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Homological realization of Nakajima varieties and Weyl group actions

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Abstract

We give a realization of Nakajima varieties and the action of the Weyl group on them using certain canonical structures of homological algebras and their natural generalization, which we develop in this paper. We consider in detail the case of an affine quiver, where we present a simple homological characterization of Nakajima varieties and its relation to moduli of sheaves on the projective plane.

Contents

Introduction

A geometric approach to the representation theory of Kac–Moody algebras was given by Nakajima in the groundbreaking work [Nak94], which was a culmination of a series of remarkable discoveries discussed in the introduction to his paper. For any simply laced Kac–Moody algebra $\mathfrak{g}$ with triples of generators $(e_a, h_a, f_a)_{a\in I}$ indexed by a finite set I, Nakajima constructed a family of complex varieties $\mathcal{M}_\zeta(\mathbf{v}, \mathbf{w})$, where $\mathbf{v}, \mathbf{w} \in \mathbb{N}^I$ and $\zeta \in \mathbb{R}^3 \otimes \mathbb{R}^I$ such that, for any *generic* ζ,

$$\dim H^{\mathrm{mid}}(\mathcal{M}_\zeta(\mathbf{v}, \mathbf{w})) = \dim L_\lambda[\lambda - \alpha],$$

where L_λ is the integrable highest weight $\mathfrak{g}$-module of highest weight λ, where $L_\lambda[\lambda - \alpha]$ is the corresponding weight space, and where

$$\alpha = \sum_a v_a \alpha_a, \quad \lambda = \sum_a w_a \omega_a,$$

for $(\alpha_a)_{a\in I}, (\omega_a)_{a\in I}$ the set of simple roots and fundamental weights, respectively. Nakajima realized the action of the generators $(e_a, h_a, f_a)_{a\in I}$ in a geometric way, the contravariant form on L_λ, and

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indicated the geometric meaning of the Weyl group action on weight spaces

$$\sigma : L_\lambda[\lambda - \alpha] \xrightarrow{\sim} L_\lambda[\sigma(\lambda - \alpha)].$$

The action of the Weyl group on the underlying quiver varieties was further developed by Lusztig [Lus00], Maffei [Maf02], and Nakajima [Nak03].

Nakajima varieties are defined in terms of certain data attached to the Dynkin diagram $\mathbf{Q}$ of the Kac–Moody algebra $\mathfrak{g}$. This data can be viewed as a generalization of the Atiyah–Drinfeld–Hitchin–Manin description of the instanton moduli spaces. Correspondingly, various structures of representation theory of Kac–Moody algebra $\mathfrak{g}$ including the action of the Weyl group on highest weight modules were described in terms of this linear data. We will review the original constructions relevant to our present work in § 1.

In this paper, we interpret the data and, consequently, Nakajima varieties, via differential graded modules over a finite-dimensional quotient $A(\mathbf{Q})$ of the double path algebra of $\mathbf{Q}$. Defining relations in this quotient algebra depend on the choice of orientation ϵ of $\mathbf{Q}$, but different orientations produce isomorphic algebras. The algebra $A(\mathbf{Q})$ is the quadratic dual of the preprojective algebra of the (oriented) graph $\mathbf{Q}$ and has a Frobenius structure. If $\mathbf{Q}$ is bipartite (see § 5), $A(\mathbf{Q})$ is isomorphic to the zigzag algebra of $\mathbf{Q}$ studied in [HK01]. Let us denote $A(\mathbf{Q})$ simply by A.

Our realization of the Nakajima varieties allows us to view their theory in the context of homological algebra. In particular, simple A-modules S_a and projective A-modules P_a are the basic building blocks in our picture. Furthermore, we give a natural interpretation of the Weyl group action on Nakajima varieties, with the simple reflection s_a acting as homological 'addition' and 'substraction' of the projective module P_a. This Weyl group action comes from a modification of the braid group action in the derived category of A-modules, see [KS02, ST01, RZ03, HK01].

The theory of Nakajima varieties also suggest certain generalizations of some classical notions and results in homological algebra. In fact, our constructions depend in an essential way on the value of the parameter $\zeta = (\zeta_\mathbb{R}, \zeta_\mathbb{C})$, where $\zeta_\mathbb{R} \in \mathbb{R}^I$ and $\zeta_\mathbb{C} \in \mathbb{C}^I$. If $\zeta_\mathbb{C} = 0$ then we use the standard theory of differential graded algebras and complexes of modules over them. On the other hand, when $\zeta_\mathbb{C} \neq 0$ we are led to consider modules over A equipped with a generalized differential d, which is a degree 1 map satisfying

$$d^2 = c$$

for a suitable central element $c \in A$. We introduce categories of (A, c)-complexes in § 2.

To define the Weyl group action we are then forced into 2-periodic generalized complexes, which we call *duplexes*. The theory of duplexes, which we outline in § 3, can be developed in parallel with some classical results of homological algebra and seems to be worthy of a deeper, independent study.

In § 4 we consider another way of deforming the derived category of A-modules: by a theorem of Happel [Hap88], this derived category is equivalent to the stable category of graded modules over the algebra $A \dot{\otimes} \mathbb{C}[d]/d^2$, where $\dot{\otimes}$ denotes the super tensor product. We introduce the stable category of $A \dot{\otimes} \mathbb{C}[d]/\langle d^2 - c\rangle$ and relate this to the category of (A, c)-duplexes.

Our interpretation of Nakajima varieties $\mathcal{M}_\zeta(\mathbf{v}, \mathbf{w})$ is given in § 5, and is set-theoretic, consisting of a bijection between the set of points of $\mathcal{M}_\zeta(\mathbf{v}, \mathbf{w})$ and certain isomorphism classes of differential A-modules. Let $(V_a)_{a\in I}$, $(W_a)_{a\in I}$ denote collections of vector spaces of dimensions $(v_a)_{a\in I}$ and $(w_a)_{a\in I}$, respectively. We consider $\mathbb{Z}/2\mathbb{Z}$-graded A-modules M equipped with a generalized differential d such that $d^2 = c$, where c is $\zeta_\mathbb{C}$, viewed as a central element of A, and

$$M \cong \bigoplus_a (P_a \otimes V_a \oplus S_a[-1] \otimes W_a), \tag{0.1}$$

where $[-1]$ denotes a grading shift, and the isomorphism is that of A-modules. When $\zeta_\mathbb{C}$ is not generic (for example, $\zeta_\mathbb{C} = 0$), we add an irreducibility condition with respect to d, analogous to a stability condition. We show that the set of classes of pairs (M, d) as above is in a natural bijection with the set of G_W-orbits of the quiver variety $\mathcal{M}_\zeta(\mathbf{v},\mathbf{w})$, where $G_W = \prod_a GL(W_a)$. The full variety $\mathcal{M}_\zeta(\mathbf{v},\mathbf{w})$ is obtained by fixing, in addition, an isomorphism (framing) $W_a \simeq \mathrm{Hom}_A(S_a[-1], M)$ for $a \in I$.

To obtain a realization of the Weyl group action, we consider a duplex of A-bimodules $C_{a,x}$ associated to a vertex a of Q:

$$\to P_a \otimes {}_aP \to A \to$$

where $x \in \mathbb{C}$ and one of the bimodule maps depends on x. We show in § 6 that duplexes $C_{a,x}$ are invertible in the homotopy category (as well as in the stable category),

$$C_{a,x} \otimes_A C_{a,-x} \cong A,$$

and satisfy Yang–Baxter relations:

$$C_{a,x} \otimes C_{b,y} \cong C_{b,y} \otimes C_{a,x}$$

for any two vertices a and b which are not joined by an edge, and

$$C_{a,x} \otimes C_{b,x+y} \otimes C_{a,y} \cong C_{b,y} \otimes C_{a,x+y} \otimes C_{b,x}$$

for a and b joined by a single edge. In the limit $x \to 0$ our duplexes degenerate into those used to categorify the Burau representation of the braid group, see [KS02].

Points of Nakajima varieties $\mathcal{M}_\zeta(\mathbf{v},\mathbf{w})$ can be identified with certain isomorphism classes of A-duplexes. The functor of the tensor product with the bimodule duplex $C_{a,x}$ acts in categories of A-duplexes and restricts to a bijection

$$\mathcal{R}_a : \mathcal{M}_\zeta(\mathbf{v},\mathbf{w}) \xrightarrow{\sim} \mathcal{M}_{s_a\cdot\zeta}(s_a \cdot (\mathbf{v},\mathbf{w})),$$

where s_a is a simple reflection. We show in § 7 that our reflection maps coincide with those in [Lus00, Maf02, Nak03].

The class of Kac–Moody algebras associated to affine Dynkin diagrams plays a very special role in representation theory. The corresponding class of quivers is also distinguished in the Nakajima theory since it appears in the study of the instanton moduli spaces. It is well known that simply-laced affine Dynkin diagrams are in a bijection with finite subgroups $\Gamma \subset SL(2,\mathbb{C})$ and it is natural to reformulate Nakajima's work entirely in terms of these finite groups. In § 8 we recast our realization of Nakajima varieties in this light. We replace A by the Morita equivalent algebra $A_\Gamma = \Lambda\rho \otimes \mathbb{C}[\Gamma]$, where ρ is the natural two-dimensional representation of Γ. To a collection of vector spaces $(V_a)_{a\in I}$ and $(W_a)_{a\in I}$ we now associate two Γ-modules

$$\mathbb{V} = \bigoplus_a \rho_a \otimes V_a, \quad \mathbb{W} = \bigoplus_a \rho_a \otimes W_a,$$

where $(\rho_a)_{a\in I}$ is the set of all irreducible representations of Γ. The module M in (0.1) is replaced by the following

$$M = \Lambda\rho \otimes \mathbb{V} \oplus \mathbb{W}[-1]. \tag{0.2}$$

The latter can be viewed as a module over $\widetilde{A}_{\Gamma,c} = A_\Gamma \mathbin{\dot\otimes} \mathbb{C}[d]/\langle d^2 - c\rangle$, where c is a central element of A_Γ which depends linearly on $\zeta_{\mathbb{C}}$. It turns out that we can characterize modules of the form (0.2) as a certain class of elements of the stable category of $\mathbb{Z}/2\mathbb{Z}$-graded $\widetilde{A}_{\Gamma,c}$-modules $\underline{\mathbf{Mod}}_2(\widetilde{A}_{\Gamma,c})$. This yields a realization of Nakajima varieties via isomorphism classes of pairs (M, u), where M is an object of $\underline{\mathbf{Mod}}_2(\widetilde{A}_{\Gamma,c})$ and $u : \mathbb{W} \to R(M)$ is a fixed isomorphism, with $R(M)$ being the restriction of the module M to A_Γ.

The relation between this realization of the Nakajima varieties by means of the stable category $\underline{\mathbf{Mod}}_2(\widetilde{A}_{\Gamma,c})$ and the realization as a moduli space of Γ-equivariant torsion-free sheaves on a *non-commutative* $\mathbb{P}^2$ with fixed framing at infinity (presented in [BGK]) is explained in § 9. In fact, one may view such torsion-free sheaves as modules over the algebra Koszul dual to $\widetilde{A}_{\Gamma,c}$, see [BGK, Appendix B]. Thus, our construction illustrates a noncommutative version of the classical theorem of [BGG78] claiming that when $\Gamma = \{e\}$ and $c = 0$ (the case of commutative $\mathbb{P}^2$), the derived category of coherent sheaves over $\mathbb{P}^2$ is equivalent to the stable category of $\widetilde{A}_{\Gamma,c}$. Finally, we reformulate the action of the Weyl group on quiver varieties via certain natural duplexes of A_Γ-bimodules.

We believe that the realization of the Nakajima varieties and Weyl group actions by means of natural categorical constructions presented in this paper is only a first step in a more general program of recasting the Nakajima geometric approach to representation theory of Kac–Moody algebras in terms of canonical structures of homological algebra. We hope that the emerging interaction between the two areas will be beneficial to both subjects. Below we will make a few remarks about further developments of both areas inspired by our constructions.

As we mentioned in the beginning of the introduction, the Nakajima varieties encode the structure of the integrable highest weight modules L_λ. The latter modules possess rich structures associated with the corresponding Weyl group $\mathbf{W}$. On the one hand, each module L_λ contains a family of Demazure submodules $L_{\lambda,w}$ defined for any $w \in \mathbf{W}$. On the other hand, each module L_λ admits the Bernšteĭn–Gel'fand–Gel'fand (BGG) resolution by Verma modules $V_{w\cdot\lambda}$, where again $w \in \mathbf{W}$ (see, e.g., [Kum02] for a review of both constructions). It is natural to expect that our realization of the action of $\mathbf{W}$ on Nakajima varieties should lead to a transparent geometric construction of the Demazure modules as well as the BGG resolution.

Concerning the applications to homological algebra, the example studied in this paper already suggests the following generalizations of algebraic structures:

$$\begin{aligned} \text{abelian} &\longrightarrow \text{triangulated} \\ \mathbb{Z}\text{-graded} &\longrightarrow \mathbb{Z}/2\mathbb{Z}\text{-graded} \\ d^2 = 0 &\longrightarrow d^2 = c. \end{aligned}$$

Starting with an abelian category of modules over a ring, we pass to the triangulated category of complexes (the top arrow in the diagram). This arrow is the familiar advancement from the classical theory of modules over a ring to homological algebra. The bottom arrows refer to two more recent developments where:

- one gains from working with periodic triangulated categories, those with $[2k] \cong \mathrm{Id}$ for some k (case $k = 1$ seems especially important);
- differential modules acquire curvature (the square of differential is no longer zero).

