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RESEARCH ARTICLE

Quasi-isolated blocks and the Alperin–McKay conjecture

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Abstract

The Alperin–McKay conjecture is a longstanding open conjecture in the representation theory of finite groups. Späth showed that the Alperin–McKay conjecture holds if the so-called inductive Alperin–McKay (iAM) condition holds for all finite simple groups. In a previous paper, the author has proved that it is enough to verify the inductive condition for quasi-isolated blocks of groups of Lie type. In this paper, we show that the verification of the iAM-condition can be further reduced in many cases to isolated blocks. As a consequence of this, we obtain a proof of the Alperin–McKay conjecture for 2-blocks of finite groups with abelian defect.

Introduction

Alperin–McKay conjecture

In the representation theory of finite groups, some of the most important conjectures predict a very strong relationship between the representations of a finite group G and certain representations of its ℓ-local subgroups, where ℓ is a prime dividing the order of G. One of these conjectures is the Alperin–McKay conjecture. For an ℓ-block b of G, we denote by $\mathrm{Irr}_0(G, b)$ the set of height zero characters of b. Then this conjecture predicts the following:

Conjecture (Alperin–McKay). Let b be an ℓ-block of G with defect group D and B its Brauer correspondent in $\mathrm{N}_G(D)$. Then

$$|\mathrm{Irr}_0(G, b)| = |\mathrm{Irr}_0(\mathrm{N}_G(D), B)|.$$

Späth [42, Theorem C] showed that the Alperin–McKay conjecture holds if the so-called inductive Alperin–McKay condition holds for all finite simple groups. In a previous article, the author has reduced the verification of the inductive Alperin–McKay condition to so-called quasi-isolated blocks of groups of Lie type [38]. The overall aim of this paper is to further reduce the verification of this condition to isolated blocks of groups of Lie type. Using this, we can then verify the inductive Alperin–McKay condition for many important classes of blocks.

Equivariant Bonnafé–Dat–Rouquier equivalence

One of our main ingredients toward such a reduction is the recent result by Bonnafé–Dat–Rouquier [4]. They have constructed a Morita equivalence, which can be seen as a first step toward a modular analogue of Lusztig's Jordan decomposition for characters. Let $\mathbf{G}$ be a simple, simply connected algebraic group

with Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ defining an $\mathbb{F}_q$-structure, where q is a power of a prime p. Fix a prime ℓ different from p, and let $(\mathcal{O}, K, k)$ be an ℓ-modular system as in 1.1 below. In the following, Λ denotes either the discrete valuation ring $\mathcal{O}$ or its residue field k. Let $(\mathbf{G}^*, F^*)$ be a group in duality with $(\mathbf{G}, F)$ and $s \in (\mathbf{G}^*)^{F^*}$ a semisimple element of ℓ'-order. Let $e_s^{\mathbf{G}^F} \in \mathrm{Z}(\Lambda\mathbf{G}^F)$ be the central idempotent associated to s as in [11, Theorem 9.12]; and we assume that $\mathbf{L}^*$ is the minimal Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$. Let N be the common stabiliser in $\mathbf{G}^F$ of the idempotent $e_s^{\mathbf{L}^F}$ and $\mathbf{L}$, and suppose that $N/\mathbf{L}^F$ is cyclic. Then according to the main result of [4], there exists a Morita equivalence between $\Lambda N e_s^{\mathbf{L}^F}$ and $\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}$. Using the methods developed in [37], we extend their result to incorporate automorphisms of $\mathbf{G}^F$.

Theorem A (see Theorem 4.2). *Let $\mathbf{G}$ be a simple, simply connected algebraic group of type B_n, C_n or E_7 such that either $n > 2$ or q is odd. Let $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding. Then there exists a Frobenius endomorphism $F_0 : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$, which commutes with F such that F_0 stabilises $\mathbf{L}$ and the image of $\tilde{\mathbf{G}}^F \rtimes \langle F_0\rangle$ in the outer automorphism group of $\mathbf{G}^F$ is $\mathrm{Out}(\mathbf{G}^F)_{e_s^{\mathbf{G}^F}}$. Moreover, there exists a Morita equivalence between $\Lambda N e_s^{\mathbf{L}^F}$ and $\Lambda\mathbf{G} e_s^{\mathbf{G}^F}$, which lifts to a Morita equivalence between $\Lambda \mathrm{N}_{\tilde{\mathbf{G}}^F\langle F_0\rangle_{\ell'}}(\mathbf{L}, e_s^{\mathbf{L}^F}) e_s^{\mathbf{L}^F}$ and $\Lambda\tilde{\mathbf{G}}^F\langle F_0\rangle_{\ell'} e_s^{\mathbf{G}^F}$.*

Quasi-isolated blocks of type A

In the next part of our paper, we focus on groups of Lie type A. According to the main result of [38], to prove the inductive Alperin–McKay condition, it suffices to consider strictly quasi-isolated block: that is, blocks of $\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}$ such that $\mathrm{C}^\circ_{\mathbf{G}^*}(s)\,\mathrm{C}_{(\mathbf{G}^*)^{F^*}}(s)$ is not contained in a proper Levi subgroup of $\mathbf{G}^*$. Therefore, we will, from now on, assume that $\mathbf{G}$ is of type A and s is a strictly quasi-isolated element. In contrast to the situation of Theorem A, the automorphism group of $\mathbf{G}^F$ is more complicated, and the quotient group $N/\mathbf{L}^F$ can become arbitrary large with the rank of $\mathbf{G}$ increasing. Thus, a direct approach along the lines of Theorem A does not seem possible.

Using the explicit description of quasi-isolated elements in groups of type A by Bonnafé [2], we instead first construct a specific F-stable Levi subgroup $\mathbf{L}'$ of $\mathbf{G}$ containing $\mathbf{L}$. This Levi subgroup has the additional property that $N'/\mathbf{L}'$ is cyclic of prime order. Again, we denote by N' the common stabiliser of $\mathbf{L}'$ and $e_s^{\mathbf{L}'^F}$ in $\mathbf{G}^F$. The main result of [4] is still applicable in this slightly more general situation, and we obtain a Morita equivalence between $\Lambda N' e_s^{\mathbf{L}'^F}$ and $\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}$. This enables us to construct a certain abelian subgroup $\mathcal{A}$ of $\mathrm{Aut}(\tilde{\mathbf{G}}^F)$ (see Definition 7.6) such that $\tilde{\mathbf{G}}^F \rtimes \mathcal{A}$ generates the stabiliser of $e_s^{\mathbf{G}^F}$ in $\mathrm{Out}(\mathbf{G}^F)$. We obtain the following result, which can be seen as a version of Theorem A for groups of type A.

Theorem B (see Corollary 7.8). *Assume that $\mathbf{G}$ is of type A, and let $s \in (\mathbf{G}^*)^{F^*}$ be a strictly quasi-isolated element of ℓ'-order. If $\ell \nmid |\mathcal{A}|$, then there exists a Morita equivalence between $\Lambda N' e_s^{\mathbf{L}'^F}$ and $\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}$, which lifts to a Morita equivalence between $\Lambda \mathrm{N}_{\tilde{\mathbf{G}}^F\mathcal{A}}(\mathbf{L}', e_s^{\mathbf{L}'^F}) e_s^{\mathbf{L}'^F}$ and $\Lambda\tilde{\mathbf{G}}^F\mathcal{A} e_s^{\mathbf{G}^F}$.*

Reduction to isolated blocks

Using as a blueprint the methods developed in the proof of the main theorem of [38], we use the result of Theorem B to obtain a reduction of the verification of the iAM-condition to unipotent blocks of type A. By the work of [12] and [8], which shows that the inductive Alperin–McKay condition holds for unipotent blocks of type A, we obtain the following:

Theorem C (see Corollary 12.5). *The inductive Alperin–McKay condition holds for all ℓ-blocks of quasi simple groups of type A, whenever $\ell \geq 5$.*

The statement of Theorem C has been obtained in special cases by [8]. As a byproduct of the reduction methods developed for the proof of Theorem C and our equivariant Jordan decomposition from Theorem A, we obtain the following:

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Theorem D (see Theorem 12.6). *Suppose that $\ell \geq 5$, and let X be one of the symbols B or C. Assume that all isolated ℓ-blocks of quasi-simple groups of type X are AM-good relative to the Cabanes group (see 11.1) of their defect group. Then all ℓ-blocks of quasi-simple groups of type X are AM-good.*

2-*Blocks with abelian defect group*

According to the classification result of [17], almost all quasi-isolated 2-blocks of groups with abelian defect arise as blocks of groups of type A. Using the arguments in Theorem C, we can then show that these blocks satisfy the inductive condition. Moreover, we can prove the inductive Alperin–McKay condition for the remaining quasi-isolated 2-blocks with the abelian defect group. In fact, we can prove this condition for the slightly larger class of 2-blocks of $\mathbf{G}^F$ whose defect group D is almost abelian in $\mathbf{G}^F$; see Definition 13.1. The advantage of working with this larger class of blocks is that we obtain a reduction theorem to quasi-simple groups of the Alperin–McKay conjecture for blocks with almost abelian defect group. Following the proof of [42], we can show the following:

Proposition E (see Proposition 13.3). *Let X be a finite group and ℓ a prime. Assume that for every nonabelian simple subquotient S of X with $\ell \mid |S|$, the following holds: Every ℓ-block of the universal covering group H of S with almost abelian defect group satisfies the iAM-condition. Then the Alperin–McKay conjecture holds for any ℓ-block of X with almost abelian defect.*

As a consequence of Proposition E and the verification of the inductive Alperin–McKay condition for the necessary ℓ-blocks, we obtain the following result.

Theorem F (see Theorem 14.10). *The Alperin–McKay conjecture holds for all 2-blocks with almost abelian defect group.*

By [22, Proposition 5.6] and [20, Theorem 1.1], we obtain the following immediate consequence of Theorem F.

Theorem G. *The Alperin weight conjecture holds for all 2-blocks with abelian defect group.*

After submitting this paper, we were informed that Y. Zhou was also working on a proof of Theorem G using different methods.

1. General properties of the Bonnafé–Dat–Rouquier equivalence

In the following, we mostly use the notation of [37] and [38]. For the convenience of the reader, we recall the most important notions.

1.1. Representation theory

Let ℓ be a prime and K be a finite field extension of $\mathbb{Q}_\ell$. We assume in the following that K is large enough for the finite groups under consideration. Let $\mathcal{O}$ be the ring of integers of K over $\mathbb{Z}_\ell$ and $k = \mathcal{O}/J(\mathcal{O})$ its residue field. We will use Λ (respectively A) to interchangeably denote $\mathcal{O}$ or k (respectively K or $\mathcal{O}$).

1.2. Groups of Lie type

Let $\mathbf{G}$ be a connected reductive group with Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ defined over $\overline{\mathbb{F}}_p$ for some prime $p \neq \ell$. Given such a group $\mathbf{G}$, it is often convenient to consider it a closed subgroup of a group whose centre is connected. Therefore, we fix a regular embedding $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ of $\mathbf{G}$ as in [11, Section 15.1], and we identify $\mathbf{G}$ with its image in $\tilde{\mathbf{G}}$. For any closed subgroup $\mathbf{M}$ of $\mathbf{G}$, we define $\tilde{\mathbf{M}} := \mathbf{M}\mathbf{Z}(\tilde{\mathbf{G}})$. Moreover, if $\mathbf{H}$ is any closed F-stable subgroup of $\tilde{\mathbf{G}}$, then we denote by H its subset of F-stable points $\mathbf{H}^F$.

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1.3. Godement resolutions and ℓ-adic cohomology

Let $\mathbf{X}$ be a variety defined over an algebraic closure of $\mathbb{F}_p$ endowed with an action of a finite group G. By work of Rickard and Rouquier, there exists an object $G\Gamma_c(\mathbf{X}, \Lambda)$ in $\mathrm{Ho}^b(\Lambda G\text{-perm})$, the bounded homotopy category of ℓ-permutation ΛG-modules. Its ith cohomology groups are denoted $H_c^i(\mathbf{X}, \Lambda)$, and we abbreviate $H_c^{\dim(X)}(\mathbf{X}, \Lambda)$ by $H_c^{\dim}(\mathbf{X}, \Lambda)$.

1.4. Deligne–Lusztig induction

Suppose that $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$ and $F(\mathbf{L}) = \mathbf{L}$. We consider the Deligne–Lusztig variety $\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$. Its cohomology groups $H_c^i(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)$ induce an additive map $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : \mathbb{Z}\,\mathrm{Irr}(\mathbf{L}^F) \to \mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^F)$, the so-called Lusztig induction. In the case under consideration, the map $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ will not depend on the choice of the parabolic subgroup $\mathbf{P}$, and thus we will write $R_{\mathbf{L}}^{\mathbf{G}}$ for $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ in the following.

1.5. The Bonnafé–Dat–Rouquier equivalence

Let $\mathbf{G}^*$ be the Langlands dual of $\mathbf{G}$ with Frobenius endomorphism $F^* : \mathbf{G}^* \to \mathbf{G}^*$ dual to F. We fix a semisimple element $s \in (\mathbf{G}^*)^{F^*}$ of ℓ'-order and, as in [11, Theorem 9.12], let $e_s^{\mathbf{G}^F} \in \mathrm{Z}(\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F})$ be the central idempotent associated to it. Assume that $\mathbf{L}^*$ is an F^*-stable Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$. Additionally, suppose that $\mathbf{L}^*\,\mathrm{C}_{\mathbf{G}^*}(s)^{F^*} = \mathrm{C}_{\mathbf{G}^*}(s)^{F^*}\mathbf{L}^*$, and define $\mathbf{N}^* := \mathrm{C}_{\mathbf{G}^*}(s)^{F^*}\mathbf{L}^*$. Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ in duality with $\mathbf{L}^*$, and denote by $\mathbf{N}$ the subgroup of $\mathrm{N}_{\mathbf{G}}(\mathbf{L})$ that corresponds to the subgroup $\mathbf{N}^*$ of $\mathrm{N}_{\mathbf{G}^*}(\mathbf{L}^*)$ under the isomorphism $\mathrm{N}_{\mathbf{G}}(\mathbf{L})/\mathbf{L} \cong \mathrm{N}_{\mathbf{G}^*}(\mathbf{L}^*)/\mathbf{L}^*$ given by duality. Throughout this paper, we assume that $\mathbf{N}/\mathbf{L} \cong \mathbf{N}^F/\mathbf{L}^F$ is cyclic.

Suppose that $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$, and consider the bimodule $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_s^{\mathbf{L}^F}$. Since $\mathbf{N}^F/\mathbf{L}^F$ is assumed to be cyclic (and of ℓ'-order by [2, Corollary 2.9]) and $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_s^{\mathbf{L}^F}$ is $\mathbf{N}^F$-stable by [4, Theorem 7.2], there exists a $\Lambda(\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{N}}^F)$-module M' extending the $\Lambda(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F)$-module $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_s^{\mathbf{L}^F}$; see [35, Lemma 10.2.13].

For the following theorem, recall that for any complex $\mathcal{C} \in \mathrm{Comp}^b(A)$, there exists (see, for instance, [4, 2.A.]) a complex $\mathcal{C}^{\mathrm{red}}$ with $\mathcal{C} \cong \mathcal{C}^{\mathrm{red}}$ in $\mathrm{Ho}^b(A)$ such that $\mathcal{C}^{\mathrm{red}}$ has no nonzero direct summand that is homotopy equivalent to 0.

Theorem 1.1 (Bonnafé–Dat–Rouquier). *There exists a complex $\mathcal{C}'$ of $\mathcal{O}\mathbf{G}^F$-$\mathcal{O}\mathbf{N}^F$-bimodules extending $G\Gamma_c(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)^{\mathrm{red}}e_s^{\mathbf{L}^F}$ such that $H^d(\mathcal{C}') \cong M'$, where $d := \dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})$. The complex $\mathcal{C}'$ induces a splendid Rickard equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$, and the bimodule M' induces a Morita equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$.*

Proof. In the proof of [4, Theorem 7.5], use the fact that M' extends $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_s^{\mathbf{L}^F}$ instead of [4, Proposition 7.3]. The rest of the proof of the theorem is as in [4, Section 7]. □

Note that the Morita equivalence in Theorem 1.1 might depend on the particular choice of the extension M'. In particular, this is also the case for the character bijection $R : \mathrm{Irr}(\mathbf{N}^F, e_s^{\mathbf{L}^F}) \to \mathrm{Irr}(\mathbf{G}^F, e_s^{\mathbf{G}^F})$ induced by the Morita bimodule M'.

