\mathcal{D})$ and $S_k(\mathcal{D})$ are homeomorphic.*

Proof: For $a \in S_k(\mathcal{D})$, define $h(a) = \{[A] : [A]_k \in a\}$. Since $A \sim \emptyset$ if and only if $A \sim_k \emptyset$, it is easy to check that h is a one-to-one mapping of $S_k(\mathcal{D})$ onto $S(\mathcal{D})$; furthermore, $h(H[A]_k) = H[A]$ for any $[A]_k \in L_k(\mathcal{D})$. Consequently, h is a homeomorphism. ∎

Lemma 1.3. *If $A, B \in \mathcal{D}$ and $(A \cap B) \sim \emptyset$, then there exists $A_1 \in \mathcal{D}$ such that $A_1 \cap B = \emptyset$ and $A \sim A_1$. The relation $\sim$ can be replaced by $\sim_k$.*

Proof: Since $A \cap B$ is compact and X is locally compact, there exist a relatively compact open set $U \subseteq X$ and a set $C \in \mathcal{D}$, such that $A \cap B \subseteq U$, $A \setminus U \subseteq C$ and $C \cap A \cap B = \emptyset$. To complete the proof, it suffices to put $A_1 = A \cap C$. ∎

Proposition 1.4. *If $\mathcal{D}$ is a Wallman base for X, then the space $S(\mathcal{D})$ is Hausdorff.*

Proof: Take any $a, b \in S(\mathcal{D})$ such that $a \neq b$. It follows from Lemma 1.3 that there exist disjoint sets $A, B \in \mathcal{D}$ such that $[A] \in a$ and $[B] \in b$. Since the base $\mathcal{D}$ is Wallman, one can find sets $C, D \in \mathcal{D}$ such that $A \subseteq X \setminus C \subseteq D \subseteq X \setminus B$. Then $a \in S(\mathcal{D}) \setminus H[C]$, $b \in S(\mathcal{D}) \setminus H[D]$ and $\big(S(\mathcal{D}) \setminus H[C]\big) \cap \big(S(\mathcal{D}) \setminus H[D]\big) = \emptyset$, which shows that $S(\mathcal{D})$ is Hausdorff. ∎

Proposition 1.5. *Let $\mathcal{D}$ be a Wallman base for X and let $K = w_{\mathcal{D}}X \setminus X$. Then, for any $D_1, D_2 \in \mathcal{D}$, we have $K \cap \operatorname{cl} D_1 = K \cap \operatorname{cl} D_2$ if and only if $[D_1] = [D_2]$ where the closure is taken in $w_{\mathcal{D}}X$.*

PROOF: Suppose that $[D_1] \neq [D_2]$. There exists a non-compact $C \in \mathcal{D}$ such that $C \subseteq D_1 \triangle D_2$. We may assume that $A = C \cap D_1$ is non-compact. Obviously, $\operatorname{cl} A \cap \operatorname{cl} D_2 = \emptyset$ and there exists $p \in K \cap \operatorname{cl} A$. Then $p \in K \cap \operatorname{cl} D_1$ and $p \notin \operatorname{cl} D_2$, which proves that if $K \cap \operatorname{cl} D_1 = K \cap \operatorname{cl} D_2$, then $[D_1] = [D_2]$.

Suppose now that $p \in (K \cap \operatorname{cl} D_1) \setminus \operatorname{cl} D_2$. There exists $E \in \mathcal{D}$ such that $p \in \operatorname{cl} E$ and $\operatorname{cl} E \cap \operatorname{cl} D_2 = \emptyset$. Put $B = E \cap D_1$. Then $p \in \operatorname{cl} B$ and $B \subseteq D_1 \triangle D_2$. Clearly, $B \in \mathcal{D}$ and B is non-compact; hence $[D_1] \neq [D_2]$, which completes the proof. $\square$

In order to prove the main result of Section 2, we shall make use of the following version of the Taimanov extension theorem (see [**15**] and [**18**, Lemma 2.5]):

THEOREM 1.6. (Taimanov) *Let X be a dense subspace of a topological space T and let f be a continuous mapping of X into a compact Hausdorff space Y. Then f is continuously extendable over T if and only if there exists a base $\mathcal{F}$ for the closed sets of Y which is stable under finite intersections and has the property that $\operatorname{cl}_T f^{-1}(A) \cap \operatorname{cl}_T f^{-1}(B) = \emptyset$ for each pair A, B of disjoint members of $\mathcal{F}$.*

2. A CONSTRUCTION OF WALLMAN-TYPE COMPACTIFICATIONS

In the sequel, we shall assume that every closed base for a space Y contains $\emptyset$ and Y.

Suppose that $\mathcal{D}$ is a Wallman base for a space X. Let K be a compact space and let $\mathcal{C}$ be a base for the closed sets of K which is stable under finite unions and finite intersections. Suppose that we are given a lattice isomorphism $\psi : \mathcal{C} \to L(\mathcal{D})$ Denote by $X \cup_\psi K$ the disjoint union $X \cup K$ equipped with the topology having the collection

$$\mathcal{F} = \{C \cup D : C \in \mathcal{C} \text{ and } D \in \psi(C)\}$$

as a base for the closed sets.

THEOREM 2.1. *The space $Y = X \cup_\psi K$ is a Hausdorff compactification of X equivalent to $w_{\mathcal{D}} X$; necessarily $w_{\mathcal{D}} X \setminus X \cong K$.*

PROOF: Obviously, X and K are subspaces of Y. To show that Y is Hausdorff, consider any pair p_1, p_2 of distinct points of Y. Suppose first that $p_1, p_2 \in K$. There exist disjoint sets $C_1, C_2 \in \mathcal{C}$ such that $p_i \in C_i$ for $i = 1, 2$. Then $\psi(C_1) \wedge \psi(C_2) = [\emptyset]$; hence, in view of Lemma 1.3, there exist disjoint sets $A_1, A_2 \in \mathcal{D}$ such that $A_i \in \psi(C_i)$ for $i = 1, 2$. Since the base $\mathcal{D}$ is Wallman, there are sets $B_i \in \mathcal{D}$ such that $B_1 \cup B_2 = X$ and $A_i \cap B_i = \emptyset$ for $i = 1, 2$. Then $\psi^{-1}([B_1]) \cup \psi^{-1}([B_2]) = K$ and $\psi^{-1}([B_i]) \cap C_i = \emptyset$ for $i = 1, 2$. Put $V_i = Y \setminus \left(\psi^{-1}([B_i])\right) \cup (B_i)$ for $i = 1, 2$. Then $p_i \in V_i$ for $i = 1, 2$ and the sets V_1, V_2 are disjoint and open in Y.