Both transformations are natural from the deformation theory viewpoint. When the cohomology ring of a symplectic manifold is deformed to the quantum cohomology ring, its $\mathbb{Z}$-grading collapses to a $\mathbb{Z}/2k\mathbb{Z}$-grading, where k is the minimal Chern number (see [MS94, Section 1.7]), and the shift functor in the A_∞ triangulated Fukaya–Floer category of the manifold is periodic, $[2k] \cong \mathrm{Id}$. Deforming $d^2 = 0$ to $d^2 = c$ is as legitimate as deforming the ring structure, when one is describing all deformations of the homotopy category of modules over a (graded) ring. We should also mention the paper of Peng and Xiao [PX00], where 2-periodic triangulated categories appear in relation to Hall algebras.

1. Nakajima varieties

We recall the definition of Nakajima quiver varieties. Let $\mathbf{Q} = (I, E)$ be an arbitrary finite graph with I the set of vertices and E the set of edges. We allow $\mathbf{Q}$ to have loops and multiple edges. Let H be the set of oriented edges of this graph (thus H is 'twice as large' as E). For any $h \in H$ we

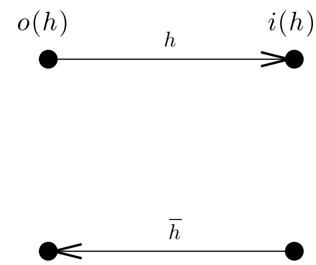

FIGURE 1. Two orientations of an edge.

denote by $o(h)$ and $i(h)$ the outgoing and incoming vertices of h, respectively, and by $\overline{h}$ the edge h with the opposite orientation, see Figure 1.

Let $(\mathbf{v}, \mathbf{w}) \in \mathbb{N}^I \times \mathbb{N}^I$ with $\mathbf{v} = (v_a)_{a \in I}$ and $\mathbf{w} = (w_a)_{a \in I}$. Fix some I-graded $\mathbb{C}$-vector spaces $V = \bigoplus V_a$ and $W = \bigoplus W_a$ such that $\dim V = \mathbf{v}$ and $\dim W = \mathbf{w}$. Set

$$E(V,V) = \bigoplus_{h \in H} \mathrm{Hom}(V_{o(h)}, V_{i(h)}),$$
$$L(V,W) = \bigoplus_a \mathrm{Hom}(V_a, W_a), \quad L(W,V) = \bigoplus_a \mathrm{Hom}(W_a, V_a)$$

and

$$\mathbf{M}(\mathbf{v}, \mathbf{w}) = E(V,V) \oplus L(W,V) \oplus L(V,W).$$

An element of $\mathbf{M}(\mathbf{v}, \mathbf{w})$ will usually be denoted by its components (B, i, j).

Let $\epsilon : H \to \{1, -1\}$ be any function satisfying $\epsilon(h) + \epsilon(\overline{h}) = 0$ for all $h \in H$. Such functions are in a bijection with orientations of $\mathbf{Q}$, the ϵ-orientation consists of all edges h with $\epsilon(h) = 1$. Consider the maps

$$\mu_{\mathbb{C}} : \mathbf{M}(\mathbf{v}, \mathbf{w}) \longrightarrow \bigoplus_a \mathfrak{gl}(V_a),$$
$$(B, i, j) \longrightarrow \left(\sum_{o(h)=a} \epsilon(h) B_{\overline{h}} B_h + i_a j_a \right)_a,$$

and

$$\mu_{\mathbb{R}} : \mathbf{M}(\mathbf{v}, \mathbf{w}) \longrightarrow \bigoplus_a \mathfrak{u}(V_a),$$
$$(B, i, j) \mapsto \frac{\sqrt{-1}}{2} \left(\sum_{o(h)=a} B_{\overline{h}} B_{\overline{h}}^\star - B_h^\star B_h + i_a i_a^\star - j_a^\star j_a \right)_a.$$

In the above, $f^\star$ denotes the Hermitian adjoint of f. Following Nakajima, to $\zeta_{\mathbb{R}} \in \mathbb{R}^I$ we associate a central element $\boldsymbol{\zeta}_{\mathbb{R}} = \sum_a (\sqrt{-1}\zeta_{\mathbb{R},a}/2)\,\mathrm{Id} \in \bigoplus_a \mathfrak{u}(V_a)$, and to $\zeta_{\mathbb{C}} \in \mathbb{C}^I$ we associate a central element $\boldsymbol{\zeta}_{\mathbb{C}} = \bigoplus_a \zeta_{\mathbb{C},a}\,\mathrm{Id} \in \bigoplus_a \mathfrak{gl}(V_a)$. The group $U_V = \prod_a U(V_a)$ acts on $\mathbf{M}(\mathbf{v}, \mathbf{w})$ by conjugation. Finally, we put

$$\mathcal{M}_\zeta(\mathbf{v}, \mathbf{w}) = (\mu_{\mathbb{R}} \times \mu_{\mathbb{C}})^{-1}(\boldsymbol{\zeta}_{\mathbb{R}}, \boldsymbol{\zeta}_{\mathbb{C}})/U_V,$$

where $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$. Different choices of ϵ yield isomorphic varieties.

1.1 When $\zeta_{\mathbb{R}} \in \mathbb{Z}^I$ there is also a purely complex-geometric description of $\mathcal{M}_\zeta(\mathbf{v}, \mathbf{w})$. Note that the group $G_V = \prod_a GL(V_a)$ acts on $\mathbf{M}(\mathbf{v}, \mathbf{w})$ by conjugation. To $\zeta_{\mathbb{R}}$ we associate the character $\chi_{\zeta_{\mathbb{R}}} : G_V \to \mathbb{C}^*, (g_a)_a \mapsto \prod_a \det(g_a)^{\zeta_{\mathbb{R},a}}$. Following Nakajima [Nak94], we have

$$\mathcal{M}_\zeta(\mathbf{v},\mathbf{w}) \simeq \mathrm{Proj}\Big(\bigoplus_{n\geqslant 0} A^n_{\zeta_{\mathbb{R}}}(\mathbf{v},\mathbf{w})\Big),$$

where

$$A^n_{\zeta_{\mathbb{R}}}(\mathbf{v},\mathbf{w}) = \{f \in \mathbb{C}[\mu_{\mathbb{C}}^{-1}(\boldsymbol{\zeta}_{\mathbb{C}})] \mid f(g\cdot(B,i,j)) = \chi_{\zeta_{\mathbb{R}}}(g)^n f((B,i,j))\ \forall g \in G_V\}.$$

There is an open subset $\mathbf{M}^{ss}_\zeta(\mathbf{v},\mathbf{w}) \subset \mu_{\mathbb{C}}^{-1}(\boldsymbol{\zeta}_{\mathbb{C}})$ of *semistable points* such that $\mathcal{M}_\zeta(\mathbf{v},\mathbf{w}) = \mathbf{M}^{ss}_\zeta(\mathbf{v},\mathbf{w})//G_V$ (categorical quotient). We will mainly be interested in the following two cases.

(i) $\zeta_{\mathbb{R}} \in (\mathbb{N}^+)^I$ and $\zeta_{\mathbb{C}} = 0$. In this case $(B,i,j) \in \mathbf{M}^{ss}_{\zeta_{\mathbb{C}}}(\mathbf{v},\mathbf{w})$ if the following condition is satisfied: the only (graded) B-invariant subspace of V contained in $\mathrm{Ker}\, j$ is $\{0\}$.

(ii) $\zeta_{\mathbb{R}}$ is arbitrary and $\zeta_{\mathbb{C}}$ satisfies the following genericity condition: for every $n_1,\ldots,n_k \in \mathbb{Z}$, we have $\sum_a n_a\zeta_{\mathbb{C},a} = 0 \Rightarrow n_a = 0$ for all a. In this case, all points (B,i,j) in $\mu_{\mathbb{C}}^{-1}(\boldsymbol{\zeta}_{\mathbb{C}})$ are semistable (and, in fact, all points (B,i,j) in $\mu_{\mathbb{C}}^{-1}(\boldsymbol{\zeta}_{\mathbb{C}})$ automatically satisfy the condition in case (i)).

In the two above cases, G_V acts freely on $\mathbf{M}^{ss}_\zeta(\mathbf{v},\mathbf{w})$ (see, e.g., [Nak94]), so that the categorical quotients are actually geometric (smooth) quotients. Note also that, in case (ii), the variety $\mathcal{M}_\zeta(\mathbf{v},\mathbf{w})$ is actually independent of $\zeta_{\mathbb{R}}$.

1.2 To the graph (I,E) we associate a symmetric $|I|\times|I|$ Borcherds matrix $A = (a_{ij})$ with

$$a_{ij} = 2\delta_{ij} - \#\{h \in H \mid i(h) = i, o(h) = j\}.$$

Let $I^{re} \subset I$ be the set of all loopless vertices (characterized by the relation $a_{ii} = 2$). To A corresponds a Borcherds algebra (or generalized Kac–Moody algebra) $\mathfrak{g}$ (see [Bor88]). Let us fix a Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and let α_a and ω_a (respectively $\alpha^\vee$ and $\omega_a^\vee$) stand for the simple root and fundamental weight (respectively simple coroot and fundamental coweight) associated to a vertex a. We put

$$Q = \bigoplus_s \mathbb{Z}\alpha_a, \quad P = \bigoplus_a \mathbb{Z}\omega_a.$$

We also set $Q^\vee = \bigoplus_a \mathbb{Z}\alpha_a^\vee$ and $P^\vee = \bigoplus_a \mathbb{Z}\omega_a^\vee$ and we denote by $\langle\cdot,\cdot\rangle$ the natural pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$. We consider $\mathbf{v},\mathbf{w},\zeta_{\mathbb{R}}$ and $\zeta_{\mathbb{C}}$ as elements of $\mathfrak{h}^*$ via the identifications

$$\mathbf{v} \mapsto \sum_a v_a\alpha_a, \quad \mathbf{w} \mapsto \sum_a w_a\omega_a, \quad \zeta_{\mathbb{R}} \mapsto \sum_a \zeta_{\mathbb{R},a}\omega_a, \quad \zeta_{\mathbb{C}} \mapsto \sum_a \zeta_{\mathbb{C},a}\omega_a.$$

1.3 We say that the parameter ζ is *generic* when

$$\text{For every } \nu \in P^\vee, \text{we have } \langle\nu,\zeta_{\mathbb{R}}\rangle \neq 0 \text{ or } \langle\nu,\zeta_{\mathbb{C}}\rangle \neq 0 \tag{1.1}$$

The variety $\mathcal{M}_\zeta(\mathbf{v},\mathbf{w})$ is smooth whenever ζ is generic. Moreover (for fixed $\mathbf{v}$ and $\mathbf{w}$), the varieties corresponding to generic parameters are all diffeomorphic [Nak94, Corollary 4.2].

1.4 The Weyl group $\mathbf{W}$ of $\mathfrak{g}$ is defined to be the subgroup of $\mathrm{Aut}(\mathfrak{h}^*)$ generated by reflections

$$s_a : \alpha \mapsto \alpha - \langle\alpha,\alpha_a^\vee\rangle\alpha_a$$

for $a \in I^{re}$. Note that $\mathbf{W}$ acts on Q. The dual action on $Q^\vee$ is given by

$$s_a : \alpha^\vee \mapsto \alpha^\vee - \langle\alpha_a,\alpha^\vee\rangle\alpha_a^\vee. \tag{1.2}$$

Moreover, $\langle\cdot,\cdot\rangle$ induces a perfect pairing between P and $Q^\vee$, and thus (1.2) gives rise to an action of $\mathbf{W}$ on P by duality.

With this convention, $\mathbf{W}$ acts on $\zeta_{\mathbb{R}}$ and $\zeta_{\mathbb{C}}$ via the identifications in § 1.2. We also define an action on pairs $(\mathbf{v},\mathbf{w})$ by $\sigma\cdot(\mathbf{v},\mathbf{w}) := (\sigma(\mathbf{v}-\mathbf{w})+\mathbf{w},\mathbf{w})$. Following [Lus00] (see also [Nak94]), Maffei [Maf02] defined, for *generic* ζ, isomorphisms

$$\kappa_\sigma : \mathcal{M}_\zeta(\mathbf{v},\mathbf{w}) \xrightarrow{\sim} \mathcal{M}_{\sigma\cdot\zeta}(\sigma\cdot(\mathbf{v},\mathbf{w})).$$

Let us consider the situation when $\zeta_{\mathbb{C}}$ is generic as in § 1.1 case (ii) (so that ζ is generic). In that situation, it is more convenient to use the purely complex description of the quiver variety given in § 1.1. In the case of a simple reflection s_a, the construction is as follows. Let $(B,i,j)\in\mathbf{M}^{ss}_\zeta(\mathbf{v},\mathbf{w})$. Define vector spaces V'_k, W'_k by $V'_k = V_k$ if $k\neq a$,

$$V'_a = \Big(W_a \oplus \bigoplus_{o(h)=a} V_{i(h)}\Big)\Big/\Big(j_a + \sum_{o(h)=a} B_h\Big)V_a$$

and $W'_k = W_k$ for all k. Let Z be the set of all $(B',i',j')\in\mathbf{M}_{s_a\zeta}(\mathbf{v}',\mathbf{w}')$ such that:

(i) $B'_h = B_h$ if $i(h)\neq a$ and $o(h)\neq a$;

(ii) $i_k = i'_k$ and $j'_k = j_k$ if $k\neq a$;

(iii) set

$$x_a = j_a \oplus \bigoplus_{o(h)=a} B_h : V_a \to W_a \oplus \bigoplus_{o(h)=a} V_{i(h)},$$
$$y_a = i_a \oplus \bigoplus_{i(h)=a} \epsilon(\overline{h})B_h : W_a \oplus \bigoplus_{i(h)=a} V_{o(h)} \to V_a$$

and define x'_a and y'_a in a similar fashion. The sequence

$$0 \to V_a \xrightarrow{x_a} W_a \oplus \bigoplus_{o(h)=a} V_{i(h)} \xrightarrow{y'_b} V'_a \to 0$$

is exact, and $x_a y_a = x'_a y'_a - \lambda_a \,\mathrm{Id}$.

