FINITE GROUPS WITH NORMAL NORMALIZERS

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We say that a finite group G has property N if the normalizer of every subgroup of G is normal in G. Such groups are nilpotent since every Sylow subgroup is normal (the normalizer of a Sylow subgroup is its own normalizer). Thus it is sufficient to study p-groups which have property N. Note that property N is inherited by subgroups and factor groups. We shall show that $P(G) \supseteq G_3$. It follows that if $p > 3$, then G is regular and $P(G') \supseteq G_4$. In particular, G' is one of the groups studied in (**5**). If G can be generated by n elements, then G has class at most $2n$. We shall find all of the 2-generator p-groups ($p > 3$) which have property N. Since property N is inherited by subgroups, it follows that any group which has property N can be generated by elements $x_1, \ldots, x_n$ where the groups $\langle x_i, x_j \rangle$ are known.

All groups considered are finite p-groups. We shall use the following notation: $h^g = g^{-1}hg$; $(h, g) = h^{-1}h^g$; (H, K) is the subgroup generated by $\{(h, k) | h \in H, k \in K\}$; $G_1 = G$, $G_{n+1} = (G_n, G)$; $G' = G_2$; $P(G)$ is the subgroup generated by pth powers; $\phi(G)$ is the Frattini subgroup of G; $N_G(H)$ is the normalizer in G of H; $H(x)$ is the (normal) subgroup generated by $\{x^g | g \in G\}$.

LEMMA 1. *Suppose that G has property* N. *Then $H(x)$ has class at most* 2. *If x has order p, then $H(x)$ is abelian.*

Proof. It follows from property N that $H(x)$ normalizes the cyclic group $\langle x \rangle$. If M is the subgroup of $H(x)$ consisting of elements which commute with x, then M is normal in $H(x)$ and $H(x)/M$ is isomorphic to a group of automorphisms of $\langle x \rangle$. Since the automorphism group of a cyclic group is abelian, it follows that M contains the commutator subgroup of $H(x)$. Thus $(x, H(x)') = 1$. Therefore $(x^g, H(x^g)') = 1$ for every $g \in G$. Since $H(x)$ is generated by $\{x^g | g \in G\}$, we see that $H(x^g) = H(x)$ and it follows that $(H(x), H(x)') = 1$.

If x has order p, then $\langle x \rangle$ is a normal subgroup of order p in $H(x)$ and hence is in the centre of $H(x)$. Since $H(x) = H(x^g)$ for every $g \in G$, it follows that x^g is also in the centre of $H(x)$. Therefore $H(x)$ is abelian.

THEOREM 1. *If G has property* N *and if G can be generated by n elements, then* $G_{2n+1} = 1$.

Received May 1, 1967. This research was supported jointly by the National Science Foundation under grant GR-5691 and by the Air Force Office of Scientific Research under contract AF-AFOSR-937-65.

Proof. Suppose that $G = \langle x_1, \ldots, x_n \rangle$. Then $G = H(x_1)\ H(x_2) \ldots H(x_n)$ where, by Lemma 1, each $H(x_i)$ has class at most 2. It is known that whenever A, B are normal subgroups of G of class a, b, respectively, then AB has class at most $a + b$. The theorem follows from a straightforward argument.

THEOREM 2. *Suppose that G has property* N. *Then $P(G) \supseteq G_3$. If $p > 3$, then G is regular.*