Suppose now that $p_1 \in X$ and $p_2 \in K$. Take a relatively compact open neighbourhood U of p_1 in X. There exist disjoint sets $D_1, D_2 \in \mathcal{D}$ such that $p_1 \in D_1 \subseteq U$

and $X \setminus U \subseteq D_2$; further, we can choose sets $E_1, E_2 \in \mathcal{D}$ such that $E_1 \cup E_2 = X$ and $E_i \cap D_i = \emptyset$ for $i = 1,2$. Put $W_i = Y \setminus (\psi^{-1}([E_i]) \cup E_i)$ for $i = 1,2$. Obviously, the sets W_1, W_2 are disjoint and open in Y, and $p_1 \in W_1$. Since $E_2 \subseteq U$, we have $[E_2] = [\emptyset]$. Hence $\psi^{-1}([E_2]) = \emptyset$ and, in consequence, $p_2 \in W_2$.

Let $p_1, p_2 \in X$. There exist sets $F_1, F_2 \in \mathcal{D}$ such that $F_1 \cup F_2 = X$ and $p_i \in X \setminus F_i$ for $i = 1,2$. The proof that Y is Hausdorff will be completed if we consider the neighbourhoods $Y \setminus (\psi^{-1}([F_i]) \cup F_i)$ of p_i for $i = 1,2$. Now, we are going to show that X is dense in Y.

Let $p \in K$, $C \in \mathcal{C}$, $D \in \psi(C)$ and $p \notin C$. There is $E \in \mathcal{C}$ such that $p \in E$ and $E \cap C = \emptyset$. Then $\psi(E) \wedge \psi(C) = [\emptyset]$. By Lemma 1.3, there is $A \in \psi(E)$ such that $A \cap D = \emptyset$. If A were empty, then $E = \psi^{-1}([A])$ would be empty, too. Therefore $A \neq \emptyset$ because $E \neq \emptyset$. Since $A \subseteq Y \setminus (C \cup D)$, we have that X is dense in Y.

We shall show that Y is compact. To this end, take a centred subfamily $\mathcal{H}$ of $\mathcal{F}$ and suppose, for a contradiction, that $\bigcap_{H \in \mathcal{H}} H = \emptyset$. The space K being compact, there exists a finite subfamily $\{H_1, \dots, H_n\}$ of $\mathcal{H}$ such that $\bigcap_{i=1}^{n} (H_i \cap K) = \emptyset$. Then $\bigwedge_{i=1}^{n} \psi(H_i \cap K) = [\emptyset]$. This implies that the set $B = \bigcap_{i=1}^{n} (H_i \cap X)$ is compact. The collection $\mathcal{H}$ being centred, it follows from the compactness of B that $B \cap \bigcap_{H \in \mathcal{H}} H \neq \emptyset$, which is absurd. The contradiction shows that Y is compact. Hence $Y \in \mathcal{K}(X)$. Since both the collections $\mathcal{F}$ and $\{\mathrm{cl}_{w_{\mathcal{D}}X} D : D \in \mathcal{D}\}$ are stable under finite intersections and serve as closed bases for Y and $w_{\mathcal{D}}X$, respectively, by Theorem 1.6, to prove that the compactifications Y and $w_{\mathcal{D}}X$ of X are equivalent, it is enough to check that $\mathrm{cl}_Y D_1 \cap \mathrm{cl}_Y D_2 = \emptyset$ whenever $D_1, D_2 \in \mathcal{D}$ and $D_1 \cap D_2 = \emptyset$.

Consider any pair D_1, D_2 of disjoint members of $\mathcal{D}$. There exist sets $A_1, A_2 \in \mathcal{D}$ such that $A_1 \cup A_2 = X$ and $D_i \cap A_i = \emptyset$ for $i = 1,2$. Take any $p \in K$. As $K = \psi^{-1}([A_1]) \cup \psi^{-1}([A_2])$, we may assume that $p \in \psi^{-1}([A_1])$. But $\psi^{-1}([D_1]) \cap \psi^{-1}([A_1]) = \emptyset$; hence $p \in V = Y \setminus (\psi^{-1}([D_1]) \cup D_1)$. Since the set V is open in Y and does not meet D_1, we have $p \notin \mathrm{cl}_Y D_1$. Therefore $\mathrm{cl}_Y D_1 \cap \mathrm{cl}_Y D_2 = \emptyset$, which concludes the proof of Theorem 2.1. $\square$

Let us observe that, in view of Proposition 1.5, by assigning to any $[D] \in L(\mathcal{D})$ the set $(w_{\mathcal{D}}X \setminus X) \cap \mathrm{cl}_{w_{\mathcal{D}}X} D$, we establish a lattice isomorphism between the lattice $L(\mathcal{D})$ and the closed base $\{(w_{\mathcal{D}}X \setminus X) \cap \mathrm{cl}_{w_{\mathcal{D}}X} D : D \in \mathcal{D}\}$ for $w_{\mathcal{D}}X \setminus X$. Therefore, $w_{\mathcal{D}}X$ is always of the form $X \cup_{\psi} K$ for suitably chosen K and ψ. Our next theorem gives an exact description of $w_{\mathcal{D}}X$ in terms of $L(\mathcal{D})$. Accordingly, Theorem 2.1 and Theorem 2.2 taken together can be regarded as a new method of constructing Wallman-type compactifications of locally compact spaces.

THEOREM 2.2. *Let $\mathcal{C} = \{H[A] : A \in \mathcal{D}\}$ where $H[A] = \{a \in S(\mathcal{D}) : [A] \in a\}$.*

Clearly, the base $\mathcal{C}$ for the closed sets of $S(\mathcal{D})$ is stable under finite unions and finite intersections. For $A \in \mathcal{D}$, let $\psi(H[A]) = [A]$. Then ψ is a lattice isomorphism of $\mathcal{C}$ onto $L(\mathcal{D})$. In consequence, $w_{\mathcal{D}}X = X \cup_{\psi} S(\mathcal{D})$ and $w_{\mathcal{D}}X \setminus X \cong S(\mathcal{D})$.