Then (see [Maf02]), Z is a principal $GL(V_a)$-homogeneous space. Thus, it corresponds to a unique point $\kappa_{s_a}(B,i,j)\in\mathcal{M}_{s_a\zeta}(\mathbf{v}',\mathbf{w}') = \mathcal{M}_{s_a\zeta}(s_a\cdot(\mathbf{v},\mathbf{w}))$.

2. Categories of (A,c)-complexes

2.1 Let A be a $\mathbb{Z}$-graded ring and c a central element of A of degree 2. We denote by $Z_2(A)$ the degree 2 summand of the center of A, so that $c\in Z_2(A)$.

Definition 1. A left (A,c)-complex is a $\mathbb{Z}$-graded left A-module M together with a degree one map $d: M\to M$ such that

$$d^2 = c,$$

and d (super)commutes with the action of A:

$$d(am) = (-1)^{|a|} a d(m), \quad a\in A, m\in M.$$

A morphism of (A,c)-complexes is a degree zero morphism of A-modules which commutes with d. The category $\mathbf{Com}(A,c)$ of left (A,c)-complexes is abelian. The translation functor [1] in the category $\mathbf{Com}(A,c)$ is defined by

$$(M[1])^i = M^{i+1}, \quad d_{[1]} = -d,$$

and the A-module structure on $M[1]$ is

$$a\circ m = (-1)^{|a|} am.$$

Alternatively, we may define a $\mathbb{Z}$-graded algebra $\widetilde{A}_c = A \mathbin{\dot{\otimes}} \mathbb{C}[d]/(d^2 - c)$ with $da = (-1)^{|a|}ad$ and $\deg(d) = 1$. Then $\mathbf{Com}(A, c)$ is nothing but the category $\mathbf{Mod}(\widetilde{A}_c)$ of graded left $\widetilde{A}_c$-modules.

2.2 Let M and N be left (A, c)-complexes. Given a morphism of graded A-modules $h : M \to N[-1]$, the map $f = hd_M + d_N h$ is a morphism $M \to N$ of (A, c)-complexes. Any such morphism is called *null-homotopic*. The following result is clear.

Proposition 2.1. *Null-homotopic morphisms form a two-sided ideal in the category* $\mathbf{Com}(A, c)$*.*

We say that morphisms $f, g : M \to N$ are homotopic and write $f \sim g$ if $f - g$ is null-homotopic. Define the homotopy category $\mathcal{K}(A, c)$ as follows. Objects are (A, c)-complexes and for any two (A, c)-complexes M and N we put

$$\mathrm{Hom}_{\mathcal{K}(A,c)}(M, N) = \mathrm{Hom}_{\mathbf{Com}(A,c)}(M, N)/\sim.$$

Categories $\mathbf{Com}(A, c)$, as well as $\mathcal{K}(A, c)$, for various $c \in Z_2(A)$, might have common objects. If $M \in \mathbf{Com}(A, c)$ and $c'M = 0$ for some $c' \in Z_2(A)$ then $M \in \mathbf{Com}(A, c + c')$.

2.3 Tensor product of left and right (A, c)-complexes

If M is a right graded A-module and N a left graded A-module, the tensor product $M \otimes_A N$ is a graded abelian group. If M is a right (A, c)-complex and N a left $(A, -c)$-complex, then $M \otimes_A N$ is a complex of graded abelian groups with the differential

$$d(m \otimes n) = dm \otimes n + (-1)^{|m|} m \otimes dn,$$

since

$$d^2(m \otimes n) = d^2 m \otimes n + m \otimes d^2 n = mc \otimes n + m \otimes (-c)n = 0.$$

2.4 Bimodules

Let $c_0, c_1 \in Z_2(A)$. An (A, c_0, c_1)-complex is a graded A-bimodule N together with a degree one map $d : N \to N$ such that $d^2 = l_{c_0} + r_{c_1}$, (where l_{c_0} is left multiplication by c_0 and r_{c_1} is right multiplication by c_1), d (super)commutes with the left action of A:

$$d(an) = (-1)^{|a|} a\,dn, \quad a \in A, n \in N,$$

and commutes with the right action of A.

If M is a left $(A, -c_1)$-complex, the tensor product $N \otimes_A M$ is a left (A, c_0)-complex. Thus, the tensor product with N is a functor from $\mathbf{Com}(A, -c_1)$ to $\mathbf{Com}(A, c_0)$, and from $\mathcal{K}(A, -c_1)$ to $\mathcal{K}(A, c_0)$.

3. Categories of duplexes

3.1 Let A be a $\mathbb{Z}/2\mathbb{Z}$-graded ring, $A = A_0 \oplus A_1$, and c a degree zero central element of A. A *duplex* over (A, c) is a $\mathbb{Z}/2\mathbb{Z}$-graded A-module $M = M^0 \oplus M^1$ with a generalized differential

$$M^0 \xrightarrow{d} M^1 \xrightarrow{d} M^0 \tag{3.1}$$

which supercommutes with the action of A and satisfies $d^2(m) = cm$ for all $m \in M$. A duplex over $(A, 0)$ is simply a 2-periodic complex of A-modules. An (A, c)-duplex for $c \neq 0$ can be viewed as a 2-periodic 'complex' with the differential satisfying $d^2 = c$ rather than $d^2 = 0$.

A homomorphism $f : M \to N$ of (A, c)-duplexes is a degree zero A-module map that commutes with the differentials:

$$\begin{array}{ccccc} M^0 & \xrightarrow{d} & M^1 & \xrightarrow{d} & M^0 \\ \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^0} \\ N^0 & \xrightarrow{d} & N^1 & \xrightarrow{d} & N^0 \end{array}$$

We denote by $\mathbf{Com}_2(A, c)$ the category of (A, c)-duplexes. The shift functor [1] in this category is 2-periodic, $[2] \cong \mathrm{Id}$. The category $\mathbf{Com}_2(A, c)$ is abelian. We will often write duplexes in the form

$$\xrightarrow{d} M^0 \xrightarrow{d} M^1 \xrightarrow{d}$$

Given maps $f : M \to N$ and $g : N \to M$ of (A, c)-duplexes such that $fg = c_1$ and $gf = c_1$ for some degree zero central element c_1 of A,

$$\begin{array}{ccccc} M^0 & \xrightarrow{d_M} & M^1 & \xrightarrow{d_M} & M^0 \\ \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^0} \\ N^0 & \xrightarrow{d_N} & N^1 & \xrightarrow{d_N} & N^0 \\ \downarrow{\scriptstyle g^0} & & \downarrow{\scriptstyle g^1} & & \downarrow{\scriptstyle g^0} \\ M^0 & \xrightarrow{d_M} & M^1 & \xrightarrow{d_M} & M^0 \end{array}$$

the *cone* of (f, g) is defined as the 'total' $(A, c + c_1)$-duplex of the above diagram,

$$\xrightarrow{d} M^0 \oplus N^1 \xrightarrow{d} M^1 \oplus N^0 \xrightarrow{d}$$

with $d = d_M - d_N + f + g$.

We will also use a less precise notation $\xrightarrow{g} M \xrightarrow{f} N \xrightarrow{g}$ for the cone of (f, g).

Note that we may once again think of $\mathbf{Com}_2(A, c)$ as the category of $\mathbb{Z}/2\mathbb{Z}$-graded left $\widetilde{A}_c$-modules, where $\widetilde{A}_c$ is defined as in § 2.

3.2 Given an A-homomorphism $h : M \to N[-1]$, the map $f = hd_M + d_N h$ is a morphism $M \to N$ of duplexes. We will say that morphisms $f, g : M \to N$ are homotopic if $f - g = hd_M + d_N h$ for some h. The following is straightforward.

Proposition 3.1. *Null-homotopic morphisms form a two-sided ideal in the category* $\mathbf{Com}_2(A, c)$.

We call the quotient category of $\mathbf{Com}_2(A, c)$ by this ideal the *homotopy category of (A, c)-duplexes* and denote it by $\mathcal{K}_2(A, c)$.

Example. For an A-module M let $M_{c,1}$ be the duplex

$$\xrightarrow{1} M \xrightarrow{c} M \xrightarrow{1}, \tag{3.2}$$

which is the cone of $(c, 1)$. The identity morphism of $M_{c,1}$ is null-homotopic, and $M_{c,1}$ is isomorphic to the zero object in the homotopy category of duplexes.

Remark. If c is invertible, any (A, c)-duplex is trivial in the homotopy category, and the category $\mathcal{K}_2(A, c)$ is trivial. The case of noninvertible c is more interesting. If A is artinian, any element of A is either invertible or nilpotent, and the only nontrivial case is that of nilpotent c.

Proposition 3.2. *If M is an (A, c)-duplex and I an injective $\mathbb{Z}/2\mathbb{Z}$-graded A-submodule of M such that d is injective on I and $I \cap dI = 0$, then M is isomorphic in the homotopy category to its quotient by the subduplex generated by I:*

$$M \cong \{\longrightarrow M^0/(I^0 \oplus d(I^1)) \longrightarrow M^1/(I^1 \oplus d(I^0)) \longrightarrow\}.$$

The proof is again straightforward.

3.3 Duplexes of bimodules

Tensor product with a duplex N of A-bimodules such that $d^2 = l_{c_0} + r_{c_1}$ is a functor from $\mathbf{Com}_2(A, -c_1)$ to $\mathbf{Com}_2(A, c_0)$ and from $\mathcal{K}_2(A, -c_1)$ to $\mathcal{K}_2(A, c_0)$.

4. Stable categories

4.1 Let A be a $\mathbb{Z}$-graded ring and $c \in Z_2(A)$ a degree two central element. Let $\underline{\mathbf{Mod}}(A)$ denote the stable category of $\mathbb{Z}$-graded A-modules (see, e.g., [Hap88]). Its objects are $\mathbb{Z}$-graded A-modules and for any modules M and N we have

$$\mathrm{Hom}_{\underline{\mathbf{Mod}}(A)}(M, N) = \mathrm{Hom}_A(M, N)/I$$

where I is the ideal of all morphisms $f : M \to N$ which admit a factorization $f = g \circ h$ where $h : M \to P$, $g : P \to N$ for some projective module P. In particular, an object M of $\underline{\mathbf{Mod}}(A)$ is isomorphic to the zero object if and only if it is projective as an A-module. We define the stable category $\underline{\mathbf{Mod}}(\widetilde{A}_c)$ in a same way. Since $\widetilde{A}_c$ is projective (in fact, free) as an A-module, there is a natural restriction functor $R : \underline{\mathbf{Mod}}(\widetilde{A}_c) \to \underline{\mathbf{Mod}}(A)$.

There is a canonical functor $\Phi : \mathbf{Mod}(\widetilde{A}_c) \simeq \mathbf{Com}(A, c) \to \mathcal{K}(A, c)$.

Lemma 4.1. *For any projective $\widetilde{A}_c$-module P we have $\Phi(P) = 0$.*

Proof. Let $\{P'_i\}$ be the collection of indecomposable projective A-modules. It is easy to check that $P_i = \widetilde{A}_c \otimes_A P'_i$ is an indecomposable projective $\widetilde{A}_c$-module, and hence that $\{P_i\}$ forms the complete collection of indecomposable projectives for $\widetilde{A}_c$. However, $P_i = P'_i \oplus P'_i[-1]$ as A-module and $d : P'_i \xrightarrow{\sim} P'_i[-1]$, so that P_i is homotopic to zero as an (A, c)-complex. □

We deduce that the functor Φ admits a factorization

$$\mathbf{Mod}(\widetilde{A}_c) \simeq \mathbf{Com}(A, c) \xrightarrow{\Phi_1} \underline{\mathbf{Mod}}(\widetilde{A}_c) \xrightarrow{\Phi_2} \mathcal{K}(A, c). \tag{4.1}$$

Similar results hold for $\mathbf{Com}_2(A, c)$, $\underline{\mathbf{Mod}}_2(\widetilde{A}_c)$ and $\mathcal{K}_2(A, c)$ if A is a $\mathbb{Z}/2\mathbb{Z}$-graded ring.

4.2 Proposition 3.2 admits the following straightforward generalization.

Proposition 4.1. *If M is an (A, c)-duplex and I an injective and projective $\mathbb{Z}/2\mathbb{Z}$-graded A-submodule of M such that d is injective on I and $I \cap dI = 0$, then M is isomorphic in the stable category $\underline{\mathbf{Mod}}_2(\widetilde{A}_c)$ to its quotient by the subduplex generated by I:*

$$M \cong \{\longrightarrow M^0/(I^0 \oplus d(I^1)) \longrightarrow M^1/(I^1 \oplus d(I^0)) \longrightarrow\}.$$

5. Homological realization of Nakajima varieties

5.1 Let $\mathbf{Q} = (I, E), H$ and ϵ be as in § 1.1. We can view (I, H) as the oriented double of the unoriented graph $\mathbf{Q}$. Consider the path algebra of (I, H). Note that in this algebra the product hh' of two length one paths is nonzero if and only if $i(h) = o(h')$.