Proof. Let $K = G/P(G)$. If $P(G)$ does not contain G_3, then $K_3 \neq 1$. By Lemma 1, $H(x)$ is abelian for every x in K. Thus, K is a p-group of exponent p in which every element commutes with all of its conjugates. Such groups are known to have class at most 2 when $p \neq 3$; see (**1**). Suppose now that $p = 3$. If $K_3 \neq 1$, there are elements a, b, c in K such that $(a, b, c) \neq 1$. Since K has exponent 3, $(a, b, c) = (b, c, a) = (c, a, b)$, and $K_4 = 1$; see (**2**, p. 322). Let T be the subgroup generated by a, b. Clearly, $b \in N_K(T)$; hence, it follows from property N that $(b, c, a) \in T$. Let $M = \langle a, b, c \rangle$. Every 2-generator subgroup of M has class at most 2 since if $x, y \in M$, then $(x, y) \in H(x) \cap H(y)$, where both of $H(x)$ and $H(y)$ are abelian. Since $M_3 \neq 1$, it follows that a, b are independent modulo M'; hence, $T \cap M' = T'$. Since $T_3 = 1$, $T' = \langle (a, b) \rangle$. We now have $(b, c, a) \neq 1$ in T', and we know that (b, c, a) is central in K, hence (a, b) is central in K. Therefore $(a, b, c) = 1$, a contradiction. Thus $K_3 = 1$ for all p and it follows that $P(G) \supseteq G_3$.

Suppose now that $p > 3$. A p-group is regular if and only if every 2-generator subgroup is regular. Let K be a 2-generator subgroup of G. Since K has property N we know that $P(K) \supseteq K_3$; consequently, $(K: P(K)) \leqq p^3$. By a theorem of P. Hall (**4**, Theorem 2.3) a p-group K is regular whenever $(K: P(K)) < p^p$. Therefore K is regular.

COROLLARY. *If G has property* N *and $p > 3$, then $P(G') \supseteq G_4$.*

Proof. By Theorem 2, G is regular and $P(G) \supseteq G_3$. Therefore $(G, P(G)) \supseteq G_4$. The result follows from the fact that in a regular group, $(G, P(G)) = P(G')$; see (**3**, Theorem 4.4).

Remark. The restrictions on p in Theorem 2 are necessary. When p is 2 or 3 there are irregular groups which have property N. The non-abelian groups of order 8 are examples for $p = 2$. An example for $p = 3$ is the group $G = \langle a, b \rangle$ defined by the relations $a^9 = 1$, $b^3 = a^6$, $a^b = ac$, $a^c = a^4$, $c^3 = (b, c) = 1$. This group has the property that if $x \in G - G'$, then $\langle x^3 \rangle = \langle a^3 \rangle = G_3$. It follows that $P(G) = \langle a^3 \rangle$; hence $(G : P(G)) = 3^3$ whereas the elements of order 3 generate a subgroup of order 3^2. This cannot happen in a regular group. On the other hand, G' normalizes every cyclic subgroup, hence G has property N.

We shall now restrict our attention to p-groups (for $p > 3$) which can be generated by two elements.

https://doi.org/10.4153/CJM-1968-121-8 Published online by Cambridge University Press

LEMMA 2. *Suppose that G has property* N *and that $p > 3$. If G can be generated by two elements, then $G_4 = 1$.*

Proof. Let $K = G/G_5$. It will suffice to show that $K_4 = 1$. By Theorem 2, K is regular; thus, we may suppose that the generators x, y of K are chosen from a *canonical basis*; see (**3**, p. 91). In particular, we may suppose that $\langle x\rangle \cap \langle y\rangle = 1$. By property N, $(x, y) \in H(x) \cap H(y)$; hence, $(x, y, x) \in \langle x\rangle$ and $(x, y, y) \in \langle y\rangle$. Therefore, $(x, y, x, x) = (x, y, y, y) = 1$. The remaining generators for K_4 are (x, y, x, y) and (x, y, y, x). We shall use the identity

$$(1)\qquad (u, v^{-1}, w)^v(v, w^{-1}, u)^w(w, u^{-1}, v)^u = 1$$

which is valid in any group. Set $u = (x, y)$, $v = x$, $w = y$. Then each term of (1) is in K_4 which is central; thus, we can omit the conjugations by v, w, u. We have that