PROOF: We must check that ψ is well-defined. To this end, suppose that $A, B \in \mathcal{D}$ and $[A] \neq [B]$. There exists a non-compact $C \in \mathcal{D}$ such that $C \subseteq A \triangle B$. We may assume that $C \cap A$ is non-compact. There exists an ultrafilter a in $L(\mathcal{D})$ which contains $[C \cap A]$. Then $a \in H[A] \setminus H[B]$, so that $H[A] \neq H[B]$. This implies that ψ is a well-defined mapping of $\mathcal{C}$ onto $L(\mathcal{D})$. Now, it is easily seen that $\psi : \mathcal{C} \to L(\mathcal{D})$ is a lattice isomorphism. The proof will be completed if we apply Proposition 1.4 and Theorem 2.1. $\square$

REMARKS. Let us note that if we consider the relation $\sim_k$ on the collection $\mathcal{D}_c = \{X \setminus D : D \in \mathcal{D}\}$, then the space $M(\mathcal{D}_c)$ of maximal ideals of the quotient lattice $L_k(\mathcal{D}_c)$ of equivalence classes of $\sim_k$ in $\mathcal{D}_c$ is homeomorphic to $S(\mathcal{D})$. In the light of Theorem 2.2, $w_{\mathcal{D}}X \setminus X \cong M(\mathcal{D}_c)$; this gives an extension of [**9**, Theorem 2.8]. Of course, if there exists a lattice isomorphism between $L_k(\mathcal{D})$ and a closed base for $S(\mathcal{D})$, using such an isomorphism and replacing $\sim$ by $\sim_k$, we can construct $w_{\mathcal{D}}X$ in a similar way as in Theorem 2.1. Unfortunately, $L_k(\mathcal{D})$ need not be isomorphic to any closed base for $w_{\mathcal{D}}X \setminus X$. Even if X is normal and $\mathcal{D}$ is the collection of all closed sets of X, to guarantee the existence of an isomorphism between $L_k(\mathcal{D})$ and a closed base for $w_{\mathcal{D}}X \setminus X$, the authors of [**9**] had to assume some additional condition (C) which played only a technical role in their paper (see [**9**, pp. 936–937]). Therefore, contrary to $\sim$, the relation $\sim_k$ does not give a sufficiently deep insight into the structure of $w_{\mathcal{D}}X$. Among other things, the relation $\sim_k$ does not lead to internal necessary and sufficient conditions for a compact space K and a space X to have $\beta X \setminus X \cong K$ (see [**9**, Proposition 2.5]). However, if we replace $\sim_k$ by $\sim$, we can get the following immediate consequences of Theorems 2.1 and 2.2:

THEOREM 2.3. *A compact space K is homeomorphic to $w_{\mathcal{D}}X \setminus X$ if and only if K has a base for the closed sets which is stable under finite unions and finite intersections and which is lattice isomorphic to $L(\mathcal{D})$.*

COROLLARY 2.4. *A compact space K is the remainder of the Čech-Stone compactification of X if and only if K has a base for the closed sets which is stable under finite unions and finite intersections and which is lattice isomorphic to $L(Z(X))$.*

If one would like to make use of Theorem 2.3 in order to check whether a given compact space K is the remainder of $w_{\mathcal{D}}X$ or not, it might be difficult to judge which one of the closed bases for K could be isomorphic to $L(\mathcal{D})$. We shall show that this problem can be overcome when we deal with compactifications of pseudocompact spaces or with compactifications having zero-dimensional remainders. Let us recall the following result brought out in [**18**, Corollary 3.4]:

THEOREM 2.5. *A Tychonoff space X is pseudocompact if and only if every compactification αX of X is the Wallman-type compactification which arises from the Wallman base $Z_\alpha(X)$.*

LEMMA 2.6. *If αX is a compactification of a pseudocompact space X, then*

$$Z(\alpha X \setminus X) = \{(\alpha X \setminus X) \cap \mathrm{cl}_{\alpha X} Z : Z \in Z_\alpha(X)\}.$$

PROOF: Consider any $f \in C(\alpha X)$. Put $A = f^{-1}(0) \cap (\alpha X \setminus X)$ and $Z = f^{-1}(0) \cap X$. Suppose for a contradiction that $y \in A$ and $y \notin \mathrm{cl}_{\alpha X} Z$. There exists $g \in C(\alpha X)$ such that $g(y) = 0$ and $\mathrm{cl}_{\alpha X} Z \subseteq g^{-1}(1)$. Then $\emptyset \neq g^{-1}(0) \cap f^{-1}(0) \subseteq \alpha X \setminus X$, which contradicts the fact that X is pseudocompact (see [**10**, 6I]). Hence $A = (\alpha X \setminus X) \cap \mathrm{cl}_{\alpha X} Z$. ∎

THEOREM 2.7. *Let αX be a compactification of a pseudocompact space X. Then a compact space K is homeomorphic to $\alpha X \setminus X$ if and only if $Z(K)$ is lattice isomorphic to $L(Z_\alpha(X))$.*

PROOF: The result follows from Theorems 2.5 and 2.3, Proposition 1.5 and Lemma 2.6. ∎

The assumption of pseudocompactness cannot be omitted in Theorem 2.7. For instance, if X is Lindelöf, then $Z_\alpha(X) = Z(X)$ for any $\alpha X \in \mathcal{K}(X)$ (see [**18**]).

The following example shows that the lattice $L(Z_\alpha(X))$ cannot be replaced by $L_k(Z_\alpha(X))$ in Theorem 2.7.