Define the $\mathbb{C}$-algebra $A(\mathbf{Q})$ as the quotient of this path algebra by relations:

(i) $hh' = 0$ if $h' \neq \overline{h}$;

(ii) $\epsilon(h)h\overline{h} = \epsilon(h')h'\overline{h'}$ if $o(h) = o(h')$.

Relations of the second type say that $\epsilon(h)h\overline{h}$ in the quotient algebra depends only on the outgoing vertex of h. We denote $X_a = \epsilon(h)h\overline{h}$ where $a = o(h)$.

If the graph has only two vertices, a and b, and one edge connecting them, we let $A(\mathbf{Q})$ be the quotient of the path algebra by relations $h\overline{h}h = 0 = \overline{h}h\overline{h}$ (where h is one of the orientations of the edge and $\overline{h}$ is the reverse of h). If the graph has only one vertex a, and no edges, define $A(\mathbf{Q})$ as the exterior algebra on one generator X_a, and place it in degree 2 to make $A(\mathbf{Q})$ graded. If the graph has more than one vertex, we grade $A(\mathbf{Q})$ by lengths of paths. The graded algebra $A(\mathbf{Q})$ is (up to isomorphism) independent of the choice of the orientation ϵ.

For simplicity, we will write A instead of $A(\mathbf{Q})$. The algebra A is finite-dimensional, $\dim(A) = 2(|I| + |E|)$. Any path of length at least 3 equals 0 in A.

Note that X_a (see above) is central, and X_a, over all vertices a, form a basis for the degree 2 subspace of A.

A length 0 path (a), for a vertex $a \in I$, is a minimal idempotent in A, and $1 = \sum_a (a)$.

Example. If $\mathbf{Q}$ has one vertex and one loop, A is isomorphic to the exterior algebra on two generators $h, \overline{h}$:

$$A \cong \mathbb{C}\langle h, \overline{h}\rangle / h^2 = \overline{h}^2 = h\overline{h} + \overline{h}h = 0.$$

The trace $tr : A \to \mathbb{C}$ defined by

$$tr(X_a) = 1, \quad tr(h) = tr((a)) = 0,$$

makes A into a graded Frobenius algebra. Note that A is symmetric (but with respect to a different trace) if and only if $\mathbf{Q}$ is bipartite (i.e. if it is possible to partition the set of vertices of $\mathbf{Q}$ into two disjoint sets in such a way that all edges go from one set to the other). In the latter case, A is isomorphic to the zigzag algebra of $\mathbf{Q}$, see [HK01]. For any $\mathbf{Q}$, the algebra A is a skew-zigzag algebra, in the terminology of [HK01, § 4.6].

Denote by P_a the indecomposable projective left A-module $A(a)$. An indecomposable projective left A-module is isomorphic to P_a, for some a. Denote by ${}_aP$ the indecomposable projective right A-module $(a)A$. Since A is Frobenius, P_a and ${}_aP$ are, in addition, injective.

Let S_a be the simple quotient of P_a (equivalently, the quotient of P_a by all paths of length greater than 0). Denote by $\widehat{a}$ the image of $(a) \in P_a$ under the quotient map. S_a is a one-dimensional complex vector space and is spanned by $\widehat{a}$. A simple left A-module is isomorphic to S_a, for some a. The modules P_a, S_a, and ${}_aP$ inherit $\mathbb{Z}$-grading from A.

Denote by $[m]$ the grading shift down by m., i.e $(M[m])^l = M^{m+l}$. Let $\mathbf{Mod}(A)$ (respectively $\mathbf{HMod}(A)$) be the category of graded A-modules (respectively the category of graded A-modules equipped with a Hermitian structure $x \mapsto x^*$ such that $h(x^*) = (hx)^*$ for all edges h). For any two graded A-modules M, N we denote by $\mathrm{Hom}_A(M, N)$ the set of grading-preserving A-homomorphisms.

The modules P_a, ${}_aP, S_a$ have unique Hermitian structures $x \mapsto x^*$ such that $(a)^* = (a)$ and $h(x^*) = (hx)^*$ for all edges h.

5.2 Let V_a, W_a, $a \in I$ be finite-dimensional $\mathbb{C}$-vector spaces. Consider the graded A-module

$$M = \bigoplus_a V_a \otimes P_a \oplus W_a \otimes S_a[-1],$$

a direct sum of a projective and a semisimple A-module. We raise the grading of simple modules S_a by 1 to 'balance' them in the middle of projective modules P_a, the latter non-zero in degrees $0, 1, 2$.

Let us equip M with a degree 1 generalized differential $d : M \to M$ with the property $d^2 = c$ for a fixed degree 2 central element c of A,

$$c = \sum_a c_a X_a, \quad c_a \in \mathbb{C}.$$

d should super-commute with A,

$$dx = (-1)^{|x|} x d, \quad x \in A.$$

The graded components of M are

$$M^0 = \bigoplus_a (V_a \otimes (a)),$$

$$M^1 = \bigoplus_a ((W_a \otimes \widehat{a}) \bigoplus_{o(h)=a} (V_{i(h)} \otimes h)),$$

$$M^2 = \bigoplus_a (V_a \otimes X_a),$$

and the differential must have the form

$$0 \longrightarrow M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^2 \longrightarrow 0.$$

Since $d(a) = (a)d$, for minimal idempotents $(a) \in A$, the generalized complex decomposes into the sum of

$$0 \longrightarrow (a)M^0 \xrightarrow{d^0} (a)M^1 \xrightarrow{d^1} (a)M^2 \longrightarrow 0,$$

over all a. We can write the latter as

$$0 \longrightarrow V_a \otimes (a) \xrightarrow{d^0} \bigoplus_{o(h)=a} (V_{i(h)} \otimes h) \oplus (W_a \otimes \widehat{a}) \xrightarrow{d^1} V_a \otimes X_a \longrightarrow 0.$$

The components of d^0 can be described as maps $B_h \in \mathrm{Hom}(V_a, V_{i(h)}), j_a \in \mathrm{Hom}(V_a, W_a)$:

$$d^0 = \biggl(\bigoplus_{o(h)=a} B_h, j_a \biggr)^{\mathrm{t}}.$$

The superscript t in the formula stands for transposing a row vector into a column vector. From $dh = -hd$, for all edges h, we derive that

$$d^1 = \biggl(\bigoplus_{o(h)=a} \epsilon(\overline{h}) B_{\overline{h}}, i_a \biggr),$$

where $i_a \in \mathrm{Hom}(W_a, V_a)$. We should have $c = d^1 d^0$, or, specializing to a vertex a,

$$c_a \,\mathrm{Id} = \sum_{o(h)=a} \epsilon(\overline{h}) B_{\overline{h}} B_h + i_a j_a. \tag{5.1}$$

The right-hand side is the a-component of the complex moment map for the Nakajima quiver varieties.

If d is given as above by the data $d = (B_h, i_a, j_a)$ we may define its Hermitian adjoint $d^* = (\epsilon(\overline{h}) B^*_{\overline{h}}, -j^*_a, i^*_a)$ to be of the same form. The real component of the moment map equation $\mu_{\mathbb{R}}(B, i, j) = \boldsymbol{\zeta}_{\mathbb{R}}$ is equivalent to the relation

$$\frac{\sqrt{-1}}{2}(dd^* + d^*d) = \sum_a \zeta_{\mathbb{R},a} X_a. \tag{5.2}$$

We can think of $\boldsymbol{\zeta}_{\mathbb{C}}$ and $\boldsymbol{\zeta}_{\mathbb{R}}$ as degree 2 central elements of A, by taking the standard bases of $\mathbb{C}^I$ and $\mathbb{R}^I$ to $\{X_a\}_{a \in I}$. Collapse the grading from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$, and write $\mathbf{Mod}_2(A)$ and $\mathbf{HMod}_2(A)$ for the corresponding categories of $\mathbb{Z}/2\mathbb{Z}$-graded modules. Equations (5.1) and (5.2) together with the definitions of quiver varieties imply the following result.

PROPOSITION 5.1. *For any $\zeta_{\mathbb{C}}, \zeta_{\mathbb{R}}$ there is a bijection between points on the Nakajima variety $\mathcal{M}_\zeta(\mathbf{v},\mathbf{w})$ and isomorphism classes of the following data (M, d, ψ):*

A graded A-module $M \in \mathbf{HMod}_2(A)$ which is a direct sum of a projective and a semisimple module, with v_a the multiplicity of P_a and w_a the multiplicity of $S_a[-1]$, with a generalized differential d such that $d^2 = \boldsymbol{\zeta}_{\mathbb{C}}$, and $(\sqrt{-1}/2)(dd^* + d^*d) = \boldsymbol{\zeta}_{\mathbb{R}}$, and isomorphisms $\psi_a : W_a \cong \mathrm{Hom}_A(S_a[-1], M)$.

Now suppose that $\zeta_{\mathbb{C}}$ is generic. The complex description of quiver varieties yield the following result.

PROPOSITION 5.2. *There is a bijection between points on the Nakajima variety $\mathcal{M}_\zeta(\mathbf{v},\mathbf{w})$ and isomorphism classes of the following data (M, d, ψ).*

A graded A-module $M \in \mathbf{Mod}_2(A)$ which is a direct sum of a projective and a semisimple module, with v_a the multiplicity of P_a and w_a the multiplicity of $S_a[-1]$, with a generalized differential d such that $d^2 = \boldsymbol{\zeta}_{\mathbb{C}}$, and isomorphisms $\psi_a : W_a \cong \mathrm{Hom}_A(S_a[-1], M)$.

It is easy to check that two nonisomorphic data (M, d, ψ) as above remain nonisomorphic after applying the functor Φ_1 (see (4.1)), and that

$$\mathrm{Hom}_A(S_a[-1], M) \cong \mathrm{Hom}_{\underline{\mathbf{Mod}}_2(A)}(S_a[-1], R\Phi_1(M))$$

for any M as above. This gives the following variant of Proposition 5.2.

PROPOSITION 5.3. *There is a bijection between points on the Nakajima variety $\mathcal{M}_\zeta(\mathbf{v},\mathbf{w})$ and isomorphism classes of the following data $(\underline{M}, \psi)$.*

An object $\underline{M} \in \underline{\mathbf{Mod}}_2(\widetilde{A}_{\boldsymbol{\zeta}_{\mathbb{C}}})$ such that $\underline{M} \simeq \Phi_1(M')$ for some $M' \in \mathbf{Com}_2(A, \boldsymbol{\zeta}_{\mathbb{C}})$ with $M' \simeq \bigoplus_s S_a[-1] \otimes \mathbb{C}^{w_a} \oplus \bigoplus_a P_a \otimes \mathbb{C}^{v_a}$ as an A-module; and isomorphisms $\psi_a : W_a \cong \mathrm{Hom}_{\underline{\mathbf{Mod}}_2(A)}(S_a[-1], R(\underline{M}))$.

Consider now the case when $\zeta_{\mathbb{C}} = 0$ and $\zeta_{\mathbb{R}} \in (\mathbb{N}^+)^I$. The description of $\mathbf{M}^{ss}_\zeta(\mathbf{v},\mathbf{w})$ given in § 1.1 case (i) yields the following.

PROPOSITION 5.4. *There is a bijection between points on $\mathcal{M}_\zeta(\mathbf{v},\mathbf{w})$ and isomorphism classes of (M, d, ψ) where $M \in \mathbf{Mod}_2(A)$ and ψ are as in Proposition 5.2, $d^2 = 0$ and no projective submodule of M is d-stable.*

Remark. All the above results also hold in the $\mathbb{Z}$-graded case.

6. Weyl group action in categories of duplexes

6.1 We use notation from § 5.1. Choose a vertex a without a loop. Since P_a is a left A-module and ${}_aP$ a right A-module, $P_a \otimes {}_aP$ is an A-bimodule (the tensor product is over $\mathbb{C}$). Denote by $m_a : P_a \otimes {}_aP \longrightarrow A$ the bimodule map which is simply the restriction of the multiplication map $A \otimes A \longrightarrow A$, so that $m_a((a) \otimes (a)) = (a)$. Denote by $\Delta_a : A \longrightarrow P_a \otimes {}_aP$ the bimodule map determined by

$$\Delta_a(1) = X_a \otimes (a) + (a) \otimes X_a + \sum_{o(h)=a} \epsilon(h)\overline{h} \otimes h,$$

the sum over all edges h that start at a.

6.2 Let $C_{a,x}$, for $x \in \mathbb{C}$, be the following duplex of bimodules:

$$\xrightarrow{\Delta_a} P_a \otimes {}_aP \xrightarrow{xm_a} A \xrightarrow{\Delta_a} . \tag{6.1}$$

If $x \neq 0$, this duplex is isomorphic to

$$\xrightarrow{x\Delta_a} P_a \otimes {}_aP \xrightarrow{m_a} A \xrightarrow{x\Delta_a} . \tag{6.2}$$

Denote by d the differential in (6.1) and (6.2). We have

$$d^2 \mid_{P_a \otimes {}_aP} = x(X_a \otimes 1 + 1 \otimes X_a),$$
$$d^2 \mid_A = x\left(2X_a - \sum_{o(h)=a} X_{i(h)}\right).$$

Here $X_a \otimes 1$ is the left multiplication by X_a, etc. Hence, as operators on $C_{a,x}$ we have

$$d^2 = x\left(X_a \otimes 1 + 1 \otimes X_a - \sum_{o(h)=a} X_{i(h)} \otimes 1\right). \tag{6.3}$$

Note that $X_b \otimes 1 - 1 \otimes X_b$ acts trivially on $P_a \otimes {}_aP$ and A if $a \neq b$. Thus, we also have

$$d^2 = x\left(X_a \otimes 1 + 1 \otimes X_a - \sum_{o(h)=a} X_{i(h)} \otimes 1\right) + \sum_{b \neq a} x_b(X_b \otimes 1 - 1 \otimes X_b) \tag{6.4}$$

for any $x_b \in \mathbb{C}$, as b ranges over all vertices other than a.