1.6. Some local properties of the Bonnafé–Dat–Rouquier Morita equivalence

Let $\mathcal{C}'$ be a complex of $\Lambda(\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}})$-modules inducing the splendid Rickard equivalence between $\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\Lambda\mathbf{N}^F e_s^{\mathbf{L}^F}$ as in Theorem 1.1 above. Then $\mathcal{C}'$ induces a bijection $c \mapsto b$ between the blocks of $\Lambda\mathbf{N}^F e_s^{\mathbf{L}^F}$ and $\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}$, where b is defined as the unique block such that $b\mathcal{C}'c$ is not homotopy equivalent to 0. We fix a block c of $\Lambda\mathbf{N}^F e_s^{\mathbf{L}^F}$, and we let (Q, c_Q) be a c-Brauer pair. There exists a unique b-Brauer pair (Q, b_Q) such that the complex $\mathrm{Br}_{\Delta Q}(\mathcal{C}')c_Q$ induces a Rickard equivalence between

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$k\mathrm{C}_{\mathbf{G}^F}(Q)b_Q$ and $k\mathrm{C}_{\mathbf{N}^F}(Q)c_Q$; see [4, Theorem 7.7]. Moreover, this implies that every defect group of the block c is a defect group of the block b; see also [37, Theorem 1.3].

Lemma 1.2. *Suppose that Q is an ℓ-subgroup of $\mathbf{L}^F$. Then the $\mathrm{N}_{\mathbf{G}^F}(Q) \times \mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}}$-bimodule $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)\mathrm{br}_Q(e_s^{\mathbf{L}^F})$ is multiplicity-free.*

Proof. According to the proof of [34, Theorem 5.2], there exists a unique complex $\mathcal{C}'_Q$ of ℓ-permutation $\Lambda(\mathrm{C}_{\mathbf{G}^F}(Q) \times \mathrm{C}_{\mathbf{N}^F}(Q)^{\mathrm{opp}})$-modules lifting the complex $\mathrm{Br}_{\Delta Q}(\mathcal{C}')$ from k to Λ. Note that

$$\mathrm{Res}^{\mathrm{C}_{\mathbf{G}^F}(Q)\times\mathrm{C}_{\mathbf{N}^F}(Q)^{\mathrm{opp}}}_{\mathrm{C}_{\mathbf{G}^F}(Q)\times\mathrm{C}_{\mathbf{L}^F}(Q)^{\mathrm{opp}}}(\mathrm{Br}_{\Delta Q}(\mathcal{C}')) \cong G\Gamma_c(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)}, k)\mathrm{br}_Q(e_s^{\mathbf{L}^F})$$

in $\mathrm{Ho}^b(k(\mathrm{C}_{\mathbf{G}^F}(Q) \times \mathrm{C}_{\mathbf{L}^F}(Q)^{\mathrm{opp}}))$; see, for example, [37, Lemma 2.9]. Let $M'_Q := H^{d_Q}(\mathcal{C}'_Q)$, where by [4, Theorem 4.14], the integer $d_Q := \dim(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)})$ is the unique degree in which $\mathrm{Br}_{\Delta Q}(\mathcal{C})$ has nonzero cohomology. Thus, $M'_Q c_Q$ induces a Morita equivalence between $\Lambda\mathrm{C}_{\mathbf{G}^F}(Q)b_Q$ and $\Lambda\mathrm{C}_{\mathbf{N}^F}(Q)c_Q$. By the proof of [37, Proposition 1.12], the bimodule $\mathrm{Ind}^{\mathrm{N}_{\mathbf{G}^F}(Q)\times\mathrm{N}_{\mathbf{N}^F}(Q)^{\mathrm{opp}}}_{\mathrm{C}_{\mathbf{G}^F}(Q)\times\mathrm{C}_{\mathbf{N}^F}(Q)^{\mathrm{opp}}\Delta\,\mathrm{N}_{\mathbf{N}^F}(Q)}(M'_Q c_Q)$ induces a Morita equivalence between $\Lambda\mathrm{N}_{\mathbf{G}^F}(Q)B_Q$ and $\Lambda\mathrm{N}_{\mathbf{N}^F}(Q)C_Q$, where B_Q (respectively C_Q) is the block covering b_Q (respectively c_Q).

In particular, $\mathrm{Ind}^{\mathrm{N}_{\mathbf{G}^F}(Q)\times\mathrm{N}_{\mathbf{N}^F}(Q)^{\mathrm{opp}}}_{\mathrm{C}_{\mathbf{G}^F}(Q)\times\mathrm{C}_{\mathbf{N}^F}(Q)^{\mathrm{opp}}\Delta\,\mathrm{N}_{\mathbf{N}^F}(Q)}(M'_Q)C_Q$ is indecomposable. Since this is true for all Brauer pairs of all blocks c of $\Lambda\mathbf{N}^F e_s^{\mathbf{L}^F}$, it follows that $\mathrm{Ind}^{\mathrm{N}_{\mathbf{G}^F}(Q)\times\mathrm{N}_{\mathbf{N}^F}(Q)^{\mathrm{opp}}}_{\mathrm{C}_{\mathbf{G}^F}(Q)\times\mathrm{C}_{\mathbf{N}^F}(Q)^{\mathrm{opp}}\Delta\,\mathrm{N}_{\mathbf{N}^F}(Q)}(M'_Q)$ is multiplicity free as well. By Mackey's formula, this module is an extension of

$$\mathrm{Ind}^{\mathrm{N}_{\mathbf{G}^F}(Q)\times\mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}}}_{\mathrm{C}_{\mathbf{G}^F}(Q)\times\mathrm{C}_{\mathbf{L}^F}(Q)^{\mathrm{opp}}\Delta\,\mathrm{N}_{\mathbf{L}^F}(Q)}(H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{C}_{\mathbf{G}}(Q)}, \Lambda)\mathrm{br}_Q(c_Q)) \cong (H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)\mathrm{br}_Q(c_Q)).$$

Since $\mathbf{N}^F/\mathbf{L}^F$ is cyclic of ℓ'-order and the Morita bimodule is indecomposable, the result therefore follows from Clifford theory. □

1.7. Extending the action of complexes

We need to slightly strengthen [4, Theorem 7.6] by including the diagonal action of $\tilde{N}$.

Lemma 1.3. *Assume that $\ell \nmid |H^1(F, \mathrm{Z}(\mathbf{G}))|$. Then there exists a complex $\mathcal{C}'$ of $\Lambda(\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{N}}^F)$-modules such that $H^d(\mathcal{C}') \cong M'$ and $\mathcal{C}'$ induces a splendid Rickard equivalence between $\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\Lambda\mathbf{N}^F e_s^{\mathbf{L}^F}$.*

Proof. Let us first assume that $\Lambda = k$. We consider the complex of ℓ-permutation $k(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\Delta(\tilde{\mathbf{L}}^F))$-modules $\mathcal{C} := G\Gamma_c(\mathbf{Y}_{\mathbf{U}}, k)^{\mathrm{red}} e_s^{\mathbf{L}^F}$. Consider an ℓ-subgroup $\tilde{R}$ of $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\Delta(\tilde{\mathbf{L}}^F)$. Let $Z := \mathrm{Z}(\tilde{\mathbf{G}}^F)_\ell$, and denote $R := \tilde{R} \cap (\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}})$. By assumption, $\ell \nmid |H^1(F, \mathrm{Z}(\mathbf{G}))| = |\tilde{\mathbf{G}}^F/\mathbf{G}^F\mathrm{Z}(\tilde{\mathbf{G}}^F)|$, and therefore $\tilde{R} \subset R\Delta(Z)$. The subgroup $\Delta(Z)$ centralises the variety $\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}$. Hence, $\mathcal{C}$ can be regarded as complex of $(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\Delta(\tilde{\mathbf{L}}^F))/\Delta(Z)$-modules. Since $\tilde{R} \subset R\Delta(Z)$, we deduce that $\mathrm{Br}_R(\mathcal{C}) \cong \mathrm{Br}_{\tilde{R}}(\mathcal{C})$. Using [4, Corollary 3.8], we deduce that $\mathrm{Br}_R(\mathcal{C})$ is acyclic unless R is $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$-conjugate to a subgroup of $\Delta\mathbf{L}^F$. Thus, $\tilde{R}$ is $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$-conjugate to a subgroup of $\Delta\tilde{\mathbf{L}}^F$. By [4, Lemma A.2], we deduce that the vertices of the indecomposable summands of the components $\mathcal{C}$ are all contained in $\Delta\tilde{\mathbf{L}}^F$.

Consider now the complex $\mathrm{End}^\bullet_{k\mathbf{G}^F}(\mathcal{C})$ of $k(\mathbf{L}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\Delta(\tilde{\mathbf{L}}^F))$-bimodules. The same argument as above (for $\mathrm{End}^\bullet_{k\mathbf{G}^F}(\mathcal{C})$ instead of $\mathcal{C}$) shows that $\mathrm{Br}_{\tilde{R}}(\mathrm{End}^\bullet_{k\mathbf{G}^F}(\mathcal{C})) \cong \mathrm{Br}_R(\mathrm{End}^\bullet_{k\mathbf{G}^F}(\mathcal{C}))$. The cohomology of the latter complex is concentrated in degree 0 only by the proof of Step 1 of [4, Theorem 7.5]. Hence, by [4, Lemma A.3], we have $\mathrm{End}^\bullet_{k\mathbf{G}^F}(\mathcal{C}) \cong \mathrm{End}_{D^b(k\mathbf{G}^F)}(\mathcal{C})$ in $\mathrm{Ho}^b(k(\mathbf{L}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F))$.

The rest of the proof is now almost identical to the proof of [4, Theorem 7.6]. For completeness, we provide most of the details here. Denote $\mathcal{C}' = \mathrm{Ind}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F}(\mathcal{C})$. Let $\mathcal{P}$ be a projective resolution of

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$k\mathbf{N}^F$ – that is, a complex of projective $k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F)$-modules such that its terms $\mathcal{P}^i = 0$ for $i > 0$ – together with a quasi-isomorphism $\mathcal{P} \to k\mathbf{N}^F$ of $k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F)$-modules.

Let $\mathcal{X}$ be a complex of $k(\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F)$-modules. Observe that we can consider the complex $\mathrm{End}^\bullet_{k\mathbf{G}^F}(\mathcal{X})$ as a complex of $k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F)$-modules. We have a natural isomorphism $\mathrm{End}^\bullet_{k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F)}(\mathcal{X}) \cong \mathrm{Hom}^\bullet_{k(\mathbf{N}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F)}(k\mathbf{N}^F, \mathrm{End}^\bullet_{k\mathbf{G}^F}(\mathcal{X}))$. The terms of $\mathcal{C}'$ are projective $k\mathbf{G}^F$-modules. Therefore, as in the proof of [4, Theorem 7.6], we can consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{End}_{\mathrm{Ho}^b(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F))}(\mathcal{C}') & \longrightarrow & \mathrm{End}_{D^b(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F))}(\mathcal{C}') \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Hom}_{\mathrm{Ho}^b(k(\mathbf{N}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F))}(k\mathbf{N}^F, \mathrm{End}^\bullet_{k\mathbf{G}^F}(\mathcal{C}')) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ho}^b(k(\mathbf{N}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F))}(\mathcal{P}, \mathrm{End}^\bullet_{k\mathbf{G}^F}(\mathcal{C}')) \end{array}$$

Using the isomorphisms of complexes in $\mathrm{Ho}^b(k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F))$

$$\mathrm{End}^\bullet_{k\mathbf{G}^F}(\mathcal{C}') \cong \mathrm{Ind}_{\mathbf{L}^F\times(\mathbf{L}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F}^{\mathbf{N}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F}(\mathrm{End}^\bullet_{k\mathbf{G}^F}(C))$$

and

$$\mathrm{End}_{D^b(k\mathbf{G}^F)}(\mathcal{C}') \cong \mathrm{Ind}_{\mathbf{L}^F\times(\mathbf{L}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F}^{\mathbf{N}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F}(\mathrm{End}_{D^b(k\mathbf{G}^F)}(C)),$$

we deduce that

$$\mathrm{End}^\bullet_{k\mathbf{G}^F}(\mathcal{C}') \cong \mathrm{End}_{D^b(k\mathbf{G}^F)}(\mathcal{C}') \text{ in } \mathrm{Ho}^b(k(\mathbf{N}^F \times (\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F)).$$

Now, the canonical map

$$\mathrm{Hom}_{\mathrm{Ho}^b(k(\mathbf{N}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(k\mathbf{N}^F, \mathrm{End}_{D^b(k\mathbf{G}^F)}(\mathcal{C}')) \to \mathrm{Hom}_{\mathrm{Ho}^b(k(\mathbf{N}^F\times(\mathbf{N}^F)^{\mathrm{opp}}))}(\mathcal{P}, \mathrm{End}_{D^b(k\mathbf{G}^F)}(\mathcal{C}'))$$

is an isomorphism, since $\mathrm{End}_{D^b(k\mathbf{G}^F)}(\mathcal{C}')$ is a complex concentrated in degree 0. It follows that the top horizontal map in the commutative diagram above is an isomorphism, hence we have canonical isomorphisms

$$\mathrm{End}_{\mathrm{Ho}^b(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta(\tilde{\mathbf{L}}^F)))}(\mathcal{C}') \xrightarrow{\cong} \mathrm{End}_{D^b(k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta(\tilde{\mathbf{L}}^F)))}(\mathcal{C}') \xrightarrow{\cong}$$

$$\mathrm{End}_{k(\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F)}(\mathrm{Ind}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}\Delta(\tilde{\mathbf{L}}^F)}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta(\tilde{\mathbf{L}}^F)} H_c^d(\mathbf{Y}_\mathbf{U}, k)e_s^{\mathbf{L}^F}).$$

Using the proof of Step 3 of [4, Theorem 7.6], we deduce that there exists a summand $\tilde{\mathcal{C}}$ of $\mathcal{C}'$ that is quasi-isomorphic to M'. As in Step 4 and 5 of the proof of [4, Theorem 7.6], we see that $\mathrm{Res}^{\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F}_{\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F}(\tilde{\mathcal{C}}) \cong \mathcal{C}$ in $\mathrm{Ho}^b(k(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F))$. Furthermore, Step 5 of the proof of [4, Theorem 7.6] shows that $\tilde{\mathcal{C}}$ induces a splendid Rickard equivalence between $k\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $k\mathbf{N}^F e_s^{\mathbf{L}^F}$.

Finally, let us consider the case $\Lambda = \mathcal{O}$. Using [34, Lemma 5.1] together with the arguments in the first paragraph of the proof of [34, Theorem 5.2], we observe that there exists a unique complex of $\mathcal{O}(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F)$-modules lifting $\tilde{\mathcal{C}}$. Moreover, by the proof of [34, Theorem 5.2], this complex induces a splendid Rickard equivalence between $\mathcal{O}\mathbf{G}^F e_s^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{N}^F e_s^{\mathbf{L}^F}$. □

1.8. The Bonnafé–Dat–Rouquier equivalence and character correspondences

Let $M := H_c^{\dim}(\mathbf{Y}_\mathbf{U}^\mathbf{G}, \Lambda)e_s^{\mathbf{L}^F}$ be considered as $X := \mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F$-module, and let M' be a $Y := \mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{N}}^F$-module extending M. Define $\tilde{M} = \mathrm{Ind}_X^{\tilde{X}}(M)$, where $\tilde{X} := \tilde{\mathbf{G}}^F \times (\tilde{\mathbf{L}}^F)^{\mathrm{opp}}$. Observe that $\tilde{M} \cong H_c^{\dim}(\mathbf{Y}_\mathbf{U}^{\tilde{\mathbf{G}}}, \Lambda)e_s^{\mathbf{L}^F}$.