EXAMPLE 2.8: For a maximal almost disjoint family $\mathcal{R}$ of subsets of the set $\mathbb{N}$ of positive integers, let $\mathbb{N} \cup \mathcal{R}$ denote the set-theoretic union of $\mathbb{N}$ and $\mathcal{R}$ equipped with the following well-known topology: the points of $\mathbb{N}$ are isolated, while a neighbourhood base for a point $\lambda \in \mathcal{R}$ is the collection $\{\{\lambda\} \cup (\lambda \setminus F) : F$ is a finite subset of $\mathbb{N}\}$ (see [**10**, 5I] and [**16**]). Suppose that $\psi : L_k(Z(\mathbb{N} \cup \mathcal{R})) \to L(Z(\mathbb{N} \cup \mathcal{R}))$ is a lattice isomorphism. Obviously, $\mathcal{R} \in Z(\mathbb{N} \cup \mathcal{R})$ and $[\mathcal{R}]_k \neq [\mathbb{N} \cup \mathcal{R}]_k$; hence $\psi([\mathcal{R}]_k) \neq [\mathbb{N} \cup \mathcal{R}]$. Let $A \in \psi([\mathcal{R}]_k)$. There exists a non-compact $B \in Z(\mathbb{N} \cup \mathcal{R})$ such that $B \subseteq (\mathbb{N} \cup \mathcal{R}) \setminus A$. Since $\psi^{-1}([A]) \wedge \psi^{-1}([B]) = [\emptyset]_k$, according to Lemma 1.3, there exists $C \in \psi^{-1}([B])$ such that $C \cap \mathcal{R} = \emptyset$. Then C is an infinite subset of $\mathbb{N}$. It follows from the maximality of $\mathcal{R}$ that $\mathcal{R} \cap \mathrm{cl}\, C \neq \emptyset$, which is absurd because C is closed in $\mathbb{N} \cup \mathcal{R}$. The contradiction proves that the lattices $L_k(Z(\mathbb{N} \cup \mathcal{R}))$ and $L(Z(\mathbb{N} \cup \mathcal{R}))$ are non-isomorphic.

Let us pass to compactifications having zero-dimensional remainders.

Recall that a π-open (or γ-open in the terminology of [**7**]) set of X is an open set $U \subseteq X$ such that $\mathrm{bd}_X U$ is compact. Denote by $\Pi(X)$ the collection of the closures of π-open subsets of X. It is well known that if a Hausdorff space Y is rimcompact, then $\Pi(Y)$ is a Wallman base for Y and the Wallman compactification of Y with respect to $\Pi(Y)$ is equivalent to the Freudenthal compactification ϕY of Y (see [**12**, p.273]; we refer the reader to [**7**] for more information about ϕY). It is easily seen that if $A \in \Pi(X)$ then $\mathrm{cl}_X(X \setminus A) \in \Pi(X)$, and $A \cap \mathrm{cl}_X(X \setminus A)$ is compact. This implies that $[\mathrm{cl}_X(X \setminus A)]$ is the complement of $[A]$ in $L(\Pi(X))$. Accordingly, the lattice $L(\Pi(X))$ is complemented.

DEFINITION: A Wallman base $\mathcal{D}$ for a space X will be called:

(i) *complemented* if the lattice $L(\mathcal{D})$ is complemented;
(ii) *π-complemented* if $\mathcal{D} \subseteq \Pi(X)$ and $\mathrm{cl}_X(X \setminus A) \in \mathcal{D}$ whenever $A \in \mathcal{D}$.

Note that $\mathcal{D}$ is complemented if and only if $L(\mathcal{D})$ is a Boolean algebra.

2.9. LEMMA. *If $\mathcal{D}$ is a complemented Wallman base for a space X, then*

$$CO(w_{\mathcal{D}}X \setminus X) = \{(w_{\mathcal{D}}X \setminus X) \cap \mathrm{cl}\, A : A \in \mathcal{D}\}$$

where the closure is taken in $w_{\mathcal{D}}X$.

PROOF: Let $Y = w_{\mathcal{D}}X$ and $K = Y \setminus X$. Consider any $C \in CO(K)$. The sets C and $K \setminus C$ being compact, there exist $A, B \in \mathcal{D}$ such that $C \subseteq \mathrm{cl}_Y A$, $K \setminus C \subseteq \mathrm{cl}_Y B$ and $\mathrm{cl}_Y A \cap \mathrm{cl}_Y B = \emptyset$. This implies that $C = K \cap \mathrm{cl}_Y A$. On the other hand, if $D \in \mathcal{D}$, then there exists $E \in \mathcal{D}$ such that $[D] \wedge [E] = [\emptyset]$ and $[D] \vee [E] = [X]$. It follows from Proposition 1.5 that $K \cap \mathrm{cl}_Y D \cap \mathrm{cl}_Y E = \emptyset$ and $K \subseteq \mathrm{cl}_Y D \cup \mathrm{cl}_Y E$; thus $K \cap \mathrm{cl}_Y D \in CO(K)$. ∎

PROPOSITION 2.10. *For a compactification αX of a space X, the following conditions are equivalent:*

(i) *$\alpha X \setminus X$ is zero-dimensional;*
(ii) *there exists a π-complemented Wallman base $\mathcal{D}$ for X such that $\alpha X = w_{\mathcal{D}}X$;*
(iii) *there exists a complemented Wallman base $\mathcal{D}$ for X such that $\alpha X = w_{\mathcal{D}}X$.*

PROOF: It is fairly easy to deduce from the proof of [**12**, Theorem 5] that $(i) \Rightarrow (ii)$. Implication $(ii) \Rightarrow (iii)$ is obvious. That $(iii) \Rightarrow (i)$ follows from Lemma 2.9. ∎

THEOREM 2.11. *Let $\mathcal{D}$ be a complemented Wallman base for a space X. Then a compact space K is homeomorphic to $w_{\mathcal{D}}X \setminus X$ if and only if $CO(K)$ is lattice isomorphic to $L(\mathcal{D})$.*

PROOF: The result is an immediate consequence of Proposition 1.5, Lemma 2.9 and Theorem 2.3. ∎

The lattice $L(\mathcal{D})$ cannot be replaced by $L_k(\mathcal{D})$ in Theorem 2.11.

EXAMPLE 2.12: Let us come back to the space $\mathbb{N} \cup \mathcal{R}$ considered in Example 2.8. It is easily seen that the collection $\mathcal{D} = \{(\mathcal{R} \cap A) \cup B : A, B \in \Pi(\mathbb{N} \cup \mathcal{R})\}$ is a complemented Wallman base for $\mathbb{N} \cup \mathcal{R}$ such that $w_{\mathcal{D}}(\mathbb{N} \cup \mathcal{R}) = \phi(\mathbb{N} \cup \mathcal{R})$. Using similar arguments to those in Example 2.8, one can check that the lattices $L(\mathcal{D})$ and $L_k(\mathcal{D})$ are not isomorphic. Accordingly, $L_k(\mathcal{D})$ cannot be lattice isomorphic to $CO(\phi(\mathbb{N} \cup \mathcal{R}) \setminus (\mathbb{N} \cup \mathcal{R}))$. Note that the base $\mathcal{D}$ is not π-complemented.