The Weyl group $\mathbf{W}$ of (I, E) has generators s_a, over all loopless vertices $a \in I$, and relations:

(i) $s_a^2 = 1$ for all a;
(ii) $s_a s_b = s_a s_b$ if a and b do not have a common edge;
(iii) $s_a s_b s_a = s_b s_a s_b$ if a and b are joined by exactly one edge.

The Weyl group acts on $Z_2(A)$, the degree 2 summand of the center of A (on the vector space with the basis $\{X_a\}_{a \in I}$) by

$$s_a(c) = c + x_a\left(\sum_{o(h)=a} X_{i(h)} - 2X_a\right)$$

for $c = \sum_{b \in I} x_b X_b$. This action is compatible with the one defined on $\zeta_{\mathbb{C}}$ in § 1.4. via the natural identification $\mathbb{C}^I \to Z_2(A)$. It follows from (6.4) that, for any $c \in Z_2(A)$, the tensor product with $C_{a,-x_a}$ is a functor from $\mathbf{Com}_2(A, c)$ to $\mathbf{Com}_2(A, s_a(c))$ and from $\mathcal{K}_2(A, c)$ to $\mathcal{K}_2(A, s_a(c))$. Denote this functor by $\mathcal{R}_a : \mathcal{K}_2(A, c) \to \mathcal{K}_2(A, s_a(c))$.

Lemma 6.1. *The functor $\mathcal{R}_a$ lifts to a functor from $\underline{\mathbf{Mod}}_2(\widetilde{A}_c)$ to $\underline{\mathbf{Mod}}_2(\widetilde{A}_{s_a(c)})$.*

Proof. Let $\tilde{P}_b = \widetilde{A}_c \otimes_A P_b$ be an indecomposable projective $\widetilde{A}_c$-module. We have to show that $C_{a,-x_a} \otimes \tilde{P}_b$ is a projective $\widetilde{A}_{s_a(c)}$-module. By definition, we have

$$C_{a,x} \otimes \tilde{P}_b = \begin{pmatrix} A \otimes \tilde{P}_b \\ \oplus \\ (P_a \otimes {}_aP)[1] \otimes_A \tilde{P}_b \end{pmatrix} = \begin{pmatrix} P_b \oplus P_b[1] \\ \oplus \\ \bigoplus_{\substack{o(h)=a \\ i(h)=b}} (P_a[1] \oplus P_a) \end{pmatrix},$$

and the action of the element $d \in A_{s_a(c)}$ is given by

$$d = \begin{pmatrix} 1 \otimes d & m_a \otimes 1 \\ \Delta_a \otimes 1 & 1 \otimes d \end{pmatrix}.$$

Since A is Frobenius, P_a and P_b are both projective and injective. Furthermore, it is easy to check that d is injective on P_b and that $P_b \cap d(P_b) = 0$. Similarly, we have d is injective on $P_a[1]$ and we have $P_a[1] \cap d(P_a[1]) = 0$. By applying Proposition 4.1 twice we see that $C_{a,x} \otimes \tilde{P}_b = 0$ in $\underline{\mathbf{Mod}}_2(\widetilde{A}_{s_a(c)})$ as desired. □

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We will denote this functor by the same symbol $\mathcal{R}_a : \underline{\mathbf{Mod}}_2(\widetilde{A}_c) \to \underline{\mathbf{Mod}}_2(\widetilde{A}_{s_a(c)})$.

6.3 We now deal with the categorical analogue of the braid relation.

PROPOSITION 6.1. *If $x \neq 0$, there is an isomorphism in the stable category of bimodule duplexes*

$$C_{a,-x} \otimes_A C_{a,x} \cong A. \tag{6.5}$$

Proof. Let $N = C_{a,-x} \otimes_A C_{a,x}$ and ∂ be the differential in N. Note that $\partial^2 = 0$, and N is a duplex of A-bimodules.

Since $x \neq 0$, we can write $C_{a,-x}$ as

$$\xrightarrow{\Delta_a} P_a \otimes {}_aP \xrightarrow{-xm_a} A \xrightarrow{\Delta_a} \tag{6.6}$$

N is the total duplex of the following diagram (which is 2-cyclic in horizontal and vertical directions, and each of the four squares anticommutes)

$$\begin{array}{ccccccc}
 & & \Big\downarrow{\scriptstyle -\mathrm{id}\otimes x\Delta_a} & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes x\Delta_a} & & \\
\xrightarrow{-xm_a\otimes\mathrm{id}} & & A \otimes_A P_a \otimes {}_aP & \xrightarrow{\mathrm{id}\otimes\Delta_a} & P_a \otimes {}_aP \otimes_A P_a \otimes {}_aP & \xrightarrow{-xm_a\otimes\mathrm{id}} & \\
 & & \Big\downarrow{\scriptstyle -\mathrm{id}\otimes m_a} & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes m_a} & & \\
\xrightarrow{-xm_a\otimes\mathrm{id}} & & A \otimes_A A & \xrightarrow{\mathrm{id}\otimes\Delta_a} & P_a \otimes {}_aP \otimes_A A & \xrightarrow{-xm_a\otimes\mathrm{id}} & \\
 & & \Big\downarrow{\scriptstyle -\mathrm{id}\otimes x\Delta_a} & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes x\Delta_a} & &
\end{array}$$

Denote by N_{ij} the four bimodules in the above diagram:

$$\begin{array}{ccccc}
 & \Big\downarrow & & \Big\downarrow & \\
\longrightarrow & N_{10} & \longrightarrow & N_{00} & \longrightarrow \\
 & \Big\downarrow & & \Big\downarrow & \\
\longrightarrow & N_{11} & \longrightarrow & N_{01} & \longrightarrow \\
 & \Big\downarrow & & \Big\downarrow &
\end{array}$$

Simplifying our notation as at the end of § 3.1, we write N as

$$\xrightarrow{\partial} N_{00} \oplus N_{11} \xrightarrow{\partial} N_{01} \oplus N_{10} \xrightarrow{\partial} .$$

Note that ${}_aP \otimes_A P_a \cong \mathbb{C}(a) \oplus \mathbb{C}X_a$ is a two-dimensional vector space. Let $\zeta : N_{11} \to N_{00}$ be the map

$$N_{11} \cong A \xrightarrow{-\Delta_a} P_a \otimes {}_aP \longrightarrow P_a \otimes {}_aP \otimes_A P_a \otimes {}_aP,$$

where the last map takes $u_1 \otimes u_2$ to $u_1 \otimes (a) \otimes u_2$. Let $N' = \{u + \zeta(u) \mid u \in N_{11}\}$. It is a subbimodule of N_{11} isomorphic to A, and $\partial N' = 0$.

Let $N'' = P_a \otimes (a) \otimes {}_aP$. It is a subbimodule of N_{00}.

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Lemma 6.2. *N is a direct sum of its three subduplexes*

$$T_{-1} = \{\longrightarrow N_{10} \longrightarrow \partial N_{10} \longrightarrow\},$$
$$T_0 = \{\longrightarrow N' \longrightarrow 0 \longrightarrow\},$$
$$T_1 = \{\longrightarrow N'' \longrightarrow \partial N'' \longrightarrow\}.$$

Since ∂ is injective on N_{10} and on N'', and both N_{10} and N'' are projective bimodules, the duplexes T_{-1} and T_1 are stably equivalent to the zero duplex. Therefore, N is equivalent in the stable category to the bimodule duplex $\{\longrightarrow A \longrightarrow 0 \longrightarrow\}$. □

Remarks. (i) Proposition 6.1 says that the functor $\mathcal{R}_a^2$

$$\mathcal{K}_2(A,c) \xrightarrow{\mathcal{R}_a} \mathcal{K}_2(A,s_a(c)) \xrightarrow{\mathcal{R}_a} \mathcal{K}_2(A,c)$$

is isomorphic to the identity functor, as long as $s_a(c) \neq c$ (equivalently, if $c_a \neq 0$).

(ii) The isomorphism (6.5) holds for $x=0$ as well, if we use (6.1), with $x=0$, to define one of the duplexes on the left-hand side of (6.5) and (6.2) to define the other.

Proposition 6.2. *If a and b are not connected by an edge, for any $x,y \in \mathbb{C}$ there is an isomorphism of bimodule duplexes*

$$C_{a,x} \otimes_A C_{b,y} \cong C_{b,y} \otimes_A C_{a,x}. \tag{6.7}$$

Proof. Since ${}_aP \otimes_A P_b \cong 0 \cong {}_bP \otimes_A P_a$, left- and right-hand sides of (6.7) are isomorphic to

$$\xrightarrow{m_a+m_b} A \xrightarrow{x\Delta_a + y\Delta_b} (P_a \otimes {}_aP) \oplus (P_b \otimes {}_bP) \xrightarrow{m_a+m_b}.$$ □

Proposition 6.3. *If a and b are connected by one edge, for any $x,y \in \mathbb{C}$ there is an isomorphism in the stable category of bimodule duplexes*

$$C_{a,y} \otimes_A C_{b,x+y} \otimes_A C_{a,x} \cong C_{b,x} \otimes_A C_{a,x+y} \otimes_A C_{b,y}. \tag{6.8}$$

Proof. Denote by N the duplex on the left-hand side of (6.8). It is built out of eight bimodules

$$N_{ijk} = C^i_{a,y} \otimes_A C^j_{b,x+y} \otimes_A C^k_{a,x}, \quad i,j,k \in \{0,1\}.$$

The differential ∂ of N is injective on N_{000} (the component $N_{000} \to N_{010}$ of ∂ is already injective). Let

$$T_{-1} = \{\xrightarrow{\partial} N_{000} \xrightarrow{\partial} \partial N_{000} \xrightarrow{\partial}\}$$

be the subduplex of N generated by N_{000}.

Let $N' = P_a \otimes (a) \otimes {}_aP \subset P_a \otimes {}_aP \otimes_A P_a \otimes {}_aP \cong N_{010}$. The differential is injective on N_{010} (since the component $N_{010} \to N_{110}$ of ∂ is injective). Let

$$T_1 = \{\xrightarrow{\partial} \partial N' \xrightarrow{\partial} N' \xrightarrow{\partial}\}$$

be the subduplex of N generated by N_{010}.

The algebra A is Frobenius, and each projective A-module is injective. In particular, P_a, P_b are injective A-modules, and N', N_{000} (both isomorphic to $P_a \otimes {}_aP$) are injective $A \otimes A^o$-modules (that is, injective A-bimodules). Moreover, $T_1 \cap T_{-1} = 0$. Applying Proposition 4.1 twice, we see that duplexes N and $\widetilde{N} = N/(T_{-1} \oplus T_1)$ are isomorphic in the stable category of duplexes of bimodules.

Let h be the edge with $o(h)=a$ and $i(h)=b$. The duplex $\widetilde{N}$ is isomorphic to

$$\xrightarrow{\tilde{\partial}^1} \begin{pmatrix} P_a \otimes {}_aP \\ \oplus \\ P_b \otimes {}_bP \end{pmatrix} \xrightarrow{\tilde{\partial}^0} \begin{pmatrix} P_a \otimes {}_bP \\ \oplus \\ A \\ \oplus \\ P_b \otimes {}_aP \end{pmatrix} \xrightarrow{\tilde{\partial}^1} \tag{6.9}$$

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with the differential $\widetilde{\partial}$ given by matrices of bimodule maps

$$\widetilde{\partial}^0 = \begin{pmatrix} \epsilon(\overline{h})y \ \mathrm{id}\otimes\overline{h} & \epsilon(h)y\overline{h}\otimes\mathrm{id} \\ m_a & -m_b \\ \epsilon(h)xh\otimes\mathrm{id} & \epsilon(\overline{h})x \ \mathrm{id}\otimes h \end{pmatrix} \tag{6.10}$$

$$\widetilde{\partial}^1 = \begin{pmatrix} \mathrm{id}\otimes h & (x+y)\Delta_a & -\overline{h}\otimes\mathrm{id} \\ h\otimes\mathrm{id} & -(x+y)\Delta_b & -\mathrm{id}\otimes\overline{h} \end{pmatrix}. \tag{6.11}$$

The following example explains our notations: the entry $\epsilon(\overline{h})y \ \mathrm{id}\otimes\overline{h}$ in the top left corner of (6.10) is a bimodule map $P_a\otimes_a P \longrightarrow P_a\otimes_b P$ which takes $u\otimes v\in P_a\otimes_a P$ to $\epsilon(\overline{h})yu\otimes\overline{h}v\in P_a\otimes_b P$.

Denote by M be the duplex on the right-hand side of (6.8). Since the right-hand side is obtained from the left-hand side by switching a with b, x with y, and h with $\overline{h}$ we see that M is isomorphic in the stable category to the duplex $\widetilde{M}$ defined by making these switchings in formulas (6.9), (6.10), (6.11). It is easy to check that duplexes $\widetilde{N}$ and $\widetilde{M}$ are isomorphic. Therefore, duplexes N and M are isomorphic in the stable category of duplexes. □

We may restate the above results in the following form.