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Recall from the proof of [4, Theorem 7.5] that there exists a central idempotent $e \in \mathrm{Z}(\Lambda\tilde{\mathbf{L}}^F)$ such that $\sum_{n\in\mathbf{N}^F/\mathbf{L}^F} {}^n e = e_s^{\mathbf{L}^F}$. Moreover, the induction functor yields a Morita equivalence between $\Lambda\tilde{\mathbf{L}}^F e$ and $\Lambda\tilde{\mathbf{N}}^F e_s^{\mathbf{L}^F}$. From this, it follows that the right action of $\tilde{\mathbf{L}}^F$ on $\tilde{M}$ extends to $\tilde{\mathbf{N}}^F$ and the extended bimodule $\tilde{M}' := \tilde{M}e \otimes_{\Lambda\tilde{\mathbf{L}}^F} \Lambda\tilde{\mathbf{N}}^F$ induces a Morita equivalence between $\Lambda\tilde{\mathbf{N}}^F e_s^{\mathbf{L}^F}$ and $\Lambda\tilde{\mathbf{G}}^F e_s^{\mathbf{G}^F}$. Writing

$$\tilde{R} : \mathrm{Irr}(\tilde{\mathbf{N}}^F, e_s^{\mathbf{L}^F}) \to \mathrm{Irr}(\tilde{\mathbf{G}}^F, e_s^{\mathbf{G}^F})$$

for the character bijection induced by the Morita bimodule $\tilde{M}'$, we therefore obtain:

Lemma 1.4. *For every $\psi \in \mathrm{Irr}(\tilde{\mathbf{N}}^F, e_s^{\mathbf{L}^F})$, there exists a unique character $\lambda \in \mathrm{Irr}(\tilde{\mathbf{L}}^F, e)$ such that $\mathrm{Ind}_{\tilde{\mathbf{L}}^F}^{\tilde{\mathbf{N}}^F}(\lambda) = \psi$, and we have $R_{\tilde{\mathbf{L}}^F}^{\tilde{\mathbf{G}}^F}(\lambda) = \tilde{R}\mathrm{Ind}_{\tilde{\mathbf{L}}^F}^{\tilde{\mathbf{N}}^F}(\lambda)$.*

The proof of [4, Theorem 7.5] shows that $\mathrm{Ind}_Y^{\tilde{Y}}(M') \cong \tilde{M}'$. This property ensures that $\tilde{R}$ restricts to a bijection $\tilde{R} : \mathrm{Irr}(\tilde{\mathbf{N}}^F \mid \chi) \to \mathrm{Irr}(\tilde{\mathbf{G}}^F \mid R(\chi))$ for every $\chi \in \mathrm{Irr}(\mathbf{N}^F, e_s^{\mathbf{L}^F})$. Here, R denotes the character bijection induced by the Morita bimodule M'.

1.9. Local character correspondences

Our aim is now to prove a local version of Lemma 1.4. For this, let $\tilde{b}$ be a block of $\Lambda\tilde{\mathbf{G}}^F e_s^{\mathbf{G}^F}$ covering b and $\tilde{c}$ be the unique block of $\Lambda\tilde{\mathbf{N}}^F e_s^{\mathbf{L}^F}$ corresponding to it under the Morita equivalence given by $\tilde{M}'$. We let $\tilde{f} := \tilde{c}e$ be the unique block of $\Lambda\tilde{\mathbf{L}}^F e$ below $\tilde{c}$.

Let D be a common defect group of the blocks c and b. We assume that Q is a characteristic subgroup of D and consider the $X_Q := \mathrm{N}_{\mathbf{G}^F}(Q) \times \mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}} \Delta\, \mathrm{N}_{\tilde{\mathbf{L}}^F}(Q)$-module $M_Q := H_c^{\dim}(\mathbf{Y}_{\mathrm{C_U}(Q)}^{\mathrm{N_G}(Q)}, \Lambda)\,\mathrm{br}_Q(e_s^{\mathbf{L}^F})$. The bimodule M_Q is $\Delta\,\mathrm{N}_{\tilde{\mathbf{N}}^F}(Q)$-stable by [37, Theorem 5.2]. Thus, there exists a $Y_Q := \mathrm{N}_{\mathbf{G}^F}(Q) \times (\mathrm{N}_{\mathbf{N}^F}(Q))^{\mathrm{opp}} \Delta\, \mathrm{N}_{\tilde{\mathbf{N}}^F}(Q)$-module M'_Q extending M_Q.

Denote $\tilde{M}_Q = \mathrm{Ind}_{X_Q}^{\tilde{X}_Q}(M_Q)$, where $\tilde{X}_Q := \mathrm{N}_{\tilde{\mathbf{G}}^F}(Q) \times \mathrm{N}_{\tilde{\mathbf{L}}^F}(Q)^{\mathrm{opp}}$. Observe that $\tilde{M}_Q \cong H_c^{\dim}(\mathbf{Y}_{\mathrm{C_U}(Q)}^{\mathrm{N_{\tilde{G}}}(Q)}, \Lambda)\,\mathrm{br}_Q(e_s^{\mathbf{L}^F})$. Let $\tilde{C}_Q := \mathrm{br}_Q(\tilde{c})$ and $\tilde{B}_Q := \mathrm{br}_Q(\tilde{b})$ be the Harris–Knörr correspondents (in the sense of [37, Corollary 1.18]) of $\tilde{c}$ and $\tilde{b}$ in $\mathrm{N}_{\tilde{\mathbf{N}}^F}(Q)$ and $\mathrm{N}_{\tilde{\mathbf{G}}^F}(Q)$, respectively. Since $\tilde{f}$ corresponds to the block $\tilde{b}$ under the Morita equivalence given by $\tilde{M}$, it follows that $\tilde{M}_Q\tilde{F}_Q$ induces a Morita equivalence between $\Lambda\,\mathrm{N}_{\tilde{\mathbf{L}}^F}(Q)\tilde{F}_Q$ and $\Lambda\,\mathrm{N}_{\tilde{\mathbf{G}}^F}(Q)\tilde{B}_Q$; see [37, Theorem 3.10]. Recall that the stabiliser of the idempotent e in $\tilde{\mathbf{N}}^F$ is $\tilde{\mathbf{L}}^F$. Consequently, the stabiliser of $\mathrm{br}_Q(e)$ in $\mathrm{N}_{\tilde{\mathbf{N}}^F}(Q)$ is $\mathrm{N}_{\tilde{\mathbf{L}}^F}(Q)$. Therefore, the bimodule $\tilde{M}'_Q := \tilde{M}_Q\,\mathrm{br}_Q(e) \otimes_{\Lambda\,\mathrm{N}_{\tilde{\mathbf{L}}^F}(Q)} \Lambda\,\mathrm{N}_{\tilde{\mathbf{N}}^F}(Q)$ is an extension of $\tilde{M}_Q$, and $\tilde{M}'_Q\tilde{C}_Q \cong \tilde{M}_Q\tilde{F}_Q \otimes_{\Lambda\,\mathrm{N}_{\tilde{\mathbf{L}}^F}(Q)} \Lambda\,\mathrm{N}_{\tilde{\mathbf{N}}^F}(Q)$ induces a Morita equivalence between $\Lambda\,\mathrm{N}_{\tilde{\mathbf{N}}^F}(Q)\tilde{C}_Q$ and $\Lambda\,\mathrm{N}_{\tilde{\mathbf{G}}^F}(Q)\tilde{B}_Q$. We write

$$\tilde{R}_Q : \mathrm{Irr}(\mathrm{N}_{\tilde{\mathbf{N}}^F}(Q), \tilde{C}_Q) \to \mathrm{Irr}(\mathrm{N}_{\tilde{\mathbf{G}}^F}(Q), \tilde{B}_Q)$$

for the associated character bijection. We obtain:

Lemma 1.5. *For every $\psi \in \mathrm{Irr}(\mathrm{N}_{\tilde{\mathbf{N}}^F}(Q), \tilde{C}_Q)$, there exists a unique character $\lambda \in \mathrm{Irr}(\mathrm{N}_{\tilde{\mathbf{L}}^F}(Q), \tilde{F}_Q)$ such that $\mathrm{Ind}_{\mathrm{N}_{\tilde{\mathbf{L}}^F}(Q)}^{\mathrm{N}_{\tilde{\mathbf{N}}^F}(Q)}(\lambda) = \psi$, and we have $R_{\mathrm{N}_{\tilde{\mathbf{L}}^F}(Q)}^{\mathrm{N}_{\tilde{\mathbf{G}}^F}(Q)}(\lambda) = \tilde{R}_Q\mathrm{Ind}_{\mathrm{N}_{\tilde{\mathbf{L}}^F}(Q)}^{\mathrm{N}_{\tilde{\mathbf{N}}^F}(Q)}(\lambda)$.*

We claim that $\mathrm{Ind}_{Y_Q}^{\tilde{Y}_Q}(M'_Q) \cong \tilde{M}'_Q$. This follows as in the proof of [4, Theorem 7.5]. Observe that it suffices to show that $\mathrm{Res}_{\tilde{X}_Q}^{\tilde{Y}_Q}(\mathrm{Ind}_{Y_Q}^{\tilde{Y}_Q}(M'_Q))\,\mathrm{br}_Q(e) \cong \mathrm{Res}_{\tilde{X}_Q}^{\tilde{Y}_Q}(\tilde{M}'_Q)\,\mathrm{br}_Q(e)$. By Mackey's formula, the left-hand side is isomorphic to $\tilde{M}_Q\,\mathrm{br}_Q(e)$. Moreover, we have $\mathrm{Res}_{\tilde{X}_Q}^{\tilde{Y}_Q}(\tilde{M}'_Q)\,\mathrm{br}_Q(e) \cong \tilde{M}_Q\,\mathrm{br}_Q(e)$, which

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proves the claim. As in the global case, we therefore obtain that $\tilde{R}_Q$ restricts to

$$\tilde{R}_Q : \mathrm{Irr}(\mathrm{N}_{\tilde{\mathbf{N}}^F}(Q), \chi) \to \mathrm{Irr}(\mathrm{N}_{\tilde{\mathbf{G}}^F}(Q), R_Q(\chi)),$$

where R_Q is the character bijection induced by M_Q and $\chi \in \mathrm{Irr}(\mathrm{N}_{\mathbf{N}^F}(Q) \mid \mathrm{br}_Q(c))$.

2. Descent of scalars

2.1. Restriction of scalars for Deligne–Lusztig varieties

We assume until Section 3 that $F_0 : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ is a Frobenius endomorphism stabilising $\mathbf{L}$ that satisfies $F_0^r = F$ for some integer r and $\gamma : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ is an automorphism commuting with F_0. In what follows, $\mathcal{A}$ will denote the subgroup of $\mathrm{Aut}(\tilde{\mathbf{G}}^F)$ generated by F_0 and γ.

Let us recall the setup from [37, Section 5]. We consider the reductive group $\underline{\mathbf{G}} = \mathbf{G}^r$ with Frobenius endomorphism $F_0 \times \cdots \times F_0 : \underline{\mathbf{G}} \to \underline{\mathbf{G}}$, which we also denote by F_0. More generally, whenever $\sigma : \mathbf{G} \to \mathbf{G}$ is a bijective morphism of $\mathbf{G}$, then we also denote by σ the induced map $\sigma \times \cdots \times \sigma : \underline{\mathbf{G}} \to \underline{\mathbf{G}}$ on $\underline{\mathbf{G}}$. We consider the automorphism

$$\tau : \underline{\mathbf{G}} \to \underline{\mathbf{G}}$$

given by $\tau(g_1, \ldots, g_r) = (g_2, \ldots, g_r, g_1)$. Consider the projection onto the first component

$$\mathrm{pr} : \underline{\mathbf{G}} \to \mathbf{G},\ (g_1, \ldots, g_r) \mapsto g_1.$$

The restriction of pr to $\underline{\mathbf{G}}^{F_0\tau}$ induces an isomorphism $\mathrm{pr} : \underline{\mathbf{G}}^{F_0\tau} \to \mathbf{G}^F$ of finite groups. For any connected subset $\mathbf{H}$ of $\mathbf{G}$, we set

$$\underline{\mathbf{H}} := \mathbf{H} \times F_0^{r-1}(\mathbf{H}) \times \cdots \times F_0(\mathbf{H}).$$

Note that if $\mathbf{H}$ is F-stable, then $\underline{\mathbf{H}}$ is τF_0-stable, and the projection map $\mathrm{pr} : \underline{\mathbf{H}} \to \mathbf{H}$ induces an isomorphism $\underline{\mathbf{H}}^{\tau F_0} \cong \mathbf{H}^F$. Conversely, one easily sees that any τF_0-stable subset of $\underline{\mathbf{G}}$ is of the form $\underline{\mathbf{H}}$ for some F-stable subset $\mathbf{H}$ of $\mathbf{G}$.

We consider the r-fold product $\underline{\mathbf{G}^*} := (\mathbf{G}^*)^r$ of the dual group $\mathbf{G}^*$ endowed with the Frobenius endomorphism $F_0^* := F_0^* \times \cdots \times F_0^* : \underline{\mathbf{G}^*} \to \underline{\mathbf{G}^*}$. Moreover, let

$$\tau^* : \underline{\mathbf{G}^*} \to \underline{\mathbf{G}^*},\ (g_1, \ldots, g_r) \mapsto (g_r, g_1 \ldots, g_{r-1}).$$

We denote by $\mathrm{pr} : \underline{\mathbf{G}^*} \to \mathbf{G}^*$ the projection onto the first coordinate.

2.2. Restriction of scalars and Jordan decomposition of characters

We let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$. Then $\underline{\mathbf{P}}$ is a parabolic subgroup of $\underline{\mathbf{G}}$ with Levi decomposition $\underline{\mathbf{P}} = \underline{\mathbf{L}} \ltimes \underline{\mathbf{U}}$ such that $\tau F_0(\underline{\mathbf{L}}) = \underline{\mathbf{L}}$. We can therefore consider the Deligne–Lusztig variety $\mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}}, F_0\tau}$, which is a $\underline{\mathbf{G}}^{F_0\tau} \times (\underline{\mathbf{L}}^{F_0\tau})^{\mathrm{opp}}$-variety. Under the isomorphism $\underline{\mathbf{G}}^{\tau F_0} \cong \mathbf{G}^F$ given by pr, we can consider it as a $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$-variety. Endowed with this structure, the projection map induces an isomorphism

$$\mathrm{pr} : \mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}}, F_0\tau} \to \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}},$$

which is $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$-equivariant; see [37, Proposition 5.3].

We consider the unipotent radical $\underline{\mathbf{U}}' := \mathbf{U}^r$ of the parabolic subgroup $\underline{\mathbf{P}}' = \mathbf{P}^r$ of $\underline{\mathbf{G}}$. Note that we have a Levi decomposition $\underline{\mathbf{P}}' = \underline{\mathbf{L}} \ltimes \underline{\mathbf{U}}'$ in $\underline{\mathbf{G}}$ and the parabolic subgroup $\underline{\mathbf{P}}'$ is τ-stable. In the following,

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we will use the language of parabolic subgroups and Levi subgroups of disconnected reductive groups as in [37, Section 2.1].

Lemma 2.1. *Assume that $e_s^{\mathbf{L}^F}$ is $\mathcal{A}$-stable. If $\mathbf{P}$ is γ-stable, then the module $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_s^{\mathbf{L}^F}$ extends to a $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\Delta(\tilde{\mathbf{L}}^F\mathcal{A})$-module. If in addition $\mathbf{N}^F/\mathbf{L}^F$ is centralised by $\mathcal{A}$, then $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda)e_s^{\mathbf{L}^F}$ extends to $\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}\Delta(\tilde{\mathbf{L}}^F\mathcal{A})$.*

Proof. The pair $(\underline{\mathbf{L}}, \underline{\mathbf{P}}')$ is $\langle\tau, \gamma\rangle$-stable. Therefore, we can consider $\mathbf{Y}_{\underline{\mathbf{U}}'}^{\underline{\mathbf{G}}}$ a $\underline{\mathbf{G}}^{\tau F_0} \times (\underline{\mathbf{L}}^{\tau F_0})^{\mathrm{opp}}\Delta(\tilde{\underline{\mathbf{L}}}^{\tau F_0}\langle\tau, \gamma\rangle)$-variety. By [4, Theorem 7.2], we have an isomorphism

$$H_c^{\dim}(\mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}}, \tau F_0}, \Lambda)e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}} \cong H_c^{\dim}(\mathbf{Y}_{\underline{\mathbf{U}}'}^{\underline{\mathbf{G}}, \tau F_0}, \Lambda)e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$$

of $\Lambda((\underline{\mathbf{G}}^{\tau F_0} \times (\underline{\mathbf{L}}^{\tau F_0})^{\mathrm{opp}})\Delta(\tilde{\underline{\mathbf{L}}}^F))$-modules. The projection $\mathrm{pr} : \underline{\mathbf{G}} \to \mathbf{G}$ onto the first coordinate defines an isomorphism

$$H_c^{\dim}(\mathbf{Y}_{\underline{\mathbf{U}}}^{\underline{\mathbf{G}}, \tau F_0}, \Lambda)e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}} \cong H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}, F}, \Lambda)e_s^{\mathbf{L}^F}$$

of $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F$-modules. By transport of structure, we can endow $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}, F}, \Lambda)e_s^{\mathbf{L}^F}$ with a $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F\langle F_0, \gamma\rangle$-structure.