$$(2)\qquad ((x, y), x^{-1}, y)(x, y^{-1}, (x, y))(y, (x, y)^{-1}, x) = 1.$$

Since $K_5 = 1$, we have that

$$(x, y, x^{-1}, y) = (x, y, x, y)^{-1}, \qquad ((x, y^{-1}), (x, y)) = 1,$$

$$(y, (x, y)^{-1}, x) = (y, (x, y), x)^{-1} = (x, y, y, x).$$

Substituting these results in (2) yields

$$(x, y, y, x) = (x, y, x, y).$$

The left-hand side of this equation is an element of $\langle x\rangle$ while the right-hand side is an element of $\langle y\rangle$, thus each side is 1. We have shown that a generating set for K_4 consists of elements which are 1, therefore $K_4 = 1$.

We shall show later that G' normalizes every cyclic subgroup of G. We observe now that G' normalizes $\langle g\rangle$ whenever $g \notin \phi(G)$.

LEMMA 3. *Suppose that G has property* N *and that $p > 3$. If G can be generated by two elements and if $g \notin \phi(G)$, then G' normalizes $\langle g\rangle$.*

Proof. It follows from the choice of g that there is an element h such that $G = \langle g, h\rangle$. Since g normalizes $\langle g\rangle$, any commutator involving g must normalize $\langle g\rangle$. Therefore, G' normalizes $\langle g\rangle$.

We can now describe the 2-generator groups which have property N.

THEOREM 3. *Suppose that G has property* N *and that $p > 3$. If G can be generated by two elements, then $G = \langle x, y\rangle$, where $\langle x\rangle \cap \langle y\rangle = 1$; $(x, y, x) = x^{kp^s}$, $(x, y, y) = y^{kp^s}$, where k is prime to p; $G_4 = 1$; if G_3 is cyclic, then we may suppose that $(x, y, y) = 1$. Conversely, any group which satisfies these relations will have property* N.

Proof. Suppose that G satisfies the relations given in the theorem. We shall show that G' normalizes every cyclic subgroup of G. It will follow immediately that G has property N. Set $c = (x, y)$. If $g \in G$, then $g = x^u y^v c^n$ for appropriate integers u, v, n. If $h \in G'$, then $h = c^w z$ for some integer w and some z in the centre of G. Since $G_4 = 1$, $(h, g) = (c, x)^{wu}(c, y)^{wv}$; thus $(h, g) = x^{uwkp^s}y^{vwkp^s}$. We must show that (h, g) is a power of g. Since $p > 3$ and $G_4 = 1$, the group G is regular; thus $g^{p^s} = x^{up^s}y^{vp^s}d^{p^s}$ for some $d \in G'$. The order of the commutator $(x, y) = c$ cannot be greater than the smallest power of x which lies in $Z(G)$; see (**3**, Theorem 4.22). Therefore, $c^{p^s} = 1$, and hence, G' has exponent p^s. Thus, $g^{p^s} = x^{up^s}y^{vp^s}$. Therefore, $(h, g) = g^{kwp^s}$.

Suppose now that G is a 2-generator p-group ($p > 3$) which has property N. We know that $G_4 = 1$ (Lemma 2) and that G is regular (Theorem 2). Pick generators x, y for G from a canonical basis. Then $G = \langle x, y\rangle$, where $\langle x\rangle \cap \langle y\rangle = 1$. It follows from Lemma 3 that $(x, y, x) = x^{kp^s}$, $(x, y, y) = y^{rp^t}$ for appropriate integers k, s, r, t, where we may assume that k and r are prime to p. We must show that we can take $k = r$, $s = t$.