Theorem 1. *The functors $\mathcal{R}_a : \underline{\mathbf{Mod}}_2(\widetilde{A}_c) \to \underline{\mathbf{Mod}}_2(\widetilde{A}_{s_a(c)})$ define a braid group action on the family of categories $\underline{\mathbf{Mod}}_2(\widetilde{A}_{w(c)})$ for $w\in\mathbf{W}$; in other words, we have, for any $c\in Z_2(A)$ isomorphisms of functors*

$$\mathcal{R}_a\circ\mathcal{R}_b\circ\mathcal{R}_a \simeq \mathcal{R}_b\circ\mathcal{R}_a\circ\mathcal{R}_b : \underline{\mathbf{Mod}}_2(\widetilde{A}_c) \to \underline{\mathbf{Mod}}_2(\widetilde{A}_{s_as_bs_a(c)})$$

if a and b are connected by one edge, and

$$\mathcal{R}_a\circ\mathcal{R}_b \simeq \mathcal{R}_b\circ\mathcal{R}_a : \underline{\mathbf{Mod}}_2(\widetilde{A}_c) \to \underline{\mathbf{Mod}}_2(\widetilde{A}_{s_as_b(c)})$$

if a and b are not connected. Moreover, if c lies in a generic orbit then this action factors through a Weyl group action, i.e. we have

$$\mathcal{R}_a\circ\mathcal{R}_a \simeq \mathrm{Id} : \underline{\mathbf{Mod}}_2(\widetilde{A}_c) \to \underline{\mathbf{Mod}}_2(\widetilde{A}_c)$$

for any $a\in I$.

Passing to the homotopy categories $\mathcal{K}_2(A,c)$ we obtain the following.

Theorem 2. *The functors $\mathcal{R}_a : \mathcal{K}_2(A,c)\to\mathcal{K}_2(A,s_a(c))$ define a braid group action on the family of categories $\mathcal{K}_2(A,w(c))$ for $w\in\mathbf{W}$. This action factors through to a Weyl group action if c lies in a generic orbit.*

Remark. In order to ensure that the above braid group action on the set of categories $\underline{\mathbf{Mod}}_2(\widetilde{A}_{w(c)})$ or $\mathcal{K}_2(A,w(c))$ factors through the Weyl group it is enough to impose the following weaker condition: no point of the orbit of c under W is fixed by any of the reflexions s_a for loopless vertices a.

7. Weyl group actions on Nakajima varieties

7.1 Let us say that $\boldsymbol{\zeta}_{\mathbb{C}} = -\sum_a \zeta_{\mathbb{C},a}X_a\in A$ is *generic* if $\zeta_{\mathbb{C}}$ is generic in the sense of § 1.1. If N is any $(A,\boldsymbol{\zeta}_{\mathbb{C}})$-duplex with $\boldsymbol{\zeta}_{\mathbb{C}}$ generic and $a\in I$ we set $\mathcal{R}_a(N) = C_{a,\zeta_{\mathbb{C},a}}\otimes N$.

It follows from the results in § 6 that this defines an action of the Weyl group $\mathbf{W}$ on the set of objects of $\underline{\mathbf{Mod}}_2(\widetilde{A}_c)$ *for all generic c.*

So let us assume that $\zeta_{\mathbb{C}}$ is generic, and let us identify the points of $\mathcal{M}_\zeta(\mathbf{v},\mathbf{w})$ with isomorphism classes of data $(\underline{M},\psi)$ as in Proposition 5.3.

Theorem 3. *The functor $\mathcal{R}_a$ induces a bijection of sets from $\mathcal{M}_\zeta(\mathbf{v},\mathbf{w})$ to $\mathcal{M}_{s_a(\zeta)}(s_a(\mathbf{v},\mathbf{w}))$, which coincides with the isomorphism κ_{s_a}.*

Proof. Let us describe the action of $\mathcal{R}_a$ in more details. For notational convenience we will write $N^{\oplus V}$ for the tensor product $N \otimes_{\mathbb{C}} V$ when N is an $\widetilde{A}_c$-module and V a $\mathbb{C}$-vector space. We will also denote by (M, d) the objects of $\underline{\mathbf{Mod}}_2(\widetilde{A}_c)$. If $(M, d) \in \underline{\mathbf{Mod}}_2(\widetilde{A}_{\underline{\zeta_{\mathbb{C}}}})$ then by definition $\mathcal{R}_a(M, d) = (M', d') \in \underline{\mathbf{Mod}}_2(\widetilde{A}_{s_a \underline{\zeta_{\mathbb{C}}}})$ where

$$M' = \begin{pmatrix} A \otimes M \\ \oplus \\ (P_a \otimes {}_aP)[1] \otimes_A M \end{pmatrix}$$

and

$$d' = \begin{pmatrix} 1 \otimes d & m_a \otimes 1 \\ \Delta_a \otimes 1 & 1 \otimes d \end{pmatrix}.$$

Let us write

$$M = \bigoplus_{k \in I} (P_k^{\oplus V_k} \oplus S_k^{\oplus W_k}[1]).$$

Observe that

$$A \otimes P_a^{\oplus V_a} \xrightarrow{\Delta_a \otimes 1} (P_a \otimes {}_aP)[1] \otimes_A P_a^{\oplus V_a}$$

is injective. This implies that $A \otimes P_a^{\oplus V_a} \oplus d'(A \otimes P_a^{\oplus V_a})$ is stably trivial. Hence, by Proposition 4.1, $(M', d') \simeq (M'', d'')$ where $M'' = M'/(A \otimes P_a^{\oplus V_a} \oplus d'(A \otimes P_a^{\oplus V_a}))$. A direct computation shows that

$$(P_a \otimes {}_aP)[1] \otimes_A M = \begin{pmatrix} P_a^{\oplus V_a}[1] \\ \oplus \\ P_a^{\oplus V_a}[1] \\ \oplus \\ P_a^{\oplus W_a} \\ \oplus \\ \bigoplus_{b - a} P_a^{\oplus V_b} \end{pmatrix} \tag{7.1}$$

and

$$M'' \simeq \begin{pmatrix} \bigoplus_{k \neq a} P_k^{\oplus V_k} \\ \oplus \\ \bigoplus_{k \neq a} S_k^{\oplus W_k} \\ \oplus \\ S_a^{\oplus W_a} \\ \oplus \\ P_a[1]^{\oplus V_a} \\ \oplus \\ P_a^{\oplus W_a} \\ \oplus \\ \bigoplus_{b - a} P_a^{\oplus V_b} \end{pmatrix}.$$

We have

$$d'' = \begin{pmatrix} B & i & 0 & 0 & -Bi & \lambda_a \operatorname{Id} - \epsilon BB \\ j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_a \operatorname{Id} - ji & -jB \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & \operatorname{Id} & j & 0 & 0 \\ \operatorname{Id} & 0 & 0 & B & 0 & 0 \end{pmatrix}. \tag{7.2}$$

From the relation $\mu(B,i,j)=\lambda$ and from the fact that λ is generic it follows that the fourth column of (7.2) is nonsingular. In particular, $d''_{|P_a[1]^{\oplus V_a}}$ is injective and $P_a[1]^{\oplus V_a}\oplus d''(P_a[1]^{\oplus V_a})$ is stably trivial.

Thus, (M'',d'') is isomorphic, in $\underline{\mathbf{Mod}}_2(\widetilde{A}_{s_a(\underline{\zeta_{\mathbb{C}}})})$, to

$$(M''',d''')=(M''/(P_a[1]^{\oplus V_a}\oplus d''(P_a[1]^{\oplus V_a}),d'')). \tag{7.3}$$

Moreover, there is a canonical isomorphism $u:R((M,d))\simeq R((M',d'))\simeq R((M''',d'''))$ and we may set $\psi'=u\circ\psi$. Let

$$(B',i',j')\in\mathcal{M}_{s_a(\zeta)}(s_a(\mathbf{v},\mathbf{w}))$$

be the element corresponding to (M''',d''',ψ'). Comparing with the construction of § 1.4 we see that $\kappa_{s_a}(B,i,j)=(B',i',j')$, which proves the theorem. □

8. Nakajima varieties for affine quivers

In this section we restrict ourselves to the case when (I,E) is an affine bipartite Dynkin diagram, and reinterpret the above construction in terms of the McKay correspondence. This section, and the following section, can be read independently of the rest of the paper, with the exception of § 9.3.

8.1 Let $\{\pm1\}\subset\Gamma\subset SL(2,\mathbb{C})$ be a finite group and let $\{\rho_a\}_{a\in I}$ be the set of its irreducible representations. We also let ρ_0 and ρ be the trivial (respectively the natural two-dimensional) representation. Let $\mathbf{Q}=(I,E)$ be the (unoriented) affine quiver associated to Γ via the McKay correspondence, with I as the set of vertices and with T_{ab} arrows between a and b, where

$$T_{ab}=\dim\operatorname{Hom}_\Gamma(\rho_a\otimes\rho,\rho_b).$$

8.2 Let us consider the algebra $A_\Gamma:=\Lambda\mathbb{C}^2\rtimes\mathbb{C}[\Gamma]$, and set $\widetilde{A}_\Gamma=A_\Gamma\dot{\otimes}\mathbb{C}[d]/d^2\simeq\Lambda\mathbb{C}^3\rtimes\Gamma$ with relations $dz=-zd$ for any $z\in\mathbb{C}^2$ and $d\gamma=\gamma d$ for any $\gamma\in\Gamma$. Both A_Γ and $\widetilde{A}_\Gamma$ are naturally $\mathbb{Z}$-graded. We denote by $\mathbf{Mod}(A_\Gamma)$ and $\mathbf{Mod}_2(A_\Gamma)$ (respectively $\mathbf{Mod}(\widetilde{A}_\Gamma)$ and $\mathbf{Mod}_2(A_\Gamma)$) the categories of $\mathbb{Z}$-graded and $\mathbb{Z}/2\mathbb{Z}$-graded A_Γ-modules (respectively $\widetilde{A}_\Gamma$-modules). We will reformulate the definition of $\mathcal{M}_\zeta(\mathbf{v},\mathbf{w})$ entirely in terms of representation theory of A_Γ and $\widetilde{A}_\Gamma$.

The link with the setting of § 5 is as follows. Let $I^\pm$ be the set of indices a such that $\rho_a(-1)=\pm1$. Then $I=I^+\sqcup I^-$ and

$$T_{ab}\neq0\Rightarrow a\in I^+,\ b\in I^-\quad\text{or}\quad a\in I^-,\ b\in I^+. \tag{8.1}$$

In particular, the Dynkin diagram (I,E) is bipartite. Write $A(\mathbf{Q})$ for the zigzag algebra corresponding to (I,E). Recall that $\{\rho_a\}_{a\in I}$ denotes the set of simple left Γ-modules. Then $\{\rho_a^*\}$ is the set of *right* simple Γ-modules. Consider the right projective A_Γ-modules ${}_a\mathbf{P}=\rho_a^*\otimes\Lambda\mathbb{C}^2$, and put $\mathbf{P}=\bigoplus_a {}_a\mathbf{P}$.

It is easy to check that $A(\Gamma)\simeq\operatorname{End}_{A_\Gamma}(\mathbf{P})$. Moreover, the functor $\mathbf{P}\otimes-$ induces a Morita equivalence

$$\mathbf{Mod}(A_\Gamma)\simeq\mathbf{Mod}(A(\mathbf{Q})). \tag{8.2}$$

8.3 Note that A_Γ and $\widetilde{A}_\Gamma$ are symmetric algebras. In particular, A_Γ and $\widetilde{A}_\Gamma$ are self-injective algebras (i.e projective and injective objects coincide). If M is a graded A_Γ (respectively $\widetilde{A}_\Gamma$)-module then the graded dual space M^* is again an A_Γ (respectively $\widetilde{A}_\Gamma$)-module.

If U is any Γ-module, we will regard $\Lambda\mathbb{C}^2\otimes U$ and $(\Lambda\mathbb{C}^2\dot{\otimes}\mathbb{C}[d]/d^2)\otimes U$ as graded A_Γ-module and $\widetilde{A}_\Gamma$-module, respectively, where $\Lambda^0\mathbb{C}^2\otimes U$ is placed in degree 0. Note that any projective

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indecomposable A_Γ-module (respectively $\widetilde{A}_\Gamma$-module) is isomorphic to $(\Lambda\mathbb{C}^2 \otimes \rho_a)[n]$ (respectively $((\Lambda\mathbb{C}^2 \dot{\otimes} \mathbb{C}[d]/d^2) \otimes \rho_a)[n])$ for some $a \in I$ and $n \in \mathbb{Z}$.

Let us fix a basis $\{x, y\}$ in $\mathbb{C}^2$. For any $a \in I$, let us fix intertwiners

$$\bigoplus_{(a,b)\in H} \varphi_{ab} : \bigoplus_{(a,b)\in H} \rho_b \xrightarrow{\sim} \mathbb{C}^2 \otimes \rho_a. \tag{8.3}$$

Define a function $\epsilon : H \to \mathbb{C}^*$ by the following condition: $\pi \circ \varphi_{ba} \circ \varphi_{ab} = \epsilon_{(a,b)} x \wedge y$, where $\pi : \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \rho_b \to \Lambda^2\mathbb{C}^2 \otimes \rho_b$ is the projection. Note that $\epsilon(h) + \epsilon(\overline{h}) = 0$ for any $h \in H$.