Assume now that $\mathbf{N}/\mathbf{L}$ is centralised by $\mathcal{A}$, and let $n \in \mathbf{N}^F$ be a generator of the quotient group $\mathbf{N}^F/\mathbf{L}^F$. Then we have $\gamma(n)n^{-1}, F_0(n)n^{-1} \in \mathbf{L}^F$. Thus, conjugation by $\underline{n}$ defines an automorphism of $\underline{\mathbf{G}}\langle\tau, \gamma\rangle$, which normalises $\underline{\mathbf{L}}\langle\tau, \gamma\rangle$. Hence, by [37, Lemma 3.1], the $\Lambda((\underline{\mathbf{G}}^{\tau F_0} \times (\underline{\mathbf{L}}^{\tau F_0})^{\mathrm{opp}})\Delta(\tilde{\underline{\mathbf{L}}}^{\tau F_0}\langle\tau, \gamma\rangle))$-module $H_c^{\dim}(\mathbf{Y}_{\underline{\mathbf{U}}'}^{\underline{\mathbf{G}}, \tau F_0})e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$ is $\underline{n}$-stable. By transport of structure, we deduce that $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}, F}, \Lambda)e_s^{\mathbf{L}^F}$ is $\mathbf{N}^F$-stable as a $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}\Delta(\tilde{\mathbf{L}}^F\mathcal{A})$-module. Thus, it extends to a $\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F\langle F_0, \gamma\rangle$-module by [35, Lemma 10.2.13]. □

In the following, we abbreviate $\mathcal{N} := \mathrm{N}_{\tilde{\mathbf{G}}^F\mathcal{A}}(\mathbf{L}, e_s^{\mathbf{L}^F})$.

Lemma 2.2. *Suppose that we are in the situation of Lemma 2.1. Let Q be an ℓ-subgroup of $\mathbf{L}^F$. If $\mathbf{N}^F/\mathbf{L}^F$ is centralised by $\mathcal{A}$, then $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)\mathrm{br}_Q(e_s^{\mathbf{L}^F})$ extends to an $\mathrm{N}_{\mathbf{G}^F}(Q) \times \mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\mathcal{N}}(Q)$-module.*

Proof. Recall that the projection map $\mathrm{pr} : \underline{\mathbf{G}}^{\tau F_0} \to \mathbf{G}^F$ is an isomorphism of finite groups. If H is a subgroup of $\mathbf{G}^F$, we let $\underline{H} := \mathrm{pr}^{-1}(H)$; and if $x \in \Lambda H$, we let $\underline{x} := \mathrm{pr}^{-1}(x) \in \Lambda\underline{H}$.

The quotient group $\mathrm{N}_{\mathcal{N}}(Q)/\mathrm{N}_{\tilde{\mathbf{L}}^F\mathcal{A}}(Q)$ is cyclic as it embeds into $\mathcal{N}/\tilde{\mathbf{L}}^F\mathcal{A} \cong \mathbf{N}^F/\mathbf{L}^F$. Let $x \in \mathrm{N}_{\mathbf{N}^F}(Q)$ be a generator of said quotient. Since $\mathbf{N}^F/\mathbf{L}^F$ is centralised by F_0, we have $xF_0(x)^{-1} \in \mathrm{N}_{\mathbf{L}^F}(Q)$. Let $\underline{x} := (x, F_0^{r-1}(x), \ldots, F_0(x)) \in \underline{\mathbf{G}}^{\tau F_0}$ such that $\mathrm{pr}(\underline{x}) = x$. Consider the bijective morphism $\underline{\phi} : \tilde{\underline{\mathbf{G}}}\langle\tau\rangle \to \tilde{\underline{\mathbf{G}}}\langle\tau\rangle$ given by conjugation with $\underline{x}$. Note that $\underline{\phi}$ stabilises $\tilde{\underline{\mathbf{G}}}$ and commutes with the Frobenius endomorphism τF_0 of $\tilde{\underline{\mathbf{G}}} \rtimes \langle\tau\rangle$. Moreover, $\underline{\phi}$ also stabilises the Levi subgroup $\tilde{\underline{\mathbf{L}}}\langle\tau\rangle$ of $\tilde{\underline{\mathbf{G}}} \rtimes \langle\tau\rangle$. We denote $e := \mathrm{br}_Q(e_s^{\mathbf{L}^F})$. By [37, Lemma 2.23], we obtain an isomorphism

$$\underline{\phi}(H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\underline{\mathbf{U}}'}(\underline{Q})}^{\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q}), \tau F_0}, \Lambda)\underline{e})^{\underline{\phi}} \cong H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\underline{\phi}(\underline{\mathbf{U}}')}(\underline{Q})}^{\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q}), \tau F_0}, \Lambda)\underline{e}$$

of $\Lambda((\mathrm{N}_{\underline{\mathbf{G}}^{\tau F_0}}(\underline{Q}) \times \mathrm{N}_{\underline{\mathbf{L}}^{\tau F_0}}(\underline{Q})^{\mathrm{opp}})\Delta(\mathrm{N}_{\tilde{\underline{\mathbf{L}}}^{\tau F_0}\langle\tau\rangle}(\underline{Q}))$-modules. We have two Levi decompositions

$$\tilde{\underline{\mathbf{P}}}\langle\tau\rangle = \tilde{\underline{\mathbf{L}}}\langle\tau\rangle \ltimes \underline{\mathbf{U}}' \text{ and } \underline{\phi}(\tilde{\underline{\mathbf{P}}}\langle\tau\rangle) = \tilde{\underline{\mathbf{L}}}\langle\tau\rangle \ltimes \underline{\phi}(\underline{\mathbf{U}}')$$

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with the same Levi subgroup $\underline{\tilde{\mathbf{L}}}\langle\tau\rangle$ of $\underline{\tilde{\mathbf{G}}}\langle\tau\rangle$. Therefore, [37, Theorem 5.2] yields

$$H_c^{\dim}(\mathbf{Y}^{\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q}),\tau F_0}_{\mathrm{C}_{\underline{\phi}(\underline{\mathbf{U}'})}(\underline{Q})},\Lambda)\underline{e} \cong H_c^{\dim}(\mathbf{Y}^{\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q}),\tau F_0}_{\mathrm{C}_{\underline{\mathbf{U}'}}(\underline{Q})},\Lambda)\underline{e}.$$

It follows from this that $H_c^{\dim}(\mathbf{Y}^{\mathrm{N}_{\underline{\mathbf{G}}}(\underline{Q}),\tau F_0}_{\mathrm{C}_{\underline{\mathbf{U}'}}(\underline{Q})},\Lambda)\underline{e}$ is $(\underline{\phi},\underline{\phi}^{-1})$-invariant. Hence, the bimodule $H_c^{\dim}(\mathbf{Y}^{\mathrm{N}_{\mathbf{G}}(Q)}_{\mathrm{C}_{\mathbf{U}}(Q)},\Lambda)e$ is by transport of structure x-invariant as $\Lambda((\mathrm{N}_{\mathbf{G}^F}(Q)\times \mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}})\Delta(\mathrm{N}_{\tilde{\mathbf{L}}^F\langle F_0\rangle}(Q)))$-module. The claim now follows from [35, Lemma 10.2.13]. □

3. Construction for twisted groups

We generalise the construction of the previous section. This is essentially necessary for working with automorphisms of twisted groups. We suppose now that F_0 is a Frobenius endomorphism with $F_0^r\rho = F$ for some integer r and $\rho : \mathbf{G} \to \mathbf{G}$ a graph automorphism of order l that commutes with F_0. We denote by $\mathcal{A}$ the subgroup of $\mathrm{Aut}(\tilde{\mathbf{G}}^F)$ generated by F_0.

The construction in this section will be done in two separate steps. We first consider the connected reductive group

$$\mathbf{G}_\rho := \{(g,\rho(g),\dots,\rho^{l-1}(g)) \mid g \in \mathbf{G}\}.$$

We note that the projection onto the first coordinate defines an isomorphism $\mathrm{pr}_1 : \mathbf{G}_\rho \to \mathbf{G}$. Denote by $\tau : \mathbf{G}_\rho \to \mathbf{G}_\rho$ the automorphism given by

$$\tau(g,\rho(g),\dots,\rho^{l-1}(g)) := (\rho(g),\rho^2(g),\dots,\rho^{l-1}(g),g),$$

and let σ be the permutation induced by τ on the coordinates of $\mathbf{G}_\rho$. With this notation, the projection onto the first coordinate defines a $(\tau F_0^r, F)$-equivariant isomorphism $\mathbf{G}_\rho \to \mathbf{G}$.

Now define the group

$$\underline{\mathbf{G}}_\rho := \mathbf{G}_\rho \times \cdots \times \mathbf{G}_\rho$$

as the r-fold product of the group $\mathbf{G}_\rho$. We define a permutation σ_0 on the set $\{1,\dots,rl\}$ as follows:

$$\sigma_0(i) = \begin{cases} i-l & \text{if } i > l,\\ rl-l+\sigma(i) & \text{if } 1 \le i \le l.\end{cases}$$

Consider the automorphism $\tau_0 : \underline{\mathbf{G}}_\rho \to \underline{\mathbf{G}}_\rho$, which for an element $(g_1,\dots,g_{lr}) \in \underline{\mathbf{G}}_\rho$ is given by $\tau_0((g_1,\dots,g_{lr})) := (g_{\sigma_0(1)},\dots,g_{\sigma_0(rl)})$. By construction, it follows that $\tau_0^r = \tau$ (here τ is understood as the permutation $\tau\times\cdots\times\tau$ on $\underline{\mathbf{G}}_\rho$). In particular, the morphism τ_0F_0 satisfies $(\tau_0F_0)^r = \tau F_0^r$. Moreover, τ_0F_0 cyclically permutes the r copies $\mathbf{G}_\rho$ of the group $\underline{\mathbf{G}}_\rho$. We deduce that the projection map $\mathrm{pr}_2 : \underline{\mathbf{G}}_\rho \to \mathbf{G}_\rho$ onto the first factor $\mathbf{G}_\rho$ induces an isomorphism $\underline{\mathbf{G}}_\rho^{\tau F_0} \cong \mathbf{G}_\rho^{\tau_0F_0^r}$.

Notation 3.1. Assume that $\mathbf{H}$ is a closed connected subgroup of $\mathbf{G}$. Then we define $\mathbf{H}_\rho := \mathrm{pr}_1^{-1}(\mathbf{H})$. We then define

$$\underline{\mathbf{H}}_\rho := \mathbf{H}_\rho \times (\tau_0F_0)^{r-1}(\mathbf{H}_\rho) \times \cdots \times (\tau_0F_0)(\mathbf{H}_\rho)$$

considered a subgroup of $\underline{\mathbf{G}}_\rho$. Furthermore, we denote $\underline{\mathbf{H}}'_\rho := (\mathbf{H}_\rho)^r$.

Remark 3.2. If the subgroup $\mathbf{H}$ is ρ and F_0-stable, then we obtain $\underline{\mathbf{H}}_\rho = \underline{\mathbf{H}}'_\rho$. Moreover, the automorphism F_0 of the finite group $\mathbf{G}^F$ corresponds to the automorphism τ_0^{-1} of $\underline{\mathbf{G}}_\rho^{\tau_0F_0}$.

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Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$. Then we observe that $\underline{\mathbf{P}}_\rho$ is a parabolic subgroup of $\underline{\mathbf{G}}_\rho$ with Levi decomposition

$$\underline{\mathbf{P}}_\rho = \underline{\mathbf{L}}_\rho \ltimes \underline{\mathbf{U}}_\rho.$$

Assume now that $\mathbf{L}$ is F-stable. It follows that $\mathbf{L}_\rho$ is τF_0^r-stable and consequently the Levi subgroup $\underline{\mathbf{L}}_\rho$ is $\tau_0 F_0$-stable. We can therefore consider the Deligne–Lusztig variety $\mathbf{Y}_{\underline{\mathbf{U}}_\rho}^{\underline{\mathbf{G}}_\rho, F_0\tau_0}$, which is a $\underline{\mathbf{G}}_\rho^{F_0\tau_0} \times (\underline{\mathbf{L}}_\rho^{F_0\tau_0})^{\mathrm{opp}}$-variety. Under the isomorphism $\mathbf{G}^F \cong \underline{\mathbf{G}}_\rho^{F_0\tau_0}$ induced by the projection map

$$\mathrm{pr} := \mathrm{pr}_1 \circ \mathrm{pr}_2 : \underline{\mathbf{G}}_\rho \to \mathbf{G},$$

we will in the following regard it as a $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$-variety.

Proposition 3.3. *With the assumptions as above, the projection* $\mathrm{pr} : \underline{\mathbf{G}}_\rho \to \mathbf{G}$ *onto the first coordinate defines an isomorphism*

$$\mathbf{Y}_{\underline{\mathbf{U}}_\rho}^{\underline{\mathbf{G}}_\rho, \tau_0 F_0} \cong \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}, F}$$

of varieties that is $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$*-equivariant.*

Proof. The projection onto the first coordinate defines a $(\tau F_0^r, F)$-equivariant isomorphism $\mathbf{G}_\rho \to \mathbf{G}$. Thus, we obtain a $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$-equivariant isomorphism $\mathbf{Y}_{\mathbf{U}_\rho}^{\mathbf{G}_\rho, \tau F_0^r} \to \mathbf{Y}_{\mathbf{U}}^{\mathbf{G}, F}$.

The morphism $\tau_0 F_0$ satisfies $(\tau_0 F_0)^r = \tau F_0^r$ and cyclically permutes the r copies of $\mathbf{G}_\rho$ of the group $\underline{\mathbf{G}}_\rho$. Therefore, by [37, Theorem 5.2], projection onto the first coordinate yields a $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}}$-equivariant isomorphism $\mathbf{Y}_{\underline{\mathbf{U}}_\rho}^{\underline{\mathbf{G}}_\rho, \tau_0 F_0} \to \mathbf{Y}_{\mathbf{U}_\rho}^{\mathbf{G}_\rho, \tau F_0^r}$. The composition of these two isomorphisms yields the required isomorphism. □

We will now explain how we can explicitly construct the dual group of $(\underline{\mathbf{G}}_\rho, \tau_0 F_0)$. For this, consider the connected reductive group $\mathbf{G}_\rho^* := \{(g, \rho^*(g), \dots, (\rho^*)^{l-1}(g))\}$. Now define the group $\underline{\mathbf{G}}_\rho^* := \mathbf{G}_\rho^* \times \cdots \times \mathbf{G}_\rho^*$ as the r-fold product of the group $\mathbf{G}_\rho^*$. On this group, we consider the automorphism $\tau_0^* : \underline{\mathbf{G}}_\rho^* \to \underline{\mathbf{G}}_\rho^*$, which for an element $(g_1, \dots, g_{lr}) \in \underline{\mathbf{G}}_\rho^*$ is given by

$$\tau_0^*((g_1, \dots, g_{lr})) := (g_{(\sigma_0)^{-1}(1)}, \dots, g_{(\sigma_0)^{-1}(lr)}).$$

We then define $\tau^* := (\tau_0^*)^r$. Again we denote by $\mathrm{pr}_1 : \mathbf{G}_\rho^* \to \mathbf{G}^*$ and $\mathrm{pr}_2 : \underline{\mathbf{G}}_\rho^* \to \mathbf{G}_\rho^*$ the projections onto the first coordinate and $\mathrm{pr} = \mathrm{pr}_1 \circ \mathrm{pr}_2$.

Lemma 3.4. *The group* $(\underline{\mathbf{G}}_\rho^*, \tau_0^* F_0^*)$ *is dual to* $(\underline{\mathbf{G}}_\rho, \tau_0 F_0)$*, and for every semisimple element* $\underline{s} \in (\underline{\mathbf{G}}_\rho^*)^{\tau_0^* F_0^*}$*, the Lusztig series* $\mathcal{E}(\underline{\mathbf{G}}^{\tau_0 F_0}, \underline{s})$ *corresponds to* $\mathcal{E}(\mathbf{G}^F, \mathrm{pr}(\underline{s}))$ *under the isomorphism* $\mathrm{pr} : \underline{\mathbf{G}}_\rho^{\tau_0 F_0} \to \mathbf{G}^F$ *of finite groups.*

Proof. Recall that we have a $(\tau F_0^r, F)$-equivariant isomorphism $\mathbf{G}_\rho \to \mathbf{G}$. From this, we deduce that $(\mathbf{G}_\rho^*, \tau^*(F_0^*)^r)$ is dual to $(\mathbf{G}_\rho, \tau F_0^r)$. The Lusztig series $\mathcal{E}(\mathbf{G}_\rho^{\tau F_0^r}, \mathrm{pr}_2(\underline{s}))$ then corresponds to $\mathcal{E}(\mathbf{G}^F, \mathrm{pr}(\underline{s}))$ via the isomorphism $\mathrm{pr}_1 : \mathbf{G}_\rho^{\tau F_0^r} \to \mathbf{G}^F$. By applying [44, Corollary 8.8], we then observe that $(\underline{\mathbf{G}}_\rho^*, \tau_0^* F_0^*)$ is dual to $(\underline{\mathbf{G}}_\rho, \tau_0 F_0)$ and moreover that the Lusztig series $\mathcal{E}(\underline{\mathbf{G}}^{\tau_0 F_0}, \underline{s})$ corresponds to $\mathcal{E}(\mathbf{G}_\rho^{\tau F_0^r}, \mathrm{pr}_2(\underline{s}))$ via $\mathrm{pr}_2 : \underline{\mathbf{G}}_\rho^{\tau_0 F_0} \to \mathbf{G}_\rho^{\tau F_0^r}$. The claim follows from this. □

Lemma 3.5. *Assume that* $\mathbf{L}$ *and* $e_s^{\mathbf{L}^F}$ *are* F_0*-stable. If* $\mathbf{P}$ *is* ρ*-stable, then the module* $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda) e_s^{\mathbf{L}^F}$ *extends to a* $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}} \Delta \langle F_0 \rangle$*-module. If in addition* $\mathbf{N}^F/\mathbf{L}^F$ *is centralised by* F_0*, then* $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}}, \Lambda) e_s^{\mathbf{L}^F}$ *extends to* $\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}} \Delta(\tilde{\mathbf{L}}^F \langle F_0 \rangle)$.