PROPOSITION 2.13. *If $\mathcal{D}$ is a π-complemented Wallman base for a space X, then the relations $\sim$ and $\sim_k$ coincide on $\mathcal{D}$.*

PROOF: Let $Y = w_{\mathcal{D}}X$ and $K = w_{\mathcal{D}}X \setminus X$. Consider any $A, B \in \mathcal{D}$. Obviously, if $A \sim_k B$, then $A \sim B$. Assume that $A \sim B$ and suppose for a contradiction that $A \setminus B$ is not relatively compact in X. If $C = A \cap \mathrm{cl}_X(X \setminus B)$, then $C \cap B$ is compact because $X \setminus B$ is π-open. This, together with the fact that $C \in \mathcal{D}$, implies that $K \cap \mathrm{cl}_Y C \cap \mathrm{cl}_Y B = \emptyset$. Clearly, $K \cap \mathrm{cl}_Y C \neq \emptyset$ because $A \setminus B \subseteq C$. Therefore, $K \cap \mathrm{cl}_Y A \neq K \cap \mathrm{cl}_Y B$, which contradicts Proposition 1.5. Hence $A \sim_k B$. ∎

COROLLARY 2.14. *Let $\mathcal{D}$ be a π-complemented Wallman base for a space X. Then a compact space K is homeomorphic to $w_{\mathcal{D}}X \setminus X$ if and only if $CO(K)$ is lattice isomorphic to $L_k(\mathcal{D})$.*

COROLLARY 2.15. *Let K be a compact space. Then $K \cong \phi X \setminus X$ if and only if $CO(K) \cong L(\Pi(X))$; of course, $L_k(\Pi(X)) = L(\Pi(X))$.*

REMARKS. Denote by $\mathcal{E}(X)$ the collection of all complemented elements of the quotient lattice of the relation $\sim_k$ on the topology of X. (See [**9**, p.932].) Faulkner and Vipera showed in [**9**, Proposition 3.2] that there exists a natural isomorphism between $CO(\phi X \setminus X)$ and $\mathcal{E}(X)$; however, their route to this isomorphism seems somewhat obscure for the form of the natural isomorphism was not described explicitly. Let us observe that, in view of [**9**, Proposition 1.1], $[V] \in \mathcal{E}(X)$ if and only if V is π-open. Therefore, with Proposition 1.5, Lemma 2.9 and Proposition 2.13 in hand, it is readily seen that, by assigning to any $[V] \in \mathcal{E}(X)$ the set $(\phi X \setminus X) \cap \mathrm{cl}_{\phi X} V$, one establishes the most natural isomorphism between $\mathcal{E}(X)$ and $CO(\phi X \setminus X)$. Evidently, the mapping $[V] \to [\mathrm{cl}_X V]$ is a lattice isomorphism of $\mathcal{E}(X)$ onto $L(\Pi(X))$.

Let us mention that the arguments of our next section are sufficient to deduce [**9**, Proposition 3.4].

3. CONDITIONS UNDER WHICH $\mathcal{R}_\alpha(X) \subseteq \mathcal{R}_\gamma(Y)$

In connection with [**9**, Theorem 2.1], let us observe that one can easily infer from the proof of [**6**, Theorem 5.27] that, for compactifications αX and γY of locally compact spaces X and Y, the inclusion $\mathcal{R}_\alpha(X) \subseteq \mathcal{R}_\gamma(Y)$ holds if and only if $\alpha X \setminus X \in \mathcal{R}_\gamma(Y)$; furthermore, if $\alpha X \setminus X \cong \gamma Y \setminus Y$, then $\mathcal{R}_\alpha(X) = \mathcal{R}_\gamma(Y)$ and there exists a lattice isomorphism $\Gamma : \mathcal{K}_\alpha(X) \to \mathcal{K}_\gamma(Y)$ such that $\Gamma(\tau X) \setminus Y \cong \tau X \setminus X$ for every $\tau X \in \mathcal{K}_\alpha(X)$. Therefore, the problem of when there exists $\delta Y \in \mathcal{K}_\gamma(Y)$ with $\alpha X \setminus X \cong \delta Y \setminus Y$ is equivalent to the problem of when there exists a map r of $\mathcal{K}_\alpha(X)$ to $\mathcal{K}_\gamma(Y)$ which preserves the natural order and has the property that, for every $\tau X \in \mathcal{K}_\alpha(X)$, $r(\tau X)$ has the same remainder as τX.

The following theorem can be regarded as a generalisation of [**9**, Theorem 2.4]:

THEOREM 3.1. *Let $\mathcal{C}, \mathcal{D}$ be Wallman bases for spaces X and Y, respectively. If $L(\mathcal{C}) \cong L(\mathcal{D})$ or $L_k(\mathcal{C}) \cong L_k(\mathcal{D})$, then $w_\mathcal{C} X \setminus X \cong w_\mathcal{D} Y \setminus Y$ and, consequently $\mathcal{R}_{w_\mathcal{C}}(X) = \mathcal{R}_{w_\mathcal{D}}(Y)$.*

PROOF: In the light of Proposition 1.2, both the conditions $L(\mathcal{C}) \cong L(\mathcal{D})$ and $L_k(\mathcal{C}) \cong L_k(\mathcal{D})$ imply that $S(\mathcal{C}) \cong S(\mathcal{D})$; thus, the proof will be completed if we use Theorem 2.2. ∎

COROLLARY 3.2. *If $L(Z(X)) \cong L(Z(Y))$ or $L_k(Z(X)) \cong L_k(Z(Y))$, then $\beta X \setminus X \cong \beta Y \setminus Y$ and, consequently $\mathcal{R}(X) = \mathcal{R}(Y)$.*

It may happen that $\mathcal{R}(X) = \mathcal{R}(Y)$ but $\beta X \setminus X$ and $\beta Y \setminus Y$ are non-homeomorphic (see [**9**, p.931]). It may also happen that $\beta X \setminus X \cong \beta Y \setminus Y$ but neither $L(Z(X)) \cong L(Z(Y))$ nor $L_k(Z(X)) \cong L_k(Z(Y))$.