8.4 Let $\underline{\mathbf{Mod}}(A_\Gamma)$ denote the stable category of $\mathbb{Z}$-graded A_Γ-modules (see § 4.).

Lemma 8.1. *Let U be a Γ-module and let us consider it as a A_Γ module by trivially extending the action to A_Γ. If $M \simeq U$ in $\underline{\mathbf{Mod}}(A_\Gamma)$ then $M \simeq U \oplus P$ in $\mathbf{Mod}(A_\Gamma)$ for some projective module P.*

Proof. Let $f : M \to U$ and $f' : U \to M$ such that $ff' = \mathrm{Id}_U$ and $f'f = \mathrm{Id}_M$ in $\underline{\mathbf{Mod}}(A_\Gamma)$. Note that, for any projective module P, any composition of morphisms $U \to P \to U$ is zero. Hence, $\mathrm{Hom}_{\underline{\mathbf{Mod}}(A_\Gamma)}(U,U) = \mathrm{Hom}_{A_\Gamma}(U,U)$ and $ff' = \mathrm{Id}_U$ in $\mathbf{Mod}(A_\Gamma)$. However, then $M \simeq U \oplus \mathrm{Ker}\, f$ in $\mathbf{Mod}(A_\Gamma)$ and the lemma follows. □

Similarly, we let $\underline{\mathbf{Mod}}(\widetilde{A}_\Gamma)$ stand for the stable category of $\mathbb{Z}$-graded $\widetilde{A}_\Gamma$-modules. Replacing $\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$, we also define the categories $\underline{\mathbf{Mod}}_2(A_\Gamma)$ and $\underline{\mathbf{Mod}}_2(\widetilde{A}_\Gamma)$. The stable categories $\underline{\mathbf{Mod}}(A_\Gamma)$ and $\underline{\mathbf{Mod}}(\widetilde{A}_\Gamma)$ are endowed with structures of triangulated categories (see [Hap88]).

Note that $\widetilde{A}_\Gamma$ is a free A_Γ-module. This gives rise to functors

$$R : \underline{\mathbf{Mod}}(\widetilde{A}_\Gamma) \to \underline{\mathbf{Mod}}(A_\Gamma), \quad R : \underline{\mathbf{Mod}}_2(\widetilde{A}_\Gamma) \to \underline{\mathbf{Mod}}_2(A_\Gamma).$$

8.5 In this section we give the realization of $\mathcal{M}_\zeta(\mathbf{v}, \mathbf{w})$ in the § 1.1 case (ii), i.e. $\zeta_\mathbb{R}$ is arbitrary and $\zeta_\mathbb{C}$ is generic.

For every $a \in I$ we let $p_a \in \mathbb{C}[\Gamma]$ be the (central) primitive idempotent corresponding to ρ_a. Set $c_a = x \wedge y \cdot p_a$. Then $\{1\} \cup \{c_a\}_{a\in I}$ forms a basis of the center of A_Γ. Consider the following deformation of $\widetilde{A}_\Gamma$:

$$\widetilde{A}_{\Gamma,\zeta_\mathbb{C}} = (\Lambda\mathbb{C}^2 \rtimes \Gamma) \dot{\otimes} \mathbb{C}[d] \Big/ \Big\langle d^2 - \sum_a \zeta_{\mathbb{C},a} c_a \Big\rangle.$$

Let $\underline{\mathbf{Mod}}_2(\widetilde{A}_{\Gamma,\zeta_\mathbb{C}})$ be the stable categories of $\mathbb{Z}/2\mathbb{Z}$-graded $\widetilde{A}_{\Gamma,\zeta_\mathbb{C}}$-modules. As in § 8.4, the embedding $A_\Gamma \subset \widetilde{A}_{\Gamma,\zeta_\mathbb{C}}$ gives rise to a restriction functor $R : \underline{\mathbf{Mod}}_2(\widetilde{A}_{\Gamma,\zeta_\mathbb{C}}) \to \underline{\mathbf{Mod}}_2(A_\Gamma)$. As in the undeformed case, the algebra $\widetilde{A}_{\Gamma,\zeta_\mathbb{C}}$ is symmetric and self-injective.

Let us fix $\mathbf{w} \in \mathbb{N}^I$, $W = \bigoplus_a W_a$ such that $\dim W = \mathbf{w}$ and let $\mathbb{W} = \bigoplus_a W_a \otimes \rho_a$ be the corresponding Γ-module. We will regard $\mathbb{W}$ as a graded A_Γ-module, where $\Lambda\mathbb{C}^2$ acts trivially, placed in degree 0. Let $\mathcal{N}_\zeta(\mathbf{w})$ be the set of pairs (u, M) where $M \in \underline{\mathbf{Mod}}_2(\widetilde{A}_{\Gamma,\zeta_\mathbb{C}})$ and $u : \mathbb{W} \to R(M)$ is an isomorphism.

Lemma 8.2. *Let M be any $\widetilde{A}_{\Gamma,\zeta_\mathbb{C}}$-module such that $R(M) \simeq 0$. Then M is projective.*

Proof. By Lemma 8.1 we have $M \simeq (\Lambda\mathbb{C}^2 \otimes \mathbb{V}_1) \oplus (\Lambda\mathbb{C}^2 \otimes \mathbb{V}_2)[-1]$ for some Γ-modules $\mathbb{V}_1$ and $\mathbb{V}_2$. We may assume that

$$d(\mathbb{V}_1) \subset \mathbb{C}^2 \otimes \mathbb{V}_1. \tag{8.4}$$

Indeed, any $v \in \mathbb{V}_1$ not satisfying (8.4) generates a projective submodule M' of M. Consider the linear map $s : \Lambda^2\mathbb{C}^2 \to \mathbb{C},\ x \wedge y \mapsto 1$. From (8.4) and the relation $dz = -zd$ for $z \in \mathbb{C}^2$ we deduce that $Tr_{|V}(s \circ d^2) = 0$. However, by definition,

$$Tr_{|V}(s \circ d^2) = \sum_a \zeta_{\mathbb{C},a} \dim \operatorname{Hom}(\rho_a, \mathbb{V}_1),$$

and the genericity of $\zeta_{\mathbb{C}}$ implies that $\mathbb{V}_1 = 0$. Similarly, $\mathbb{V}_2 = 0$ and we are done. □

Theorem 4. *There is a natural bijection between the set of isomorphism classes of elements in $\mathcal{N}_\zeta(\mathbf{w})$ and the set of points of $\bigsqcup_{\mathbf{v}} \mathcal{M}_\zeta(\mathbf{v}, \mathbf{w})$.*

Proof. Let $(u, M) \in \mathcal{N}_\zeta(\mathbf{w})$. By Lemma 8.1, we may assume that $M = (\Lambda\mathbb{C}^2 \otimes \mathbb{V}_1[1]) \oplus \mathbb{W} \oplus (\Lambda\mathbb{C}^2 \otimes \mathbb{V}_0)$ as an A_Γ-module. We can also assume that

$$d(\mathbb{V}_0) \subset (\mathbb{C}^2 \otimes \mathbb{V}_0) \oplus (\Lambda^2\mathbb{C}^2 \otimes \mathbb{V}_1). \tag{8.5}$$

Indeed if not then any element $v_0 \in \mathbb{V}_0$ not satisfying (8.5) will generate a projective $\widetilde{A}_{\Gamma,\zeta_{\mathbb{C}}}$-module N_1 and $M \simeq M/N_1$ in $\underline{\mathbf{Mod}}_2(\widetilde{A}_{\Gamma,\zeta_{\mathbb{C}}})$. Similarly, we can assume that

$$d(\mathbb{V}_1) \subset (\mathbb{C}^2 \otimes \mathbb{V}_1) \oplus (\Lambda^2\mathbb{C}^2 \otimes \mathbb{V}_0) \oplus \mathbb{W}. \tag{8.6}$$

However, then $d(\Lambda^2\mathbb{C}^2 \otimes \mathbb{V}_1) = 0$ and, in particular, the composition of maps $\mathbb{V}_1 \xrightarrow{d} \mathbb{C}^2 \otimes \mathbb{V}_1 \xrightarrow{d} \Lambda^2\mathbb{C}^2 \otimes \mathbb{V}_1$ endows $\Lambda\mathbb{C}^2 \otimes \mathbb{V}_1$ with a structure of $\widetilde{A}_{\Gamma,\zeta_{\mathbb{C}}}$-module. By Lemma 8.2, this implies that $\mathbb{V}_1 = 0$.

Let us fix some decomposition

$$\mathbb{V} = \bigoplus_a V_a \otimes \rho_a \tag{8.7}$$

and set $V = \bigoplus_a V_a$. Let us split the map $d : \mathbb{V} \to \mathbb{C}^2 \otimes \mathbb{V} \oplus \mathbb{W}$ as $d = d_0 + d_1$ where $d_0 : \mathbb{V} \to \mathbb{C}^2 \otimes \mathbb{V}$ and $d_1 : \mathbb{V} \to \mathbb{W}$. Then the maps d_0 and d_1 give rise, via the identification (8.7), the fixed intertwiners (8.3) and the map u, to elements $B = \bigoplus_{h \in H} x_h \in E(V, V)$ and $j \in L(V, W)$, respectively. Similarly, the map $d : \mathbb{C}^2 \otimes \mathbb{V} \oplus \mathbb{W} \to \Lambda^2\mathbb{C}^2 \otimes \mathbb{V} \simeq \mathbb{V}$ gives rise to elements $C = \bigoplus_{h \in H} y_h \in E(V, V)$ and $i \in L(W, V)$. From the relation $dz = -zd$ for $u \in \mathbb{C}^2$ we deduce that $y_h = \epsilon(h)x_h$ where $\epsilon : H \to \mathbb{C}^*$ is defined in § 8.3. Similarly, from $d^2 = \sum_a \zeta_{\mathbb{C},a} c_a$ we deduce the relation $\mu(B, i, j) = \boldsymbol{\zeta}_{\mathbb{C}}$.

Note that the assignment $M \to (B, i, j)$ depends on a choice of the decomposition (8.7), but that two such decompositions give rise to the same element in $\mathcal{M}_\zeta(\mathbf{v}, \mathbf{w})$. Hence we have obtained in this way a well-defined map from the set of isomorphism classes of objects in $\mathcal{N}_\zeta(\mathbf{w})$ to $\bigsqcup_{\mathbf{v}} \mathcal{M}_\zeta(\mathbf{v}, \mathbf{w})$. Conversely, it is clear that any point $(B, i, j) \in \mathcal{M}_\zeta(\mathbf{v}, \mathbf{w})$ gives rise, via the above construction, to an $\widetilde{A}_{\Gamma,\zeta_{\mathbb{C}}}$-module structure on the A_Γ-module $M = \Lambda(\mathbb{C}^2 \otimes \mathbb{V})[1] \oplus \mathbb{W}$. Moreover, this module M is equipped with a canonical isomorphism $u : \mathbb{W} \xrightarrow{\sim} R(M)$. Thus, $M \in \mathcal{N}_\zeta(\mathbf{w})$. This map assigns distinct points in $\mathcal{M}_\zeta(\mathbf{v}, \mathbf{w})$ to nonisomorphic objects in $\mathcal{N}_\zeta(\mathbf{w})$ and the theorem follows. □

Remark. Theorem 4 is equivalent to Proposition 5.3.

9. Koszul duality and sheaves on $\mathbb{P}^2$

9.1 In this section we clarify the relation between the construction of § 8 and the moduli space of coherent sheaves on some noncommutative deformations of the projective plane, as studied in [BGK].

Let $\underline{\mathbf{Mod}}^0(\widetilde{A}_\Gamma)$ be the full subcategory of $\underline{\mathbf{Mod}}(\widetilde{A}_\Gamma)$ consisting of objects V satisfying

$$Ext^{l+n}_{\widetilde{A}_\Gamma}(\mathbb{C}, V[-l]) \neq 0 \Rightarrow n = 0, \tag{9.1}$$

where $\mathbb{C}$ denotes the trivial module. The category $\underline{\mathbf{Mod}}^0(A_\Gamma)$ is defined in a similar way.

To any $\widetilde{A}_\Gamma$ module $V = \bigoplus_{n\in\mathbb{Z}} V_n$ is associated a complex of Γ-equivariant coherent sheaves on $\mathbb{P}^2$

$$\cdots \xrightarrow{d} L_i(V^*) \xrightarrow{d} L_{i+1}(V^*) \xrightarrow{d} \cdots$$

where $L_i(V^*) = (V^*)_i \otimes \mathcal{O}(i)$ and where the differential is defined by $(d\zeta)(x) = x \cdot \zeta(x)$ for $x \in \mathbb{P}^2$ and any section $\zeta \in \Gamma(L_i(V))$. Here V^* denotes the $\widetilde{A}_\Gamma$-module dual to V. A well-known theorem of Bernšteĭn *et al.* [BGG78] generalizing the classical Koszul duality between $\mathbb{C}[x,y,z]$ and $\Lambda\mathbb{C}^3$ asserts that the assignment $\Phi : V \to L_\bullet(V^*)$ induces an equivalence between $\underline{\mathbf{Mod}}(\widetilde{A}_\Gamma)^{op}$ and the Γ-equivariant derived category $D^b_\Gamma(\mathrm{Coh}(\mathbb{P}^2))$ of coherent sheaves on $\mathbb{P}^2$. In particular, under this equivalence condition (9.1) corresponds to $H^i(L_\bullet(V^*)) \neq 0 \Rightarrow i = 0$ (see [BGS96, § 2.13]), and $\underline{\mathbf{Mod}}^0(\widetilde{A}_\Gamma)^{op}$ is equivalent to the category $\mathrm{Coh}_\Gamma(\mathbb{P}^2)$ of Γ-equivariant coherent sheaves on $\mathbb{P}^2$. Therefore, it is an abelian category. Similar statements hold for $\underline{\mathbf{Mod}}(A_\Gamma)$ and $D^b_\Gamma(\mathbb{P}^1)$.

The functor R restricts to a functor $\underline{\mathbf{Mod}}^0(\widetilde{A}_\Gamma) \to \underline{\mathbf{Mod}}^0(A_\Gamma)$. It corresponds to the functor of restriction

$$D^b_\Gamma(\mathrm{Coh}(\mathbb{P}^2)) \to D^b_\Gamma(\mathrm{Coh}(\mathbb{P}^1))$$

induced by the embedding $\mathbb{P}^1 \simeq \mathbb{P}((\mathbb{C}^2)^*) \hookrightarrow \mathbb{P}((\mathbb{C}^2 \oplus \mathbb{C}d)^*) \simeq \mathbb{P}^2$.