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Proof. Since $\mathbf{L}$ is F_0-stable, we have $\underline{\mathbf{L}}_\rho = \underline{\mathbf{L}}'_\rho$ and therefore $\underline{\mathbf{P}}'_\rho$ is a parabolic subgroup with Levi decomposition

$$\underline{\mathbf{P}}'_\rho = \underline{\mathbf{L}}_\rho \ltimes \underline{\mathbf{U}}'_\rho.$$

We observe that $\mathrm{C}^\circ_{\underline{\mathbf{G}}^*_\rho}(\underline{s}) \subseteq \underline{\mathbf{L}}^*_\rho$. Therefore, [4, Theorem 7.2] shows that we have an isomorphism

$$H_c^{\dim}(\mathbf{Y}^{\underline{\mathbf{G}}_\rho,\tau_0F_0}_{\underline{\mathbf{U}}_\rho})e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}} \cong H_c^{\dim}(\mathbf{Y}^{\underline{\mathbf{G}}_\rho,\tau_0F_0}_{\underline{\mathbf{U}}'_\rho})e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau F_0}}$$

of $\Lambda((\underline{\mathbf{G}}_\rho^{\tau F_0}\times(\underline{\mathbf{L}}_\rho^{\tau F_0})^{\mathrm{opp}})\Delta(\tilde{\underline{\mathbf{L}}}_\rho^{\tau F_0}))$-modules. According to Proposition 3.3 and Lemma 3.4, the projection $\mathrm{pr}: \underline{\mathbf{G}}_\rho \to \mathbf{G}$ onto the first coordinate induces an isomorphism

$$H_c^{\dim}(\mathbf{Y}^{\underline{\mathbf{G}}_\rho,\tau_0F_0}_{\underline{\mathbf{U}}_\rho},\Lambda)e_{\underline{s}}^{\underline{\mathbf{L}}_\rho^{\tau_0F_0}} \cong H_c^{\dim}(\mathbf{Y}^{\mathbf{G},F}_{\mathbf{U}},\Lambda)e_s^{\mathbf{L}^F}$$

of $\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}\Delta(\tilde{\mathbf{L}}^F)$-modules.

Note that since $\mathbf{U}$ is ρ-stable, the automorphism τ_0 stabilises the unipotent radical $\underline{\mathbf{U}}'_\rho$. Therefore, we can consider $H_c^{\dim}(\mathbf{Y}^{\underline{\mathbf{G}}_\rho,\tau_0F_0}_{\underline{\mathbf{U}}'_\rho})e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau_0F_0}}$ a $\Lambda((\underline{\mathbf{G}}_\rho^{\tau_0F_0}\times(\underline{\mathbf{L}}_\rho^{\tau_0F_0})^{\mathrm{opp}})\Delta(\tilde{\underline{\mathbf{L}}}_\rho^{\tau_0F_0}\langle\tau_0\rangle))$-module. By transport of structure, we can endow $H_c^{\dim}(\mathbf{Y}^{\mathbf{G},F}_{\mathbf{U}},\Lambda)e_s^{\mathbf{L}^F}$ with a $\mathbf{G}^F\times(\mathbf{L}^F)^{\mathrm{opp}}\Delta\tilde{\mathbf{L}}^F\langle F_0\rangle$-structure.

Assume now that $\mathbf{N}^F/\mathbf{L}^F$ is centralised by F_0. Let $n\in\mathbf{N}^F$ be a generator of the quotient group $\mathbf{N}^F/\mathbf{L}^F$. Since $F_0(n)n^{-1}\in\mathbf{L}^F$, we conclude that conjugation by $\underline{n}$ defines an automorphism of $\underline{\mathbf{G}}_\rho\langle\tau_0\rangle$, which stabilises the Levi subgroup $\underline{\mathbf{L}}_\rho\langle\tau_0\rangle$. Thus, conjugation by $\underline{n}$ yields an isomorphism

$$(H_c^{\dim}(\mathbf{Y}^{\underline{\mathbf{G}},\tau_0F_0}_{\underline{\mathbf{U}}'_\rho})e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau_0F_0}})^{\underline{n}} \cong H_c^{\dim}(\mathbf{Y}^{\underline{\mathbf{G}},\tau_0F_0}_{\underline{n}\underline{\mathbf{U}}'_\rho})e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau_0F_0}}$$

of $\Lambda((\underline{\mathbf{G}}^{\tau_0F_0}\times(\underline{\mathbf{L}}^{\tau F_0})^{\mathrm{opp}})\Delta(\tilde{\underline{\mathbf{L}}}^F\langle\tau_0\rangle))$-modules. We conclude that $H_c^{\dim}(\mathbf{Y}^{\underline{\mathbf{G}}_\rho,\tau_0F_0}_{\underline{\mathbf{U}}'})e_{\underline{s}}^{\underline{\mathbf{L}}^{\tau_0F_0}}$ is $\underline{n}$-stable. By transport of structure and [35, Lemma 10.2.13], we deduce that $H_c^{\dim}(\mathbf{Y}^{\mathbf{G},F}_{\mathbf{U}},\Lambda)e_s^{L}$ extends to a $\mathbf{G}^F\times(\mathbf{N}^F)^{\mathrm{opp}}\Delta(\tilde{\mathbf{L}}^F\langle F_0\rangle)$-module. □

Lemma 3.6. *In the situation of Lemma 3.5, let Q be an ℓ-subgroup of $\mathbf{L}^F$. If $\mathbf{N}^F/\mathbf{L}^F$ is centralised by $\mathcal{A}$, then $H_c^{\dim}(\mathbf{Y}^{\mathrm{N}_{\mathbf{G}}(Q)}_{\mathrm{C}_{\mathbf{U}}(Q)},\Lambda)\mathrm{br}_Q(e_s^{\mathbf{L}^F})$ extends to a $\mathrm{N}_{\mathbf{G}^F}(Q)\times\mathrm{N}_{\mathbf{L}^F}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\mathcal{N}}(Q)$-module, where $\mathcal{N}:=\mathrm{N}_{\tilde{\mathbf{G}}^F\langle F_0\rangle}(\mathbf{L},e_s^{\mathbf{L}^F})$.*

Proof. The proof is the same as in Lemma 2.2 with the necessary modifications made in Lemma 3.5. □

4. An equivariant Bonnafé–Dat–Rouquier equivalence

In this section, we give a partial answer to the question of whether the Morita equivalence constructed by Bonnafé–Dat–Rouquier is automorphism-equivariant. We often use the following well-known fact.

Lemma 4.1. *Let $\mathbf{G}$ be a connected reductive group and $\phi:\mathbf{G}\to\mathbf{G}$ a Frobenius endomorphism. The norm map $N_{\phi^r/\phi}:\mathbf{G}\to\mathbf{G}$, $x\mapsto\prod_{i=0}^{r-1}\phi^i(x)$, is surjective.*

Proof. By Lang's theorem, we can write $y\in\mathbf{G}$ as $y=a^{-1}\phi^r(a)$. Then for $x:=a^{-1}\phi(a)$, we have $N_{\phi^r/\phi}(x)=y$. □

Denote by $\mathrm{Bij}(\mathbf{G})$ the set of bijective maps on $\mathbf{G}$. In what follows, we will often use that for $x\in\mathbf{G}$, we have $(x\phi)^r=N_{\phi^r/\phi}(x)\phi^r$ in $\mathbf{G}\rtimes\mathrm{Bij}(\mathbf{G})$.

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Theorem 4.2. *Let* $\mathbf{G}$ *be a simple, simply connected algebraic group of type* B_n, C_n *or* E_7 *such that either* $n > 2$ *or* q *is odd. Then there exists a Frobenius endomorphism* $F_0 : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$, *which commutes with* F *such that* F_0 *stabilises* $\mathbf{L}$ *and the image of* $\tilde{\mathbf{G}}^F \rtimes \langle F_0 \rangle$ *in the outer automorphism group of* $\mathbf{G}^F$ *is* $\mathrm{Out}(\mathbf{G}^F)_{e_s^{\mathbf{G}^F}}$. *There exists a Morita equivalence between* $\Lambda \mathbf{N}^F e_s^{\mathbf{L}^F}$ *and* $\Lambda \mathbf{G}^F e_s^{\mathbf{G}^F}$, *which lifts to a Morita equivalence between* $\Lambda \mathrm{N}_{\tilde{\mathbf{G}}^F \langle F_0 \rangle_{\ell'}}(\mathbf{L}, e_s^{\mathbf{L}^F}) e_s^{\mathbf{L}^F}$ *and* $\Lambda \tilde{\mathbf{G}}^F \langle F_0 \rangle_{\ell'} e_s^{\mathbf{G}^F}$.

Proof. As in the proof of [37, Corollary 4.2], we see that there exists a field endomorphism $\phi : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ such that $\phi^r = F$ for some integer r and such that ϕ together with $\tilde{\mathbf{G}}^F$ generates $\mathrm{Out}(\mathbf{G}^F)_{e_s^{\mathbf{G}^F}}$. Since $e_s^{\mathbf{G}^F}$ is ϕ-stable, it follows that the Levi subgroup $\phi(\mathbf{L})$ is $\mathbf{G}^F$-conjugate to $\mathbf{L}$. Thus, there exists some $y \in \mathbf{G}^F$ such that $\mathbf{L}$ is $y\phi$-stable. Since $e_s^{\mathbf{G}^F}$ is ϕ-stable, there exists some $x \in \mathrm{N}_{\mathbf{G}^F}(\mathbf{L})$ such that $e_s^{\mathbf{L}^F}$ is $xy\varphi$-stable. Denoting $z := xy$, we have $(z\phi)^r = N_{F/\phi}(z)F$, and therefore $N_{F/\phi}(z) \in \mathrm{N}_{\mathbf{G}^F}(\mathbf{L}, e_s^{\mathbf{L}^F})$.

Suppose first that $z_0 := N_{F/\phi}(z) \in \mathbf{L}^F$. By construction, the Levi subgroup $\mathbf{L}$ is stable under the Frobenius endomorphism $z\phi$. Therefore, by Lemma 4.1, there exists $l \in \mathbf{L}$ such that $z_0^{-1} = N_{z_0 F/z\phi}(l)$. We define $F_0 := lz\phi$. Then we have

$$F_0^r = (lz\phi)^r = (l(z\phi))^r = N_{z_0 F/z\phi}(l) z_0 F = F.$$

Since $F_0 = lz\phi$ and ϕ commute with F, we have $F(lz)(lz)^{-1} = F(l)l^{-1} = 1$: that is, $l \in \mathbf{L}^F$.

Thus, ϕ and F_0 induce the same outer automorphism in $\mathrm{Out}(\mathbf{G}^F)$. From this, we conclude that F_0 together with $\tilde{\mathbf{G}}^F$ generates the stabiliser of $e_s^{\mathbf{G}^F}$ in $\mathrm{Out}(\mathbf{G}^F)$. Note that $(\mathbf{N}/\mathbf{L})^{F_0} = \mathbf{N}/\mathbf{L}$ since by the remarks following [36, Lemma 3], the group $\mathbf{N}/\mathbf{L}$ is isomorphic to a subgroup of $\mathrm{Z}(\mathbf{G})$ that has at most order 2. We can now apply Lemma 2.1 and conclude that there exists a $\Lambda[\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}} \Delta(\langle F_0 \rangle)]$-module M extending $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}, \Lambda) e_s^{\mathbf{L}^F}$, which induces a Morita equivalence between $\Lambda \mathbf{N}^F e_s^{\mathbf{L}^F}$ and $\Lambda \mathbf{G}^F e_s^{\mathbf{G}^F}$. In particular, the bimodule $\tilde{M} := \mathrm{Ind}_{\mathbf{G}^F \times (\mathbf{N}^F)^{\mathrm{opp}} \Delta \tilde{\mathbf{N}}^F \langle F_0 \rangle}^{\tilde{\mathbf{G}}^F \langle F_0 \rangle \times (\tilde{\mathbf{N}}^F \langle F_0 \rangle)^{\mathrm{opp}}}(M)$ induces a Morita equivalence between $\Lambda \tilde{\mathbf{N}}^F \langle F_0 \rangle e_s^{\mathbf{L}^F}$ and $\Lambda \tilde{\mathbf{G}}^F \langle F_0 \rangle e_s^{\mathbf{G}^F}$.

Assume now that $N_{F/\phi}(z) \notin \mathbf{L}^F$. Since $\mathbf{N}/\mathbf{L}$ is isomorphic to a subgroup of $\mathrm{Z}(\mathbf{G})$, we conclude that $N_{F/\phi}(z)$ generates the quotient group $\mathbf{N}/\mathbf{L}$. Thus, in this case the bijective morphism $F_0 := \phi$ has the property that the element $zF_0 \in G\langle F_0 \rangle$ generates the quotient group $\mathrm{N}_{G\langle F_0 \rangle}(\mathbf{L}, e_s^{\mathbf{L}^F})/\mathbf{L}^F$, which is in particular cyclic. We conclude by [35, Lemma 10.2.13] that in this case, the $\Lambda(\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}} \Delta(\tilde{\mathbf{L}}^F))$-module $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}}, \Lambda) e_s^{\mathbf{L}^F}$ extends to a $\mathbf{G}^F \times (\mathbf{L}^F)^{\mathrm{opp}} \Delta(\mathrm{N}_{\tilde{\mathbf{G}}^F \langle F_0 \rangle_{\ell'}}(\mathbf{L}, e_s^L))$-module. The claim is now a consequence of [28, Theorem 3.4] (see also [37, Theorem 1.7]). □

5. Quasi-isolated elements for groups of type *A*

5.1. *Groups of type A*

From now until Section 12, we assume that $\mathbf{G} = \mathrm{SL}_{n+1}(\overline{\mathbb{F}_p})$ is of type A_n. We let $\tilde{\mathbf{G}} = \mathrm{SL}_{n+1}(\overline{\mathbb{F}_p})$ and $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ the natural inclusion. For an integer $q = p^f$, we let $F_q : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ be the Frobenius endomorphism that raises every matrix entry to its qth power. For $\varepsilon \in \{\pm 1\}$, we define $F = F_q(\gamma')^{\frac{1-\varepsilon}{2}}$. Here, $\gamma' : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$, $g \mapsto g^{-\mathrm{tr}}$, denotes the graph automorphism given by transpose inversion. We let $\mathbf{T}_0$ be the torus of diagonal matrices and $\mathbf{B}_0$ the Borel subgroup of upper triangular matrices. Let Φ denote the root system of $\mathbf{G}$ with base Δ with respect to $(\mathbf{T}_0, \mathbf{B}_0)$. For a subset $I \subset \Delta$, we denote by $\mathbf{L}_I$ the standard Levi subgroup associated to $I \subset \Delta$.

We denote by $W := \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ the Weyl group of $\mathbf{G}$ and identify W with the symmetric group S_{n+1}. Observe that our Frobenius endomorphism F acts trivially on the Weyl group W.

Note that with our choices of F, the torus $\mathbf{T}_0$ is not maximally split in the twisted case. This is because the graph automorphism γ' doesn't stabilise the Borel subgroup $\mathbf{B}_0$. We also define $\gamma := \mathrm{ad}(n_0)\gamma'$, where $n_0 \in \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)$ is the signed permutation matrix with entry $(-1)^{l+1}$ at position $(l, n+1-l)$ with $1 \le l \le n$ and 0 elsewhere. In particular, γ stabilises the pair $(\mathbf{T}_0, \mathbf{B}_0)$. Observe that the image of n_0 in $w_0 \in W$ is the longest element of W.