EXAMPLE 3.3: If $X = (\beta\mathbb{N} \setminus \mathbb{N}) \times [0, \omega_1)$, then $\beta X \setminus X \cong \beta\mathbb{N} \setminus \mathbb{N}$. It follows from [**19**, 1.1 and 2.1] that, for every $Z \in Z(X)$, there exist $\alpha < \omega_1$, $A \in Z(X)$ and $B \in Z(\beta\mathbb{N} \setminus \mathbb{N})$, such that $A \subseteq (\beta\mathbb{N} \setminus \mathbb{N}) \times [0, \alpha]$ and $Z = A \cup \big(B \times [\alpha, \omega_1)\big)$. This implies that $L_k(Z(X)) = L(Z(X))$. Obviously, $L_k(Z(\mathbb{N})) = L(Z(\mathbb{N}))$. In view of Theorem 2.7, $L(Z(X)) \cong Z(\beta\mathbb{N} \setminus \mathbb{N})$. It is evident that $L(Z(\mathbb{N})) \cong CO(\beta\mathbb{N} \setminus \mathbb{N})$ (see Corollary 2.15). The lattices $Z(\beta\mathbb{N} \setminus \mathbb{N})$ and $CO(\beta\mathbb{N} \setminus \mathbb{N})$ are not isomorphic because $\beta\mathbb{N} \setminus \mathbb{N}$ contains non-P-points (see [**6**, 4.35]). Hence, the lattices $L(Z(X))$ and $L(Z(\mathbb{N}))$ cannot be isomorphic.

PROPOSITION 3.4. *Let αX and γY be compactifications of pseudocompact spaces X and Y. Then $\alpha X \setminus X \cong \gamma Y \setminus Y$ if and only if $L(Z_\alpha(X)) \cong L(Z_\gamma(Y))$.*

PROOF: Clearly, if $\alpha X \setminus X \cong \gamma Y \setminus Y$, then $Z(\alpha X \setminus X) \cong Z(\gamma Y \setminus Y)$; therefore, the result follows from Theorem 2.7 and Theorem 3.1. ∎

PROPOSITION 3.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be complemented Wallman bases for spaces X and Y, respectively. Then $w_\mathcal{C} X \setminus X \cong w_\mathcal{D} Y \setminus Y$ if and only if $L(\mathcal{C}) \cong L(\mathcal{D})$.*

PROOF: It is enough to use Theorem 2.11 and Theorem 3.1.

Clearly, the relation $\sim$ can be replaced by $\sim_k$ neither in Proposition 3.4 nor in Proposition 3.5.

Our next theorem is, in a sense, related to [**9**, Theorem 2.1]; however, it seems difficult to adopt the proof of [**9**, Theorem 2.1] to our needs. Therefore we shall give quite different arguments.

Theorem 3.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be Wallman bases for spaces X and Y, respectively. Suppose that there exists a lattice homomorphism $\psi : L(\mathcal{C}) \to L(\mathcal{D})$ such that:*

(*I*) $\psi([A]) = [\emptyset]$ *if and only if* $[A] = [\emptyset]$;

(*II*) $\psi([X]) = [Y]$.

Then $\mathcal{R}_{w_\mathcal{C}}(X) \subseteq \mathcal{R}_{w_\mathcal{D}}(Y)$.

Proof: According to Theorem 2.2 and to the proof of Magill's theorem (see [**6**, 7.2]), it suffices to show that $S(\mathcal{C})$ is a continuous image of $S(\mathcal{D})$. In order to find a map $F : S(\mathcal{D}) \to S(\mathcal{C})$, define $f(d) = \{[A] \in L(\mathcal{C}) : \psi([A]) \in d\}$ for $d \in S(\mathcal{D})$. One readily verifies that $f(d)$ is a filter in $L(\mathcal{C})$. Suppose for a contradiction that there exist $B_1, B_2 \in \mathcal{C}$ such that $[B_1] \wedge [B_2] = [\emptyset]$ but $[B_1] \wedge [A] \neq [\emptyset] \neq [B_2] \wedge [A]$ for each $[A] \in f(d)$. By Lemma 1.3, we may assume that $B_1 \cap B_2 = \emptyset$. Take $C_1, C_2 \in \mathcal{C}$ such that $B_1 \subseteq X \setminus C_1 \subseteq C_2 \subseteq X \setminus B_2$. Since d is an ultrafilter and $\psi([C_1]) \vee \psi([C_2]) = [Y]$, without loss of generality, we may suppose that $\psi([C_1]) \in d$. Then $[C_1] \in f(d)$, which is absurd because $[C_1] \wedge [B_1] = [\emptyset]$. The contradiction proves that $f(d)$ cannot be contained in two different ultrafilters in $L(\mathcal{C})$. Denote by $F(d)$ the unique ultrafilter in $L(\mathcal{C})$ which contains $f(d)$. In this way, we define a map $F : S(\mathcal{D}) \to S(\mathcal{C})$. If $c \in S(\mathcal{C})$, then $\{\psi([A]) : [A] \in c\}$ is a filter base in $L(\mathcal{D})$. It is easy to check that if d is an ultrafilter in $L(\mathcal{D})$ that contains $\{\psi([A]) : [A] \in c\}$, then $F(d) = c$. Hence $F(S(\mathcal{D})) = S(\mathcal{C})$. We shall show that F is continuous. To this end, consider any $d \in S(\mathcal{D})$ and $A_0 \in \mathcal{C}$, such that $F(d) \notin H[A_0]$ (see Section 1). By Lemma 1.3, there exists $A_1 \in \mathcal{C}$ such that $[A_1] \in F(d)$ and $A_0 \cap A_1 = \emptyset$. Choose sets $E_0, E_1 \in \mathcal{C}$ such that $A_0 \subseteq X \setminus E_0 \subseteq E_1 \subseteq X \setminus A_1$. Clearly, $[E_1] \notin F(d)$. This implies that $\psi([E_1]) \notin d$, so that the set $U = \{a \in S(\mathcal{D}) : \psi([E_1]) \notin a\}$ is an open neighbourhood of d in $S(\mathcal{D})$. Take any $a \in U$. Since a is an ultrafilter and $\psi([E_0]) \vee \psi([E_1]) = [Y]$, we have $\psi([E_0]) \in a$. Then $[E_0] \in F(a)$, which gives that $[A_0] \notin F(a)$. Therefore, $F(U) \subseteq S(\mathcal{C}) \setminus H[A_0]$ and hence the map F is continuous. ∎

Note that the homomorphism ψ of Theorem 3.6 is an injection. Indeed; if, for instance, $C \subseteq A \setminus B$ where $A, B, C \in \mathcal{C}$, and C is non-compact, then $\psi([C]) \wedge \psi([B]) = [\emptyset]$, while $\psi([C]) \wedge \psi([A]) \neq [\emptyset]$; hence $\psi([A]) \neq \psi([B])$.