Let us denote by Π_Γ the preprojective algebra of the affine quiver (I,E) (see, e.g., [Maf02]). It is well known and easy to check that $A(\Gamma)$ is Koszul and that its quadratic dual is Π_Γ (see, e.g., [HK01]). Thus, altogether we get the following diagram relating various algebras.

$$\begin{array}{ccc}
\Lambda\mathbb{C}^2 \rtimes \mathbb{C}[\Gamma] & \xrightarrow{\text{Koszul duality}} & \mathbb{C}[x,y] \rtimes \mathbb{C}[\Gamma] \\
{\scriptstyle\text{Morita eq.}}\downarrow & & \downarrow{\scriptstyle\text{Morita eq.}} \\
A(\mathbf{Q}) & \xrightarrow{\text{Koszul duality}} & \Pi_\Gamma
\end{array}$$

Similarly, there is the following diagram.

$$\begin{array}{ccc}
\Lambda\mathbb{C}^3 \rtimes \mathbb{C}[\Gamma] & \xrightarrow{\text{Koszul duality}} & \mathbb{C}[x,y,z] \rtimes \mathbb{C}[\Gamma] \\
{\scriptstyle\text{Morita eq.}}\downarrow & & \downarrow{\scriptstyle\text{Morita eq.}} \\
\widetilde{A(\mathbf{Q})} & \xrightarrow{\text{Koszul duality}} & \Pi_\Gamma[z]
\end{array} \tag{9.2}$$

Our construction in § 8 is based on a deformation of the left column of diagram (9.2). The corresponding right column consists of the homogeneous coordinate ring of the noncommutative $\mathbb{P}^2$ studied in [BGK], and the (graded) deformed preprojective algebra (see [CH98]).

9.2 In this section we show how to recover the description of Nakajima varieties as moduli space of torsion-free sheaves on $\mathbb{P}^2$ with fixed framing at ∞, using the representation theory of $\widetilde{A}_\Gamma$. This corresponds to § 1.1 case (i), i.e. $\zeta_\mathbb{C} = 0$ and $\zeta_\mathbb{R} \in (\mathbb{N}^+)^I$.

Let $\mathcal{T}$ be the full subcategory of $\underline{\mathbf{Mod}}^0(\widetilde{A}_\Gamma)$ consisting of modules T such that $\Phi(T)$ is a torsion sheaf on $\mathbb{P}^2$. We will say that an object M of $\underline{\mathbf{Mod}}^0_\mathbb{Z}(\widetilde{A}_\Gamma)$ is *torsion-free* if for any $T \in \mathcal{T}$ we have $\mathrm{Hom}_{\underline{\mathbf{Mod}}^0(\widetilde{A}_\Gamma)}(M,T) = 0$.

Lemma 9.1. *Let N be a graded $\widetilde{A}_\Gamma$-module such that $N = \Lambda\mathbb{C}^2 \otimes \mathbb{V}$ as a A_Γ-module. If $\mathbb{V} \neq \{0\}$ then $H^0(\Phi(N))$ is a nontrivial torsion sheaf.*

Proof. This follows from a direct computation. □

Lemma 9.2. *Let $N \in \underline{\mathbf{Mod}}^0(\widetilde{A}_\Gamma)$ such that $N_i = 0$ for $i > 0$ and such that $R(N) \simeq 0$. Then $N \simeq 0$.*

Proof. Suppose $N \not\simeq 0$. By Lemma 8.1, N decomposes as a A_Γ-module as $N = \bigoplus_{i=r}^{-2} \Lambda\mathbb{C}^2 \otimes \mathbb{V}_i$ for some Γ-modules $\mathbb{V}_i$ and some $r \leqslant -2$. We can assume

$$d(\mathbb{V}_r) \subset \mathbb{C}^2 \otimes \mathbb{V}_r. \tag{a}$$

Indeed, any $v_r \in \mathbb{V}_r$ not satisfying (a) generates a projective $\widetilde{A}_\Gamma$-module N' and $N \simeq N/N'$ in $\underline{\mathbf{Mod}}^0(\widetilde{A}_\Gamma)$. However, then it follows from the previous lemma that $H^{-r}(\Phi(N)) \neq 0$, in contradiction with the assumption that $N \in \underline{\mathbf{Mod}}^0(\widetilde{A}_\Gamma)$. □

As in § 8.4 we fix $\mathbf{w} \in \mathbb{N}^I$ and associate to it a Γ-module $\mathbb{W}$. We will regard also $\mathbb{W}$ as a graded A_Γ-module, where $\Lambda\mathbb{C}^2$ acts trivially, placed in degree 0. Note that $\mathbb{W}$ is naturally an object of $\underline{\mathbf{Mod}}^0(A_\Gamma)$. Let $\mathcal{N}(\mathbf{w})$ denote the set of pairs (u, M) where $M \in \underline{\mathbf{Mod}}^0(\widetilde{A}_\Gamma)$ is torsion-free and $u : \mathbb{W} \xrightarrow{\sim} R(M)$ is an isomorphism. We say two elements (u_1, M_1) and (u_2, M_2) are isomorphic if there exists an isomorphism $j : M_1 \to M_2$ such that $u_2 = R(j) \circ u_1$.

THEOREM 5. *There is a natural bijection between the set of isomorphism classes of elements in $\mathcal{N}(\mathbf{w})$ and the set of points of $\bigsqcup_{\mathbf{v}} \mathcal{M}_\zeta(\mathbf{v}, \mathbf{w})$.*

Proof. Let $(u, M) \in \mathcal{N}(\mathbf{w})$. We first show the following.

LEMMA 9.3. *There exists a Γ-module $\mathbb{V}$ and $M' \in \underline{\mathbf{Mod}}(\tilde{A}_\Gamma)$ such that $M \simeq M'$ and $M' = \Lambda\mathbb{C}^2 \otimes \mathbb{V}[+1] \oplus \mathbb{W}$ as an A_Γ-module.*

Proof. By Lemma 8.1, there exists Γ-modules $\mathbb{V}_i$, $i \in \mathbb{Z}$ such that $M \simeq \bigoplus_i \Lambda\mathbb{C}^2 \otimes \mathbb{V}_i[-i] \oplus \mathbb{W}$ as an A_Γ-module. Observe that

$$d(\mathbb{V}_{-1}) \subset \Lambda^2\mathbb{C}^2 \otimes \mathbb{V}_{-2} \oplus \mathbb{C}^2 \otimes \mathbb{V}_{-1} \oplus \mathbb{W}. \tag{9.3}$$

Indeed, if not, then any $v_{-1} \in \mathbb{V}_{-1}$ such that (9.3) does not hold generates a projective submodule N of M, and $M \simeq M/N$ in $\underline{\mathbf{Mod}}^0(\tilde{A}_\Gamma)$. Let T be the $\widetilde{A}_\Gamma$-module obtained by restricting the $\widetilde{A}_\Gamma$-action to $\bigoplus_{i \geqslant 0} \Lambda\mathbb{C}^2 \otimes \mathbb{V}_i[-i]$. Note that $T \in \mathcal{T}$. Indeed, we have $H^i(\Phi(T)) = 0$ for $i > 0$ and $H^i(\Phi(T)) = H^i(\Phi(M)) = 0$ for $i < 0$, so that $T \in \underline{\mathbf{Mod}}^0(\widetilde{A}_\Gamma)$, and $H^0(\Phi(T))$ is torsion by Lemma 9.1. However, M is assumed to be torsion-free. This forces $T \simeq 0$.

A reasoning similar to (9.3) shows that

$$d(\mathbb{V}_{-2}) \subset \mathbb{C}^2 \otimes \mathbb{V}_{-2} \oplus \Lambda^2\mathbb{C}^2 \otimes \mathbb{V}_{-3}. \tag{b}$$

Now observe that $N = \bigoplus_{i<-1} \Lambda\mathbb{C}^2 \otimes \mathbb{V}_i$ is in $\underline{\mathbf{Mod}}^0(\widetilde{A}_\Gamma)$ and that $R(N) \simeq 0$. Thus $N \simeq 0$ by Lemma 9.2, and the lemma follows. □

Note that by Lemma 9.2 again, the following holds:

$$\text{for all } N \subset M, \quad R(N) = 0 \Rightarrow N \simeq 0 \tag{9.4}$$

and that condition (9.4) is equivalent to the stability condition of § 1.1, case (i).

The rest of the proof of the theorem now closely follows the proof of Theorem 4. □

Remark. The above description of torsion-free sheaves on $\mathbb{P}^2$ with fixed framing at infinity is equivalent to the classical one in terms of Beilinson's monads (see, e.g., [Nak99, Ch. 2]) but its derivation does not use spectral sequences.

From the above proof one easily deduces the following result.

COROLLARY 1. *The variety $\mathcal{M}_\zeta(\mathbf{v}, \mathbf{w})$ is isomorphic to the set of all $\Lambda\mathbb{C}^2 \rtimes \mathbb{C}[\Gamma]$-derivations (of degree one) of the module $\Lambda\mathbb{C}^2 \otimes \mathbb{V} \oplus \mathbb{W}$ satisfying the following condition: if $\mathbb{V}' \subset \mathbb{V}$ is a Γ-submodule such that $\Lambda\mathbb{C}^2 \otimes \mathbb{V}'$ is d-stable then $\mathbb{V}' = 0$.*

9.3 Denote by $\iota_a : \mathrm{End}(\rho_a) \to \mathbb{C}[\Gamma] \simeq \bigoplus_a \mathrm{End}(\rho_a)$ the canonical embedding, and let $\pi_a : \mathbb{C}[\Gamma] \to \mathrm{End}(\rho_a)$ be the canonical projection. We call

$$m : \Lambda\mathbb{C}^2 \otimes \mathbb{C}[\Gamma] \otimes \Lambda\mathbb{C}^2 \to \Lambda\mathbb{C}^2 \otimes \mathbb{C}[\Gamma]$$

the multiplication map and we define

$$\Delta : \Lambda\mathbb{C}^2 \otimes \mathbb{C}[\Gamma] \to \Lambda\mathbb{C}^2 \otimes \mathbb{C}[\Gamma] \otimes \Lambda\mathbb{C}^2$$

to be its adjoint. Consider the following maps of A_Γ-bimodules:

$$d_1 : \Lambda\mathbb{C}^2 \otimes \rho_a \otimes_{\mathbb{C}} \rho_a^* \otimes \Lambda\mathbb{C}^2 \simeq \Lambda\mathbb{C}^2 \otimes \mathrm{End}(\rho_a) \otimes \Lambda\mathbb{C}^2 \xrightarrow{m\circ(1\otimes\iota_a\otimes 1)} \Lambda\mathbb{C}^2 \otimes \mathbb{C}[\Gamma]$$
$$d_2 : \Lambda\mathbb{C}^2 \otimes \mathbb{C}[\Gamma] \xrightarrow{(1\otimes\pi_a\otimes 1)\circ\Delta} \Lambda\mathbb{C}^2 \otimes \mathrm{End}(\rho_a) \otimes \Lambda\mathbb{C}^2.$$

As in § 6, this gives rise, for any $x \neq 0$ to a duplex of A_Γ-bimodules

$$\mathbf{C}_{a,x} : \xrightarrow{d_2} \Lambda\mathbb{C}^2 \otimes \mathrm{End}(\rho_a) \otimes \Lambda\mathbb{C}^2 \xrightarrow{d_1} \Lambda\mathbb{C}^2 \otimes \mathbb{C}[\Gamma] \xrightarrow{d_2} .$$

One checks that the A_Γ bimodule duplex $\mathbf{C}_{a,x}$ corresponds to the $A(\mathbf{Q})$-bimodule duplex $C_{a,x}$ under the equivalence $\mathbf{Mod}(A_\Gamma) \simeq \mathbf{Mod}(A(\mathbf{Q}))$. In particular, the collection of duplexes $\mathbf{C}_{a,x}$ satisfy the braid relations of § 6.3 (in the stable category of bimodule duplexes). Thus, as in § 7, tensoring by $\mathbf{C}_{a,x}$ for $a \in I$ and generic x defines an action of the Weyl group $\mathbf{W}$ on the set of objects of $\underline{\mathbf{Mod}}_2(A_{\Gamma,\zeta_\mathbb{C}})$ for all generic $\zeta_\mathbb{C}$.

In other words, (for generic $\zeta_\mathbb{C}$) and $a \in I$ we have a functor

$$\mathcal{R}_a : \underline{\mathbf{Mod}}_2(\widetilde{A}_{\Gamma,\zeta_\mathbb{C}}) \to \underline{\mathbf{Mod}}_2(\widetilde{A}_{\Gamma,s_a(\zeta_\mathbb{C})}),$$

and the collection of such functors satisfy the braid relations. Moreover, there is a canonical natural transformation $R\circ\mathcal{R}_a \to R$, and for any given fixed $\mathbf{w} \in \mathbb{N}^I$, $\mathcal{R}_a$ acts on the set of objects of $\mathcal{N}_\zeta(\mathbf{w})$. The following proposition is a consequence of Theorem 1.

Proposition 9.1. *The action of $\mathcal{R}_a$ on $\mathcal{N}_\zeta(\mathbf{w})$ coincides with the action of κ_a under the identification $\mathcal{N}_\zeta(\mathbf{w}) \simeq \bigsqcup_\mathbf{v} \mathcal{M}_\zeta(\mathbf{v},\mathbf{w})$.*

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