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5.2. Dual for groups of type A

To efficiently compute with dual groups, we will use some simplifications specific to the situation in type A. Since $\mathbf{G}^*$ is adjoint of type A, there exists a surjective morphism $\pi : \mathbf{G} \to \mathbf{G}^*$. For any closed subgroup $\mathbf{H}$ of $\mathbf{G}$, we denote $\mathbf{H}^* := \pi(\mathbf{H})$. Suppose that $f : \mathbf{G} \to \mathbf{G}$ is a combination of a graph automorphism (either γ' or γ) and a field endomorphism F_p as above. Then we denote by $f^* : \mathbf{G}^* \to \mathbf{G}^*$ the unique morphism satisfying $\pi \circ f = f^* \circ \pi$. Observe that the group $(\tilde{\mathbf{G}}, F)$ is self-dual. Therefore, by identifying its dual with $\tilde{\mathbf{G}}$, we obtain a duality isomorphism $Y(\tilde{\mathbf{T}}_0) \to X(\tilde{\mathbf{T}}_0)$ that descends via $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ and $\pi : \tilde{\mathbf{G}} \to \mathbf{G}^*$ to an isomorphism $Y(\mathbf{T}_0) \to X(\mathbf{T}_0^*)$, which puts the groups $(\mathbf{G}, \mathbf{T}_0, F)$ and $(\mathbf{G}, \mathbf{T}_0^*, F^*)$ in duality with each other. We observe that $W^* := \mathrm{N}_{\mathbf{G}^*}(\mathbf{T}_0^*)/\mathbf{T}_0^*$ is the Weyl group of $\mathbf{G}^*$, and we have a natural anti-isomorphism $^* : W \to W^*$, $w \mapsto w^* = \pi(w)^{-1}$, of Weyl groups induced by duality.

The notion of duality between $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F^*)$ can be further extended to any F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$. For this, we say that $W_I w$ is the type of the Levi subgroup $\mathbf{L}$ if there exists some $g \in \mathbf{G}$ with image $w := g^{-1}F(g)\mathbf{T}_0 \in W$ in the Weyl group such that $\mathbf{L} = {}^g\mathbf{L}_I$. Note that the type of a Levi subgroup is usually defined with respect to a maximally split torus of $(\mathbf{G}, F)$ (see, for example, [11, Section 8.2]), but to use similar arguments in the twisted case, it makes more sense to define the type with respect to the torus $\mathbf{T}_0$. The Levi subgroup $\mathbf{L}^* = {}^{\pi(g)}\mathbf{L}_I^*$ is then of type $W_I^*\pi(w)$. Note that $\mathrm{N}_W(W_I^*)/W_I^*$ is isomorphic to a product of symmetric groups, and hence the coset $W_I^* w$ is $\mathrm{N}_W(W_I^*)/W_I^*$-conjugate to $F^*(w^*)W_I^* = \pi(w)^{-1}W_I^*$. By conjugation with g (resp. $\pi(g)$), the duality isomorphism $Y(\mathbf{T}_0) \to X(\mathbf{T}_0^*)$ therefore yields a duality isomorphism $Y(\mathbf{T}) \to X(\mathbf{T}^*)$, where $\mathbf{T} := {}^g\mathbf{T}_0$, between $(\mathbf{L}, F)$ and $(\mathbf{L}^*, F^*)$.

5.3. Strictly quasi-isolated elements

We recall the notion of strictly quasi-isolated semisimple elements introduced in [38, Definition 3.1]. This notion will become important when dealing with actual blocks of groups of Lie type.

Definition 5.1. We say that a semisimple element $s \in (\mathbf{G}^*)^{F^*}$ is *strictly quasi-isolated in* $(\mathbf{G}^*, F^*)$ if $\mathrm{C}_{\mathbf{G}^*}(s)^{F^*}\mathrm{C}^\circ_{\mathbf{G}^*}(s)$ is not contained in a proper Levi subgroup of $\mathbf{G}^*$.

We denote by $A(s)$ the component group $\mathrm{C}_{\mathbf{G}^*}(s)/\mathrm{C}^\circ_{\mathbf{G}^*}(s)$. The proof of the following lemma is similar to the proof of [2, Corollary 2.9].

Lemma 5.2. *Recall that* $\mathbf{G}$ *is simple, simply connected of type A. If* $s \in (\mathbf{G}^*)^{F^*}$ *is a semisimple element that is strictly quasi-isolated, then we have* $A(s)^{F^*} = A(s)$.

Proof. Let $\iota^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ be the map dual to the map $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$, and let $\tilde{s} \in (\tilde{\mathbf{G}}^*)^{F^*}$ such that $\iota^*(\tilde{s}) = s$. Consider the injective morphism $\omega_s : A(s) \to \mathrm{Z}(\tilde{\mathbf{G}}^*)$ as in [2, Corollary 2.8]. Let e denote the exponent of the subgroup $A(s)^{F^*}$ of $A(s)$. Let $g \in \mathrm{C}^\circ_{\mathbf{G}^*}(s)\mathrm{C}_{(\mathbf{G}^*)^{F^*}}(s)$ so that $g^e \in \mathrm{C}^\circ_{\mathbf{G}^*}(s)$. Then $1 = \omega_s(g)^e = \omega_{s^e}(g)$, and therefore $g \in \mathrm{Ker}(\omega_s^e) = \iota^*(\mathrm{C}_{\tilde{\mathbf{G}}^*}(\tilde{s}^e))$. Consequently, $\iota^*(\mathrm{C}_{\tilde{\mathbf{G}}^*}(\tilde{s}^e)) = \mathrm{C}^\circ_{\mathbf{G}^*}(s^e)$ is a Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}^\circ_{\mathbf{G}^*}(s)\mathrm{C}_{(\mathbf{G}^*)^{F^*}}(s)$. This is a proper Levi subgroup of $\mathbf{G}^*$ unless $s^e = 1$. By the classification of quasi-isolated elements of $\mathbf{G}^* = \mathrm{PGL}_{n+1}(\overline{\mathbb{F}_p})$ in [2, Proposition 5.2], we have $\mathrm{o}(s) = |A(s)|$, which implies that $e = |A(s)|$. Since e is the exponent of $A(s)^{F^*}$, we must necessarily have $A(s) = A(s)^{F^*}$. □

Remark 5.3. We consider $\mathbf{G}$ with the Frobenius F as defined above, and we let $s \in (\mathbf{G}^*)^{F^*}$ be a strictly quasi-isolated element. The aim of this remark is to compute the F^*-type of the Levi subgroup $\mathbf{L}^* := \mathrm{C}^\circ_{\mathbf{G}^*}(s)$. We fix a primitive $n+1$th root of unity $\zeta_{n+1} \in \overline{\mathbb{F}_p}^\times$; and for an integer m dividing $n+1$, we let $\zeta_m := \zeta_{n+1}^e$, where $me = n+1$, and define

$$t_m := (1, \zeta_m, \dots, \zeta_m^{m-1}) \otimes I_e.$$

According to [2, Proposition 5.2], there exists $g \in \mathbf{G}^*$ and m dividing $n+1$ such that $t_m = {}^g s \in \mathbf{T}_0^*$. As the integer m will be fixed throughout we often abbreviate $t := t_m$ and $\zeta := \zeta_m$.

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Since s is F^*-stable, we have $F^*(t) = {}^{F^*(g)}s = {}^{F^*(g)g^{-1}}t$. Hence, t is w^*F^*-stable, where w^* is the image of $gF^*(g)^{-1}$ in W^*. Since s is assumed to be strictly quasi-isolated, Lemma 5.2 yields $A(s)^{F^*} = A(s)$. This implies that $A(t)^{w^*F^*} = A(t)$. By [2, Proposition 5.2], the component group $A(t)$ has order m. On the other hand, $A(s) = A(s)^{F^*}$ is isomorphic to a subgroup of $\mathrm{Z}(\mathbf{G}^F)$ that has order $(n+1, q-\varepsilon)$. Therefore m divides $(n+1, q-\varepsilon)$. In particular, $\zeta^{q-\varepsilon} = \zeta$, and we deduce that the element t is F^*-stable. From this, it follows that $w^* \in W(t) := \{v \in W^* \mid {}^v t = t\}$. Let $v_m \in W^*$ be the permutation defined by

$$v_m(i) :\equiv i + e \bmod (n+1) \text{ for } i = 1, \dots, n+1$$

such that $A(t)$ is generated by v_m and has order m.

Denote by $W^\circ(t)$ the Weyl group of $\mathbf{L}_m^* := \mathrm{C}^\circ_{\mathbf{G}^*}(t)$ relative to the maximal torus $\mathbf{T}_0^*$ such that we have $W(t) = W^\circ(t) \rtimes A(t)$; see [2, Proposition 1.3(c)]. To determine the type of $\mathbf{L}^*$, we can change w by an element of $W^\circ(t)$, and we therefore conclude that $\mathbf{L}^*$ is of F^*-type $W^\circ(t)w$, where $w \in \langle v_m \rangle$.

Corollary 5.4. *Let s be a strictly quasi-isolated element. Then the Levi subgroup $\mathbf{L}^* = \mathrm{C}^\circ_{\mathbf{G}^*}(s)$ is d-split for some integer d dividing $2|A(s)|$.*

Proof. We keep the notations of Remark 5.3. Assume first that $\varepsilon = 1$. Let $w \in \langle v_m \rangle$ as in Remark 5.3 such that $\mathbf{L}^*$ is of type $W^\circ(t)w$. Then w has order d with d dividing $m = |A(s)|$, and we can decompose $w = w_1 \cdots w_r$ into disjoint cycles, each w_i of order d with $d \mid m$ and $n+1 = rd$. Note that $\mathbf{L}^*$ is d-split if and only if $\tilde{\mathbf{L}}^*$ is d-split; see [11, Proposition 13.2].

Furthermore, $\tilde{\mathbf{L}}^F \cong \tilde{\mathbf{L}}_m^{wF} \cong \mathrm{GL}_r(\mathbb{F}_{q^d}) \times \dots \mathrm{GL}_r(\mathbb{F}_{q^d})$. Hence $\tilde{\mathbf{L}}$ and therefore also $\mathbf{L}$ is d-split; see [11, Example 13.4]. By Ennola-duality, we obtain that if $\varepsilon = -1$, then $\mathbf{L}$ is d'-split where

$$d' = \begin{cases} 2d & \text{if } d \text{ is odd,} \\ d/2 & \text{if } d \equiv 2 \bmod 4, \\ d & \text{if } d \equiv 0 \bmod 4. \end{cases}$$

This also shows the result in the twisted case. □

6. Block stabiliser of quasi-isolated blocks

Until Section 12, s denotes as in Remark 5.3 a fixed strictly quasi-isolated ℓ'-element in $(\mathbf{G}^*)^{F^*}$. As before, we denote by $\mathbf{L}$ the Levi subgroup of $\mathbf{G}$ dual to $\mathbf{L}^* = \mathrm{C}^\circ_{\mathbf{G}^*}(s)$ defined as its preimage under the projection map $\pi : \mathbf{G} \to \mathbf{G}^*$.

Assume that we are given a different Levi subgroup $\mathbf{L}'$ of $\mathbf{G}$ containing $\mathbf{L}$ such that $\mathrm{C}_{\mathbf{G}^*}(s)(\mathbf{L}')^* = (\mathbf{L}')^*\mathrm{C}_{\mathbf{G}^*}(s)$. Denote by N' the common stabiliser of $\mathbf{L}'$ and $e_s^{L'}$ in $\mathbf{G}^F$. By the results of Bonnafé–Dat–Rouquier (see Theorem 1.1), there exists a Morita equivalence between $\Lambda N' e_s^{L'}$ and $\Lambda G e_s^G$. We first make the following observation:

Lemma 6.1. *With the notation as above, N'/L' is naturally isomorphic to a quotient of N/L.*

Proof. We have $\mathbf{N}^*/\mathbf{L}^* = \mathrm{C}_{\mathbf{G}^*}(s)/\mathrm{C}^\circ_{\mathbf{G}^*}(s)$ and, on the other hand,

$$(\mathbf{N}')^*/(\mathbf{L}')^* = (\mathbf{L}')^*\mathrm{C}_{\mathbf{G}^*}(s)/(\mathbf{L}')^* \cong \mathrm{C}_{\mathbf{G}^*}(s)/\mathrm{C}_{\mathbf{L}'^*}(s).$$

Since $\mathrm{C}^\circ_{\mathbf{G}^*}(s) \subset \mathrm{C}_{\mathbf{L}'^*}(s)$, we obtain a surjection $\mathbf{N}^*/\mathbf{L}^* \to (\mathbf{N}')^*/(\mathbf{L}')^*$. The statement follows by taking F^*-fixed points and using the duality between $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F^*)$. □

Recall from Remark 5.3 that s is $\mathbf{G}^*$-conjugate to the element t_m, where m is some positive integer dividing $n+1$. Therefore, the geometric conjugacy class of a (strictly) quasi-isolated element depends only on the parameter m. We now use this uniformity of description to our advantage.

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Lemma 6.2. *There exists a proper F-stable Levi subgroup* $\mathbf{L}'$ *of* $\mathbf{G}$ *containing* $\mathbf{L}$ *such that* N'/L' *is cyclic of prime order.*

Proof. Recall that for a given m with $m \mid (n+1)$, we denote $t_m := (1, \zeta, \dots, \zeta^{m-1}) \otimes I_e$, where $\zeta \in \mathbb{F}_q^\times$ is a fixed primitive mth-root of unity with $me = n+1$. Consider the Levi subgroup $\mathbf{L}_m^* := \mathrm{C}^\circ_{\mathbf{G}^*}(t_m)$ of $\mathbf{G}^*$, and let $\mathbf{L}_m := \pi^{-1}(\mathbf{L}_m^*)$ be the standard Levi subgroup of $\mathbf{G}$ in duality with $\mathbf{L}_m^*$.

Assume that s is $\mathbf{G}^*$-conjugate to t_m: that is, there exists $g \in \mathbf{G}^*$ such that $t = {}^g s$. We let m' be a prime divisor of m so that $m = m'e'$ for some integer e'. Our definition of t_m and $t_{m'}$ in Remark 5.3 implies that $t_{m'} = t_m^{e'}$, so in particular $\mathrm{C}_{\mathbf{G}^*}(t_m) \subset \mathrm{C}_{\mathbf{G}^*}(t_{m'})$.

Denote by $W^\circ(t_m)$ the Weyl group of $\mathrm{C}^\circ_{\mathbf{G}^*}(t_m)$ relative to the maximal torus $\mathbf{T}_0^*$. By [2, Proposition 5.2], we obtain that $W^\circ(t_m) \cong (S_e)^m$, where v_m transitively permutes the m copies of S_e. Since $W^\circ(t_{m'}) \cong (S_{e'})^{m'}$ and $m' \mid m$, it follows that v_m normalises $W^\circ(t_{m'})$. We define $\mathbf{L}'^* := {}^{g^{-1}}\mathbf{L}^*_{m'}$. Recall from Remark 5.3 that the image of $gF^*(g)^{-1}$ in W^* is in $W^\circ(t_m)w$ with $w \in \langle v_m \rangle$. Since w normalises $W^\circ(t_{m'})$ and F^* acts trivially on W^*, we have ${}^{wF}W^\circ(t_{m'}) = W^\circ(t_{m'})$. It follows that $\mathbf{L}'^*$ is F^*-stable of type $W^\circ(t_{m'})w$. Furthermore, $A(t_{m'})^{wF^*} \cong A(t_{m'}) \cong C_{m'}$, so we observe that N'/L' is cyclic of m'-order. □

The upcoming sections will provide the necessary knowledge on strictly quasi-isolated elements in type A.

6.1. Some computations in the Weyl group

Recall that $\gamma : \mathbf{G} \to \mathbf{G}$ denotes the graph automorphism stabilising $(\mathbf{T}_0, \mathbf{B}_0)$. In this section, we collect some properties of the Levi subgroup $\mathbf{L}_m$, which was defined in the proof of Lemma 6.2.

Lemma 6.3. *The Levi subgroup* $\mathbf{L}_m$ *is a* γ*-stable Levi subgroup and contained in a* γ*-stable parabolic subgroup* $\mathbf{P}_m$ *of* $\mathbf{G}$*. In particular,* $\mathbf{L}_m^\gamma$ *is a Levi subgroup in the connected reductive group* $\mathbf{G}^\gamma$ *with parabolic subgroup* $\mathbf{P}_m^\gamma$*.*

Proof. Observe that γ is a quasi-central morphism in the sense of [16, Definition-Theorem 1.15]. According to [16, Remark 1.30], the group $\mathbf{G}^\gamma$ is connected. The Levi subgroup $\mathbf{L}_m$ is a standard Levi subgroup of $\mathbf{G}$ relative to the γ-stable pair $(\mathbf{T}_0, \mathbf{B}_0)$ whose associated set of simple roots is γ-stable. Thus, $\mathbf{L}_m$ is a γ-stable Levi subgroup contained in a γ-stable parabolic subgroup $\mathbf{P}_m$ of $\mathbf{G}$. From [16, Proposition 1.11], it therefore follows that $\mathbf{L}_m^\gamma$ is a Levi subgroup in the connected reductive group $\mathbf{G}^\gamma$ with parabolic subgroup $\mathbf{P}_m^\gamma$. □

The subgroup V constructed in the proof of the following lemma is sometimes also referred to as the extended Weyl group; see also [40, Section 2.3].