Of course, an analogous version of Theorem 3.6 for the relation $\sim_k$ also holds.

Let us observe that [**9**, Theorem 2.1], cannot lead to internal necessary and sufficient conditions for pseudocompact spaces X and Y to have $\mathcal{R}(X) \subseteq \mathcal{R}(Y)$. However, with Theorem 3.6 in hand, we can easily deduce such conditions. Namely, we can establish the following

Theorem 3.7. *Let αX and γY be compactifications of pseudocompact spaces X and Y. Then $\mathcal{R}_\alpha(X) \subseteq \mathcal{R}_\gamma(Y)$ if and only if there exists a lattice homomorphism*

$\psi : L(Z_\alpha(X)) \to L(Z_\gamma(Y))$ satisfying conditions (I) and (II) of Theorem 3.6.

PROOF: Suppose that $\mathcal{R}_\alpha(X) \subseteq \mathcal{R}_\gamma(Y)$. There exists $\delta Y \in \mathcal{K}_\gamma(Y)$ such that $\alpha X \setminus X = \delta Y \setminus Y = K$. Let $q : \gamma Y \to \delta Y$ be the natural quotient map witnessing that $\delta Y \leqslant \gamma Y$. It follows from Proposition 1.5, Theorem 2.5 and Lemma 2.6 that, for each $[A] \in L(Z_\alpha(X))$, the collection $\psi([A]) = \{B \in Z_\gamma(Y) : (\gamma Y \setminus Y) \cap \mathrm{cl}_{\gamma Y} B = q^{-1}(K \cap \mathrm{cl}_{\alpha X} A)\}$ is a member of $L(Z_\gamma(Y))$. It is easy to verify that $\psi : L(Z_\alpha(X)) \to L(Z_\gamma(Y))$ is a lattice homomorphism which satisfies conditions (I) and (II) of Theorem 3.6. To conclude the proof, it is enough to use Theorem 2.5 and Theorem 3.6. ▯

THEOREM 3.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be complemented Wallman bases for spaces X and Y, respectively. Then $\mathcal{R}_{w_\mathcal{C}}(X) \subseteq \mathcal{R}_{w_\mathcal{D}}(Y)$ if and only if there exists a lattice homomorphism $\psi : L(\mathcal{C}) \to L(\mathcal{D})$ which satisfies conditions (I) and (II) of Theorem 3.6.*

PROOF: Suppose that $\delta Y \in \mathcal{K}_{w_\mathcal{D}}(Y)$ is such that $w_\mathcal{C} X \setminus X = \delta Y \setminus Y = K$. Let $q : w_\mathcal{D} Y \to \delta Y$ be the natural quotient map showing that $\delta Y \leqslant w_\mathcal{D} Y$. By virtue of Proposition 1.5 and Lemma 2.9, we can define a lattice homomorphism $\psi : L(\mathcal{C}) \to L(\mathcal{D})$ such that $(w_\mathcal{D} Y \setminus Y) \cap \mathrm{cl}_{w_\mathcal{D} Y} B = q^{-1}\big(K \cap \mathrm{cl}_{w_\mathcal{C} X} A\big)$ for any $[A] \in L(\mathcal{C})$ and $B \in \psi([A])$. The homomorphism ψ satisfies conditions (I) and (II) of Theorem 3.6, which, along with Theorem 3.6, completes the proof. ▯

Let us mention that, in view of Theorem 2.11 and Proposition 2.13, [**9**, Proposition 3.4] is an immediate consequence of Theorem 3.8.

The following problem seems interesting:

PROBLEM. Suppose we are given two spaces X and Y such that $L(Z(X)) \cong L(Z(Y))$. What topological properties of X are shared by Y?

Of course, we can pose analogous questions for the relation $\sim_k$, as well as for other Wallman bases. However, for instance, if $L(\varPi(X)) \cong L(\varPi(Y))$, the spaces X and Y can be quite different. Indeed, according to Proposition 3.5, we have $L(\varPi(\mathbb{N})) \cong L(\varPi(Y))$ for every space Y with $\beta Y \setminus Y \cong \beta\mathbb{N} \setminus \mathbb{N}$.

Almost compactness (see [**10**, 6J]) is one of those properties which are preserved under lattice isomorphisms, that is, if a space X has exactly one compactification and $L(Z(X)) \cong L(Z(Y))$, then the space Y has exactly one compactification, too.

Denote by $\mathcal{D}_X$ and $\mathcal{D}_Y$ the collections of all closed subsets of spaces X and Y, respectively. Faulkner and Vipera proved in [**9**, Corollary 2.7] that if X is a normal space which satisfies condition (C) (that is, in which every non-relatively compact set contains a closed non-compact subset), then $L_k(\mathcal{D}_X) \cong L_k(\mathcal{D}_Y)$ implies the normality of Y. It is worth noticing that condition (C) is unnecessary in [**9**, Corollary 2.7]; furthermore, [**9**, Corollary 2.7] has a simple direct proof which does not require any references to [**9**, Theorem 2.4] nor to [**9**, Proposition 2.5–2.6]. Namely, we can state the following

Proposition 3.9. *If $L_k(\mathcal{D}_X) \cong L_k(\mathcal{D}_Y)$ and X is a normal space, then the space Y is also normal.*

Proof: Suppose that $\psi : L_k(\mathcal{D}_X) \to L_k(\mathcal{D}_Y)$ is a lattice isomorphism. To show that Y is normal, take a pair A_1, A_2 of disjoint closed subsets of Y. By Lemma 1.3, there exist $B_i \in \psi^{-1}([A_i]_k)$ such that $B_1 \cap B_2 = \emptyset$. Since X is normal, we can find sets $C_1, C_2 \in \mathcal{D}_X$ such that $B_1 \subseteq X \setminus C_1 \subseteq C_2 \subseteq X \setminus B_2$. By Lemma 1.3, there exist $D_i \in \psi([C_i]_k)$ such that $A_i \cap D_i = \emptyset$ for $i = 1, 2$. The set $D = Y \setminus (D_1 \cup D_2)$ being relatively compact, there exists a relatively compact open set $U \subseteq Y$ such that $\mathrm{cl}_Y D \subseteq U$. Then $U_1 = Y \setminus (D_1 \cup \mathrm{cl}_Y D)$ and $U_2 = Y \setminus D_2$ are disjoint open neighbourhoods of $A_1 \setminus U$ and A_2, respectively. Since U is relatively compact, there exist disjoint open sets $V_1, V_2 \subseteq Y$ such that $A_1 \cap \mathrm{cl}_Y U \subseteq V_1$ and $A_2 \subseteq V_2$. Then $W_1 = U_1 \cup V_1$ and $W_2 = U_2 \cap V_2$ are disjoint open neighbourhoods of A_1 and A_2, respectively. ∎