We first consider the case where G_3 is non-cyclic. Thus, $x^{p^s} \neq 1$ and $y^{p^t} \neq 1$. We know that (x, y) normalizes xy (Lemma 3); thus, $(x, y, xy) = (xy)^n$ for some n. Since $G_4 = 1$, $(x, y, xy) = (x, y, x)(x, y, y)$; thus, $(xy)^n = x^{kp^s}y^{rp^t}$. Since $(xy)^n$ is in the centre of G, and $G = \langle x, xy\rangle$, we know that G' has exponent at most n; hence, $(g\cdot xy)^n = g^n(xy)^n$ for every $g \in G$. In particular, $y^n = (x^{-1}\cdot xy)^n = x^{-n}(xy)^n$; thus, $y^{n-rp^t} = x^{kp^s-n}$. Therefore, $rp^t = xp^s$ modulo d, where d is the minimum of $|x|$, $|y|$. There is no loss of generality if we suppose that $s \leqq t$. Since G_3 is not cyclic, we know that p^{s+1} divides d. If $s < t$, we have that $(rp^{t-s} - k)p^s$ is divisible by p^{s+1}, a contradiction. Therefore, $s = t$. Thus, $(r - k)p^s = 0$ modulo d. If $d = |x|$, then $x^{rp^s} = k^{kp^s}$. If $d = |y|$, then $y^{rp^s} = y^{kp^s}$. In either case we may suppose that r and k are equal. This completes the proof when G_3 is non-cyclic.

If G_3 is cyclic but non-trivial, then we may suppose that $(x, y, x) = x^{kp^s} \neq 1$, $(x, y, y) = 1$. We must show that $y^{p^s} = 1$. As above, $(x, y, xy) = x^{kp^s}$ and hence $(xy)^n = x^{kp^s}$ for some n. Since $(xy)^n$ is central, $(xy)^n = x^ny^n$. It follows from $x^ny^n = x^{kp^s}$ that $y^n = 1$. Thus, $x^n = x^{kp^s}$. Therefore, p^s divides n but p^{s+1} does not divide n. Therefore, $y^{p^s} = 1$.

Finally, if $G_3 = 1$, we see that G satisfies the necessary relations if we set p^s equal to the maximum of $|x|$, $|y|$. This completes the proof.

If G is a group with property N, then every pair of generators of G must give one of the groups described in Theorem 3. Since property N is inherited by subgroups, one might conjecture that a group G has property N if and only if every 2-generator subgroup has property N. Unfortunately, this conjecture is false. (A counterexample is given below.) However, the corresponding conjecture for 3-generator subgroups is true for all primes p.

THEOREM 4. *A group G has property* N *if and only if every* 3-*generator subgroup of G has property* N.

Proof. It will suffice to show that if G fails to have property N, then there is a 3-generator subgroup which fails to have property N. Suppose that H is a

https://doi.org/10.4153/CJM-1968-121-8 Published online by Cambridge University Press

subgroup of G such that $N_G(H)$ is not normal in G. Then there is an element x in $N_G(H)$ and an element g in G such that x^g does not normalize H. Thus, there is an element h in H such that h^{x^g} does not belong to H. In particular, h^{x^g} does not belong to H_1, the normal subgroup of $\langle h, x\rangle$ generated by all conjugates of h obtained from elements of $\langle h, x\rangle$. Let $G_1 = \langle h, x, g\rangle$. Then x normalizes H_1 but x^g does not normalize H_1; thus, G_1 does not have property N.

We shall now give an example of a 3-generator p-group ($p > 3$) which does not have property N but in which every 2-generator subgroup does have property N. Let $\langle a, b\rangle$ be the non-abelian group of order p^3 and exponent p. Let H be the direct product of $\langle a, b\rangle$ with $\langle u, v\rangle$, an elementary abelian group of order p^2. Form K by adjoining to H an element x of order p such that $a^x = au$, $u^x = u$, $b^x = b$, $v^x = vc$, where c denotes (a, b). The required group G is formed by adjoining to K an element g such that $g^p = c$, $x^g = xb$, $b^g = b$, $a^g = av$, $v^g = v$, $u^g = uc^2$. Clearly, $G = \langle a, x, g\rangle$. G does not have property N since x normalizes $\langle a, u\rangle$ but $(x, g, a) \notin \langle a, u\rangle$. A long but routine calculation shows that every 2-generator subgroup does have property N.

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