Lemma 6.4. *There exist a subgroup* $V \subseteq \mathrm{N}_{\mathbf{G}^{F_p}}(\mathbf{T}_0)$ *and an injective map* $r : W \to V$ *such that* $r(W^\gamma) \subset V^\gamma$*.*

Proof. Define $V := \langle n_{(i,i+1)} \mid i = 1, \dots, n \rangle$, where $n_{(i,i+1)}$ is the matrix obtained by taking the permutation matrix corresponding to $(i, i+1)$ and multiplying its ith row with -1. We have a group epimorphism $V \to W$, $n_{(i,i+1)} \mapsto (i, i+1)$. By construction, we have $\gamma(n_{(i,i+1)}) = n_{\gamma((i,i+1))}$ and every $n_{(i,i+1)}$ is F_p-stable. We can write $\sigma \in W$ as a reduced expression $\sigma = s_1 \dots s_r$ with $s_i \in \{(1,2), \dots, (n-1, n)\}$. We define a map $r : W \to V$, $\sigma \mapsto n_{s_1} \dots n_{s_r}$. By Matsumoto's Lemma, the map r is well-defined and injective; see also the proof of [40, Lemma 2.23]. Since the map r is bijective onto its image and γ-equivariant, it follows that σ is γ-stable if and only if $r(\sigma)$ is γ-stable. □

Lemma 6.5. *Every* γ*-stable element of* $\mathrm{N}_{\mathbf{G}}(\mathbf{L}_m)/\mathbf{L}_m$ *has a* $\langle F_p, \gamma \rangle$*-stable preimage in* $\mathrm{N}_{\mathbf{G}}(\mathbf{L}_m)$*.*

Proof. Let $W_{\mathbf{L}_m}$ be the Weyl group of $\mathbf{L}_m$ with respect to the maximal torus $\mathbf{T}_0$. By duality, $W_{\mathbf{L}_m}$ is (anti-)isomorphic to the Weyl group $W^\circ(t_m)$ of $\mathbf{L}_m^*$. Since $W^\circ(t_m) \cong (S_e)^m$, by [2, Proposition 5.2], a computation in $W \cong S_{n+1}$ shows that $\mathrm{N}_W(W_{\mathbf{L}_m}) \cong S_e \wr S_m$. Using [27, Corollary 12.11], we thus obtain

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$\mathrm{N}_{\mathbf{G}}(\mathbf{L}_m)/\mathbf{L}_m \cong \mathrm{N}_W(W_{\mathbf{L}_m})/W_{\mathbf{L}_m} \cong S_m$. The last isomorphism is given by mapping $\sigma \in \mathrm{N}_W(W_{\mathbf{L}_m})$ to the associated permutation $\pi \in S_m$ on the set of the m simple components of the Levi subgroup $\mathbf{L}_m$. For any such permutation $\pi \in S_m$, we define an element $\sigma_\pi \in W \cong S_{n+1}$ by defining

$$\sigma_\pi(j) := \pi(j - km) + km \text{ for } j \in \{km + 1, \dots, (k + 1)m\}.$$

By direct computation we obtain

$$\sigma_{\gamma\pi}(j) = (m + 1) - \pi((m + 1) - (j - km)) + km$$

and

$${}^{\gamma}\sigma_\pi(j) = n + 2 - \pi((n + 2 - j) - (e - k - 1)m) - m(e - k - 1).$$

Comparing these two expressions, we obtain $\sigma_{\gamma\pi} = {}^{\gamma}\sigma_\pi$. In particular, σ_π is γ-invariant whenever π is. The claim now follows from Lemma 6.4. □

Denote by $\mathbf{N}_m/\mathbf{L}_m$ the subgroup of $\mathrm{N}_{\mathbf{G}}(\mathbf{L}_m)/\mathbf{L}_m$ generated by v_m.

Lemma 6.6. *Let $n \in \mathrm{N}_{\mathbf{G}}(\mathbf{L}_m)$ such that $\gamma(n)n^{-1} \in \mathbf{N}_m$. Then there exists some $y \in \mathbf{N}_m$ such that yn is $\langle F_p, \gamma\rangle$-stable.*

Proof. We abbreviate $v := v_m$.According to Lemma 6.5, it is enough to show that if $w \in \mathrm{N}_W(W_{\mathbf{L}_m})$ with $\gamma(w)w^{-1} \in \langle v\rangle W_{\mathbf{L}_m}$, then there exists some $y \in \langle v\rangle$ such that $\gamma(yw)(yw)^{-1} \in W_{\mathbf{L}_m}$.

Denote by $\bar{\ } : \mathrm{N}_W(W_{L_m}) \to \mathrm{N}_W(W_{L_m})/W_{L_m}$ the projection map. In what follows, we identify $\mathrm{N}_W(W_{L_m})/W_{L_m}$ with S_m as in the proof of Lemma 6.5. Note that the element $\overline{v}$ corresponds to an m-cycle of S_m. Assume first that $\overline{v}$ has odd order. Then the map $\langle\overline{v}\rangle \to \langle\overline{v}\rangle, \overline{v} \mapsto \gamma(\overline{v})\overline{v}^{-1} = \overline{v}^{-2}$, is surjective. Thus, there exists $\overline{x} \in \langle\overline{v}\rangle$ such that $\gamma(\overline{w})\overline{w}^{-1} = \gamma(\overline{x}^{-1})\overline{x}$. In other words, $\overline{xw}$ is γ-stable.

Assume now that $\overline{v}$ has even order. We denote by sgn the sign map on the symmetric group $\mathrm{N}_W(W_{L_m})/W_{L_m} \cong S_m$ and by A_m its kernel. Since $\overline{v}$ has even order, we have $\mathrm{sgn}(\overline{v}) = -1$. On the other hand, $\mathrm{sgn}(\gamma(\overline{w})\overline{w}^{-1}) = 1$, and thus $\gamma(\overline{w})\overline{w}^{-1} \in \langle\overline{v}^{-2}\rangle = \langle\overline{v}\rangle \cap A_m$. However, $\gamma(\overline{v})\overline{v}^{-1} = \overline{v}^2$. Thus, there exists $\overline{x} \in \langle\overline{v}\rangle$ such that $\gamma(\overline{w})\overline{w}^{-1} = \gamma(\overline{x}^{-1})\overline{x}$. In other words, $\overline{xw}$ is γ-stable. □

6.2. Stabilisers of blocks

Suppose that $f : \mathbf{G} \to \mathbf{G}$ is a combination of a graph automorphism γ' or a field automorphism F_p as above. Recall that we defined $f^* : \mathbf{G}^* \to \mathbf{G}^*$ as the unique morphism satisfying $f^* \circ \pi = \pi \circ f$, where $\pi : \mathbf{G} \to \mathbf{G}^*$ is the natural surjective map. We denote $\mathcal{B}' = \langle\gamma', F_p\rangle \subset \mathrm{Aut}(\mathbf{G}^F)$ and $\mathcal{B}'^* = \langle\gamma'^*, F_p^*\rangle \subset \mathrm{Aut}((\tilde{\mathbf{G}}^*)^{F^*})$.

The following observation will be used throughout this section:

Lemma 6.7. *Duality induces a natural isomorphism between the quotient groups $\mathrm{N}_{G\mathcal{B}'}(\mathbf{L}, e_s^L)/L$ and $\mathrm{C}_{(\mathbf{G}^*)^{F^*}\mathcal{B}'^*}(s)/(\mathbf{L}^*)^{F^*}$.*

Proof. Recall that $s = {}^g t \in (\mathbf{G}^*)^{F^*}$ is a semisimple element with connected centraliser $\mathbf{L}^* = \mathrm{C}^\circ_{\mathbf{G}^*}(s)$. We assume that the element $g \in \mathbf{G}$ has the property that $\mathbf{T} = {}^g\mathbf{T}_0$ is a maximally split torus of $\mathbf{L} = {}^g\mathbf{L}_I$. Denote $w := g^{-1}F(g)\mathbf{T}_0 \in W$.

Let $\phi \in \mathrm{N}_{\mathbf{G}^F\mathcal{B}}(\mathbf{L}, e_s^{\mathbf{L}^F})$. Since $\phi(\mathbf{T})$ is a second maximally split torus of $\mathbf{L}$, we can by possibly multiplying ϕ with an element of $\mathbf{L}^F$ assume that ϕ stabilises $\mathbf{T}$. Denote $\phi_0 := {}^{g^{-1}}\phi$, and observe that $\phi_0 = n_v f$ for some $f \in \mathcal{B}'$ and $n_v \in \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)$ with image $v = n_v\mathbf{T}_0$ in W. Since $[{}^g\phi, {}^gF] = {}^g[F, \phi] = 1$ and f, F centralise W, we conclude that $v \in \mathrm{C}_W(w)$. We observe that the morphism ϕ_0 of $\mathbf{L}_I$ is dual to the morphism $\phi_0^* := f^* n_{v^*}$ of $\mathbf{L}_I^*$. Let $\phi^* := {}^{\pi(g)}\phi_0^*$. Since $v^* \in \mathrm{C}_W(w^*)$, we find that $[F^*, \phi^*] \in \mathbf{T}^*$. Hence, by applying Lang's theorem, we find $t' \in \mathbf{T}^*$ such that $t'\phi^*$ and F^* commute. Observe that such an element t' is unique up to multiplication with $(\mathbf{T}^*)^{F^*}$. Therefore, replacing ϕ^* by $t'\phi^*$, we can

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assume that ϕ^* and F^* commute. The bijective morphisms $\phi : \mathbf{L} \to \mathbf{L}$ and $\phi^* : \mathbf{L}^* \to \mathbf{L}^*$ are therefore in duality. Since $\phi(e_s^{\mathbf{L}^F}) = e_s^{\mathbf{L}^F}$, it follows that ϕ^* stabilises the $(\mathbf{L}^*)^{F^*}$-conjugacy class of s: that is, $\phi^*(s) = s$. Hence, $\phi^* \in \mathrm{C}_{(\mathbf{G}^*)^{F^*}\mathcal{B}'^*}(s)$. This defines a map between the two quotient groups by sending the $\mathbf{L}^F$-coset of ϕ to the $(\mathbf{L}^*)^{F^*}$-coset of ϕ^*.

Assume conversely that $\phi^* \in \mathrm{C}_{(\mathbf{G}^*)^{F^*}\mathcal{B}'^*}(s)(\mathbf{L}^*)^{F^*}$, and we assume again that $\phi^*(\mathbf{T}^*) = \mathbf{T}^*$. We set $\phi_0^* = {}^{\pi(g)^{-1}}\phi^*$ and write $\phi_0^* = f^* n_v^*$ such that $n_v^* \mathbf{T}_0 = v^*$. Since $[\phi^*, F^*] = 1$, we obtain again $v \in \mathrm{C}_W(w)$. Thus, denoting $\phi := {}^g(n_v f)$, we find again some $t' \in \mathbf{T}$ such that $t'\phi$ commutes with F. □

The most striking property of strictly quasi-isolated blocks (concerning the action of group automorphisms) is the following:

Lemma 6.8. *Assume that $e_s^{\mathbf{G}^F}$ is γ-stable. Then there exist some $y \in \mathbf{G}^F$ and a parabolic subgroup $\mathbf{P}$ with Levi complement $\mathbf{L}$ such that $y\gamma$ stabilises $(\mathbf{L}, \mathbf{P})$ and the idempotent $e_s^{\mathbf{L}^F}$.*

Proof. Let $\gamma^* : \mathbf{G}^* \to \mathbf{G}^*$ be the dual of the graph automorphism γ. There exists some $g \in \mathbf{G}^*$ such that ${}^g s = t$. Consider the element

$$\tilde{t} = \tilde{t}_m := (1, \zeta, \dots, \zeta^{m-1}) \otimes I_e \in \tilde{\mathbf{G}}^* = \mathrm{GL}_{n+1}(\overline{\mathbb{F}_p}),$$

a preimage of t under the natural map $\iota^* : \mathrm{GL}_{n+1}(\overline{\mathbb{F}_p}) \to \mathrm{PGL}_{n+1}(\overline{\mathbb{F}_p})$. This element satisfies $\iota^*(\tilde{t}) = t$, and we observe that $\gamma^*(\tilde{t}) = tz$, where $z := \zeta I_{n+1} \in \mathrm{Z}(\tilde{\mathbf{G}})$. So, t is γ^*-stable, and we conclude that ${}^{g^{-1}\gamma^*(g)\gamma^*}s = {}^{g^{-1}\gamma^* g}s = s$. Consider the element $x := g^{-1}\gamma^*(g)$. By assumption, there exists some $y \in (\mathbf{G}^*)^{F^*}$ such that s is $y\gamma^*$-stable. Therefore $xy^{-1} \in \mathrm{C}_{\mathbf{G}^*}(s)$. Since $A(s)^{F^*} = A(s)$ by Lemma 5.2, we deduce that

$$F^*(xy^{-1})(xy^{-1})^{-1} = F^*(x)x^{-1} \in \mathrm{C}^\circ_{\mathbf{G}^*}(s) = \mathbf{L}^*.$$

Thus, by Lang's theorem, there exists some $l \in \mathrm{C}^\circ_{\mathbf{G}^*}(s)$ such that lx is F^*-stable. We have $(g^{-1}\gamma^* g)^2 = 1$, which implies that $(lx\gamma^*)^2 \in \mathrm{C}^\circ_{\mathbf{G}^*}(s)$. Recall that $\mathbf{L}^* = {}^{g^{-1}}\mathbf{L}_m^*$ and that $W^\circ(t_m)$ denotes the Weyl group of $\mathrm{C}^\circ_{\mathbf{G}^*}(t_m)$ relative to the maximal torus $\mathbf{T}_0^*$. Let $\mathbf{P}_m^*$ be the standard parabolic subgroup associated with the parabolic subsystem $W^\circ(t_m)$ of W^* with Levi complement $\mathbf{L}_m^*$. We define $\mathbf{P}^* := {}^{g^{-1}}\mathbf{P}_m^*$, and we observe that the pair $(\mathbf{L}_m^*, \mathbf{P}_m^*)$ is γ^*-stable. Consequently, $x\gamma^*$ stabilises the pair $(\mathbf{L}^*, \mathbf{P}^*)$; and since $l \in \mathbf{L}^*$, we observe that $lx\gamma^*$ also stabilises this pair. The statement follows now by using duality; see Lemma 6.7. □

We observe that the conclusion of Lemma 6.8 remains valid for the Levi subgroup $\mathbf{L}'$ constructed in Lemma 6.2.

Corollary 6.9. *With the assumption of Lemma 6.8, there exists a parabolic subgroup $\mathbf{P}'$ whose Levi complement is the Levi subgroup $\mathbf{L}'$ from Lemma 6.2 such that $y\gamma$ stabilises $(\mathbf{L}', \mathbf{P}')$ and the idempotent $e_s^{\mathbf{L}'}$.*

Proof. By construction, we have $((\mathbf{L}')^*, (\mathbf{P}')^*) = {}^{g^{-1}}(\mathbf{L}_{m'}, \mathbf{P}_{m'})$, where g is as in the proof of Lemma 6.8. The graph automorphism γ stabilises $((\mathbf{L}_{m'})^*, (\mathbf{P}_{m'})^*)$, and we have $\mathbf{L}^* \subseteq (\mathbf{L}')^*$. It follows that the element $lg^{-1}\gamma^*(g)\gamma^*$, where $l \in (\mathbf{L}^*)^{F^*}$ is as in the proof of Lemma 6.8, stabilises $((\mathbf{L}')^*, (\mathbf{P}')^*)$ and the $((\mathbf{L}')^*)^{F^*}$-conjugacy class of s. This yields the claim. □

6.3. Untwisted groups of type A

In the following section, we assume that $(\mathbf{G}, F)$ is untwisted of type A. Consider the subgroup $\mathcal{B}$ of $\mathrm{Aut}(\tilde{\mathbf{G}}^F)$ generated by the field automorphism F_p and the graph automorphism γ. Denote by $\mathcal{B}_{e_s^{\mathbf{G}^F}}$ the stabiliser of $e_s^{\mathbf{G}^F}$ in $\mathcal{B}$. We have $\mathcal{B}_{e_s^G} = \langle \gamma_0, \phi \rangle$, where $\phi : \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism (a

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power of F_p or $F_p\gamma$) that satisfies $\phi^r = F$ for some r and $\gamma_0 \in \{\mathrm{id}_{\mathbf{G}}, \gamma\}$ is a (possibly trivial) graph automorphism. As in the proof of Lemma 6.8, we use some explicit properties of the element s to say something about their structure.