4. Remarks on ESH-compactifications

According to [5], a compactification αX of a non-compact space X is an ESH-compactification if and only if there exists a base $\mathcal{C}$ for the closed sets of $K = \alpha X \setminus X$ which is stable under finite intersections and has the property that there exists a map η of $\mathcal{C}$ to the collection of closed sets of X satisfying the following conditions:

(E1) $\eta(\emptyset)$ is compact in X;

(E2) $X \setminus \eta(C)$ is not relatively compact in X for every $C \in \mathcal{C} \setminus \{K\}$;

(E3) for any $C_1, C_2 \in \mathcal{C}$, $\eta(C_1 \cap C_2) \triangle \big(\eta(C_1) \cap \eta(C_2)\big)$ is relatively compact in X;

(E4) if $C_1, C_2 \in \mathcal{C}$ and $\mathrm{int}_K(C_1) \cup \mathrm{int}_K(C_2) = K$, then $X \setminus \big(\eta(C_1) \cup \eta(C_2)\big)$ is relatively compact in X.

Theorem 4.1. *If $\mathcal{D}$ is a Wallman base for a non-compact space X such that $L_k(\mathcal{D}) \cong L(\mathcal{D})$, then $w_{\mathcal{D}}X$ is an ESH-compactification.*

Proof: Consider the base $\mathcal{C} = \{(w_{\mathcal{D}}X \setminus X) \cap \mathrm{cl}_{w_{\mathcal{D}}X} D : D \in \mathcal{D}\}$ for the closed sets of $w_{\mathcal{D}}X \setminus X$. It follows from Proposition 1.5 that if $L_k(\mathcal{D}) \cong L(\mathcal{D})$, then there exists a lattice isomorphism $\psi : \mathcal{C} \to L_k(\mathcal{D})$. For each $C \in \mathcal{C}$, pick an arbitrary $\eta(C) \in \psi(C)$. The map $\eta : \mathcal{C} \to \mathcal{D}$ satisfies conditions (E1)-(E4). ∎

The authors of [5] posed the following question:

Q. If X is a non-compact locally compact space, must βX be an ESH- compactification?

Caterino, Faulkner, Vipera and the referee of [5] gave a positive answer to the above question for a paracompact or realcompact X. Evidently, the Čech-Stone compactification of a non-compact space X with $L_k(Z(X)) \cong L(Z(X))$ is an ESH-compactification. We shall show that $L_k(Z(X)) = L(Z(X))$ for every strongly isocompact space X. Since the class of strongly isocompact spaces contains all paracompact and all realcompact spaces, our partial solution of (Q) is better than the answer to (Q) given in [**5**, Theorems 6 and 7].

Let us recall that a subset A of a space X is relatively pseudocompact in X if every function $f \in C(X)$ is bounded on A (see [**2, 3**]). A Tychonoff space X is strongly isocompact (or hyperisocompact) if and only if every relatively pseudocompact closed subset of X is compact (see [**3**, Proposition 3.0] and [**2**, p.81]). We refer the reader to [**4**] and [**17**] for examples of spaces that are strongly isocompact but are neither realcompact nor paracompact. We mention that, for instance, every normal weakly $[\omega_1, \infty)^r$ - refinable space is strongly isocompact.

Lemma 4.2. *If a space X is strongly isocompact, then the relations $\sim$ and $\sim_k$ coincide on $Z(X)$.*

Proof: It is enough to check that if $A, B \in Z(X)$ and $A \setminus B$ is not relatively compact, then there exists a non-compact $Z \in Z(X)$ such that $Z \subseteq A \setminus B$.

The space X being strongly isocompact, the set $S = \mathrm{cl}_X(A \setminus B)$ cannot be relatively pseudocompact. By [**2**, Proposition 2.6], there exists $p \in (\mathrm{cl}_{\beta X} S) \setminus vX$. Mimicking the proof of [**14**, Lemma 1.2], take a non-negative function $f \in C(\beta X)$ such that $p \in f^{-1}(0) \subseteq \beta X \setminus X$. Let $g = 1/f$. We can find a discrete collection of closed intervals $[a_n, b_n]$ in $\mathbb{R}$ with $b_n \nearrow \infty$ such that $g^{-1}([a_n, b_n]) \cap (A \setminus B)$ contains a non-void set $Z_n \in Z(X)$. Then $Z = \bigcup_{n=1}^{\infty} Z_n$ is a non-compact zero-set contained in $A \setminus B$. ∎

As an immediate consequence of Theorem 4.1 and Lemma 4.2, we get the following

Proposition 4.3. *If a non-compact space X is strongly isocompact, then βX is an ESH-compactification of X.*

One should not expect that if βX is an ESH-compactification, then $L_k(Z(X)) \cong L(Z(X))$. Indeed; by [**16**, Theorem 2.1], there exists a maximal almost disjoint family $\mathcal{R}$ in $\mathbb{N}$ such that $\beta(\mathbb{N} \cup \mathcal{R}) \setminus (\mathbb{N} \cup \mathcal{R})$ is zero-dimensional. It follows from [**5**, Theorem 4] or from Propositions 2.10 and 2.13 that $\beta(\mathbb{N} \cup \mathcal{R})$ is an ESH-compactification; however, as we have shown in Example 2.8, the lattices $L_k(Z(\mathbb{N} \cup \mathcal{R}))$ and $L(Z(\mathbb{N} \cup \mathcal{R}))$ are non-isomorphic.

Finally, let us note that, in view of Propositions 2.10, 2.13 and 4.3, Theorem 4.1 is a common generalisation of [**5**, Theorems 4, 6 and 7].

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