Lemma 6.10. *The quotient group* $\mathrm{N}_{G\langle\phi\rangle}(\mathbf{L}', e_s^{L'})/L'$ *is cyclic unless* $\mathbf{L}'$ *is* 1*-split.*

Proof. We have $t = {}^g s$ for some $g \in \mathbf{G}^*$. By the construction in Lemma 6.2, the quotient group $\mathrm{C}_{\mathbf{G}^*}(s)(\mathbf{L}')^*/(\mathbf{L}')^*$ is cyclic of prime order. In particular, $n' := g^{-1}F^*(g) \in \mathrm{C}_{\mathbf{G}^*}(s)$, so either n' generates $\mathrm{C}_{\mathbf{G}^*}(s)(\mathbf{L}')^*/(\mathbf{L}')^*$ or $n' \in (\mathbf{L}')^*$.

Recall that ϕ is a power of F_p or $F_p\gamma$ and γ fixes t; see the proof of Lemma 6.8. Consequently, ϕ acts as a permutation on the different eigenvalues of $\tilde{t} \in (\tilde{\mathbf{G}}^*)^{F^*}$. Therefore, there exists some $z \in W^*$ of order dividing r such that ${}^{z\phi}t = t$. We let $z_0 \in \mathrm{N}_{\mathbf{G}^*}(\mathbf{T}_0^*)$ be the permutation matrix corresponding to z.

We deduce that s is $x\phi^*$-stable, where $x := g^{-1}z_0\phi^*(g)$. By assumption, there exists some $y \in (\mathbf{G}^*)^{F^*}$ such that s is $y\phi^*$-stable. Therefore $xy^{-1} \in \mathrm{C}_{\mathbf{G}^*}(s)$. Since $A(s)^{F^*} = A(s)$ by Lemma 5.2, we deduce that $F^*(xy^{-1})(xy^{-1})^{-1} = F^*(x)x^{-1} \in \mathrm{C}^\circ_{\mathbf{G}^*}(s)$. Thus, by Lang's theorem, there exists some $l \in \mathrm{C}^\circ_{\mathbf{G}^*}(s)$ such that lx is F^*-stable. We have

$$(x\phi^*)^r = (g^{-1}z_0\phi^*g)^r = g^{-1}(z_0\phi^*)^r g = g^{-1}F^*(g)F^* = n'F^*.$$

If n' generates $(\mathbf{N}')^*/(\mathbf{L}')^* = \mathrm{C}_{(\mathbf{G}^*)^{F^*}}(s)(\mathbf{L}')^*/(\mathbf{L}')^*$, then we can conclude by duality that $\mathrm{N}_{G\langle\phi\rangle}(\mathbf{L}', e_s^{L'})/L'$ is cyclic. On the other hand, if $g^{-1}F^*(g) \in (\mathbf{L}')^*$, then there exists $l' \in (\mathbf{L}')^*$ such that gl' is F^*-stable. We conclude that $(\mathbf{L}')^* = {}^{gl}(\mathbf{L}_{m'})^*$ is $(\mathbf{G}^*)^{F^*}$-conjugate to $(\mathbf{L}_{m'})^*$ and therefore maximally split. □

Lemma 6.11. *If* $\mathbf{L}'$ *is not* 1*-split, then the quotient group* $\mathrm{N}_{G\mathcal{B}}(\mathbf{L}', e_s^{L'})/L'$ *is abelian.*

Proof. We keep the notation of Lemma 6.10. As in Lemma 6.10, there exists some permutation $z_0 \in \mathrm{N}_{\mathbf{G}^*}(\mathbf{T}_0^*)$ of order dividing r such that ${}^{z_0\phi^*}t = t$. The element $\gamma^*(z_0)z_0^{-1}$ stabilises t, so $\gamma^*(z_0)z_0^{-1} \in \mathbf{N}_m^*$. By Lemma 6.6, we first see that there exists some $y \in \mathbf{N}_m^*$ such that yz_0 is $\langle\phi^*, \gamma^*\rangle$-stable. Observe that we still have ${}^{yz_0\phi^*}t = t$, and therefore $N_{(\phi^*)^r/\phi^*}(yz_0) = (yz_0)^r \in \mathbf{N}_m^* \subset \mathbf{N}_{m'}^*$. Furthermore, since $t_{m'}$ is a power of t_m, the element $t_{m'}$ is $yz_0\phi^*$-stable as well so that $yz_0 \in \mathrm{N}_{\mathbf{G}^*}(\mathbf{L}_{m'}^*)$. Note that $v_{m'}$ corresponds to an m'-cycle under the isomorphism $\mathrm{N}_{\mathbf{G}^*}(\mathbf{L}_{m'}^*)/\mathbf{L}_{m'}^* \cong S_{m'}$. Since m' is a prime number, it follows that if some power of $\sigma \in S_{m'}$ is an m'-cycle, then σ itself must be an m'-cycle. Since $(yz_0)^r \in \mathbf{N}_{m'}^*$ this argument shows that we must have either $yz_0 \in \mathbf{N}_{m'}^*$ or $(yz_0)^r \in \mathbf{L}_{m'}^*$. In the first case, we deduce that $z_0 \in \mathbf{N}_{m'}^* = \mathbf{L}_{m'}^*\mathbf{N}_m^*$. In particular, there exists some $z_0' \in \mathbf{L}_{m'}$ such that ${}^{z_0'\phi^*}t = t$. In the second case, we define $z_0' := yz_0$. Consider $x_\phi := g^{-1}z_0'\phi^*(g)$ and $x_{\gamma_0} := g^{-1}\gamma_0^*(g)$ so that $x_\phi\phi^*$ and $x_{\gamma_0}\gamma_0^*$ stabilise the semisimple element s. By the construction of z_0', we have $N_{(\phi^*)^r/\phi^*}(z_0') \in \mathbf{L}_{m'}^*$. Therefore, as in the proof of Lemma 6.10, we deduce that $(x_\phi\phi^*)^r = n'F^*$ for some element n' generating the quotient group $\mathbf{N}'^*/\mathbf{L}'^*$. Since $\gamma_0^*(z_0')z_0'^{-1} \in \mathbf{L}_{m'}^*$, it follows moreover that

$$[x_\phi\phi^*, x_{\gamma_0}\gamma_0^*] = {}^g[z_0'\phi^*, \gamma_0^*] \in {}^g\mathbf{L}_{m'} = \mathbf{L}'.$$

As in the proof of Lemma 6.10, we find $l_\phi, l_\gamma \in \mathbf{L}^*$ such that $x'_\phi := l_\phi x_\phi$ and $x'_{\gamma_0} := l_{\gamma_0}x_{\gamma_0}$ are F^*-stable. We deduce that the morphisms dual to $x'_\phi\phi^*$ and $x'_{\gamma_0}\gamma_0^*$ generate the quotient group $\mathrm{N}_{G\mathcal{B}}(\mathbf{L}', e_s^{L'})/L'$. The statement therefore follows by duality. □

Lemma 6.12. *Assume that* $\mathbf{L}'$ *is* 1*-split and* $\mathrm{N}_{G\mathcal{B}}(\mathbf{L}', e_s^{L'})$ *is noncyclic. By possibly replacing* s *by a* $(\mathbf{G}^*)^{F^*}$*-conjugate, the following hold:*

a) *The pair* $(\mathbf{L}', \mathbf{P}')$ *and the block* $e_s^{L'}$ *are* γ_0*-stable.*
b) *There exists a Frobenius endomorphism* F_0 *of* $\tilde{\mathbf{G}}$ *with* $F_0^r = F$ *such that* F_0 *commutes with* γ_0 *and stabilises both* $\mathbf{L}'$ *and* $e_s^{L'}$*. Moreover,* $\tilde{\mathbf{G}}^F\langle F_0, \gamma_0\rangle$ *is the stabiliser of* $e_s^{\mathbf{G}^F}$ *in* $\mathrm{Out}(\mathbf{G}^F)$.

Proof. As in Lemma 6.10, let $g \in \mathbf{G}^*$ such that ${}^g s = t$. Our assumption implies that $g^{-1}F^*(g) \in (\mathbf{L}')^*$. In particular, there exists some $l' \in (\mathbf{L}')^*$ such that s and ${}^{l'}t$ are $(\mathbf{G}^*)^{F^*}$-conjugate. Therefore, we can

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replace s by ${}^{l'}t$. By doing so, we can assume that there exists some $g \in (\mathbf{L}_{m'})^*$ such that ${}^g s = t$. In particular, we have $\mathbf{L}' = \mathbf{L}_{m'}$.

Corollary 6.9 implies the existence of $m_{\gamma_0} \in G$ such that $m_{\gamma_0}\gamma_0$ stabilises $(\mathbf{L}', \mathbf{P}')$ and $e_s^{L'}$. Since $(\mathbf{L}', \mathbf{P}')$ is γ_0-stable, it follows that $m_{\gamma_0} \in \mathrm{N}_G(\mathbf{L}', \mathbf{P}')$. Since parabolic subgroups are self-normalising (see, for example, [27, Exercise 20.3]), we obtain $\mathrm{N}_{\mathbf{G}}(\mathbf{L}', \mathbf{P}') = \mathbf{L}'\mathrm{N}_{\mathbf{U}'}(\mathbf{L}')$. By the proof of [15, Corollary 1.18], we obtain $\mathrm{N}_{\mathbf{U}'}(\mathbf{L}') = 1$, so $m_{\gamma_0} \in \mathrm{N}_G(\mathbf{L}', \mathbf{P}') = L'$. This yields the first claim.

Now, since $\mathbf{L}'$ and $e_s^{\mathbf{G}^F}$ are ϕ-stable, there exists some $z \in \mathrm{N}_G(\mathbf{L}')$ such that $z\phi$ stabilises $e_s^{L'}$. If $m' = 2$, then $\mathrm{N}_G(\mathbf{L}') = \mathrm{N}_G(\mathbf{L}', e_s^{L'})$, and thus $e_s^{L'}$ is ϕ-stable so that we can define $F_0 := \phi$. Assume now that m' is odd. We define $z_0 := N_{F/\phi}(z) \in \mathrm{N}_{\mathbf{G}^F}(\mathbf{L}', e_s^{L'})$ and claim that we can assume $z_0 \in (L')^{\gamma_0}$. If $e_s^{L'}$ is not γ-stable, then we necessarily must have $z_0 \in L'$ since otherwise $\mathrm{N}_{G\mathcal{B}}(\mathbf{L}', e_s^{L'})$ is generated by $z\phi$ and thus cyclic. On the other hand, if $e_s^{L'}$ is γ-stable, then we have $\gamma(z)z^{-1} \in N'$. By Lemma 6.6, there exists $y \in \mathbf{N}'$ such that $yz \in N'$ is γ-stable. Replacing z by yz, we may assume that z is γ-stable. Since m' is odd, however, we have $(N_{m'}/L_{m'})^\gamma = 1$, so $z_0 \in (L')^\gamma$.

By Lemma 6.3, the group $(\mathbf{L}')^{\gamma_0}$ is connected reductive. Consequently, by Lemma 4.1, there exists $l \in (\mathbf{L}')^{\gamma_0}$ such that $z_0^{-1} = N_{z_0F/z\phi}(l)$. From this, we deduce that $F_0 := \mathrm{ad}(lz)\phi$ commutes with γ_0 and satisfies $F_0^r = F$. □

The previous lemma, as well as Theorem 4.2, suggest that we should distinguish the cases whether $\mathbf{L}'$ is 1-split or not.

6.4. Twisted groups

We will now assume that $(\mathbf{G}, F)$ is twisted of type A_n. We consider now the subgroup $\mathcal{B}$ of $\mathrm{Aut}(\tilde{\mathbf{G}}^F)$ generated by the field automorphism F_p and the graph automorphism γ'. Since $F = F_q\gamma'$, we observe that $\mathcal{B}$ is generated by F_p. As the presence of a nontrivial graph automorphism causes some additional difficulties, we will restrict ourselves to describing only a Sylow m'-subgroup of the quotient $\mathrm{N}_{G\mathcal{B}}(\mathbf{L}', e_s^{L'})/L'$. For this, let ϕ be a generator of the Sylow m'-subgroup of $\mathcal{B}_{e_s^{\mathbf{G}^F}}$ such that $\phi : \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism (a power of $F_p\gamma_0$) that satisfies $\phi^r = F$ if $m' \neq 2$ and $\phi^r = F\gamma'$ if $m' = 2$. Furthermore, we let $P_{m'}$ be the unique Sylow m'-subgroup of the quotient group $\mathrm{N}_{G\langle\phi\rangle}(\mathbf{L}', e_s^{L'})/L'$.

In the twisted case, we also refine our choice of m'. For this, we choose m' to be an odd prime whenever possible: that is, whenever m is not a power of 2; see the proof of Lemma 6.2.

Lemma 6.13. *The Sylow m'-subgroup $P_{m'}$ of $\mathrm{N}_{G\langle\phi\rangle}(\mathbf{L}', e_s^{L'})/L'$ is cyclic except possibly when $\mathbf{L}'$ is 2-split or $m' = 2$.*

Proof. Suppose that m' is odd. Hence, ϕ is a power of $F_p\gamma'$ and $\phi^r = F$ for some integer r. These properties allow us to use a similar proof as in Lemma 6.10.

We have $t = {}^g s$ for some $g \in \mathbf{G}^*$. In particular, $n' := g^{-1}F^*(g) \in \mathrm{C}_{\mathbf{G}^*}(s)$, so either n' generates $\mathrm{C}_{(\mathbf{G}^*)^{F^*}}(s)(\mathbf{L}')^*/(\mathbf{L}')^*$ or $n' \in (\mathbf{L}')^*$. As in the proof of Lemma 6.10, we find $z \in W^*$ of order dividing r such that ${}^{z\phi^*}t = t$. We deduce that s is $x_\phi\phi$-stable, where $x_\phi := g^{-1}z\phi(g)$. By assumption, there exists some $y \in (\mathbf{G}^*)^F$ such that s is $y\phi^*$-stable. Therefore, $x_\phi y^{-1} \in \mathrm{C}_{\mathbf{G}^*}(s)$. Since $A(s)^{F^*} = A(s)$ by Lemma 5.2, we deduce that $F^*(x_\phi y^{-1})(x_\phi y^{-1})^{-1} = F^*(x_\phi)x_\phi^{-1} \in \mathrm{C}^\circ_{\mathbf{G}^*}(s)$. Thus, there exists some $l_\phi \in \mathrm{C}^\circ_{\mathbf{G}^*}(s)$ such that $x'_\phi := l_\phi x_\phi$ is F^*-stable.

We conclude that if $g^{-1}F^*(g) \notin (\mathbf{L}')^*$, then the quotient group is generated by $x'_\phi\phi^*$ and is thus cyclic. On the other hand, if $g^{-1}F^*(g) \in (\mathbf{L}')^*$, then there exists $l' \in (\mathbf{L}')^*$ such that gl' is F^*-stable. We conclude that $(\mathbf{L}')^* = {}^{(gl')^{-1}}(\mathbf{L}'_m)^*$ is 2-split. □

Lemma 6.14. *Assume that the Sylow m'-subgroup $P_{m'}$ is not cyclic. Then one of the following holds:*

a) *There exists $z \in \mathbf{G}^F$ such that $F_0 := \mathrm{ad}(z)\phi$ satisfies $F_0^r = F\rho$ for some automorphism ρ of $\mathbf{G}$ such that F_0 stabilises $\mathbf{L}'$ and $e_s^{L'}$ and ρ stabilises $(\mathbf{L}', \mathbf{P}')$.*
b) *There exists $m_\phi, m_{\gamma'_0} \in \mathbf{G}^F$ such that $P_{m'} = \langle m_\phi\phi, m_{\gamma'_0}\gamma'_0\rangle$ and $m_{\gamma'_0}\gamma'_0$ stabilises $(\mathbf{L}', \mathbf{P}')$.*

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Proof. Assume first that m' is odd. By Lemma 6.13, we can assume that $\mathbf{L}' = \mathbf{L}_{m'}$ and thus ϕ with $\phi^r = F$ stabilises $\mathbf{L}'$. Arguing as in the proof of Lemma 6.12, we find some $z \in \mathrm{N}_G(\mathbf{L}')$ such that $z\phi$ stabilises $e_s^{L'}$ and such that $z_0 := N_{F/\phi}(z) \in L'$. Consequently, by Lemma 4.1, there exists $l \in \mathbf{L}'$ such that $z_0^{-1} = N_{z_0F/z\phi}(l)$. From this, we deduce that $F_0 := \mathrm{ad}(lz_0)\phi$ satisfies $F_0^r = F$. Hence, we are in the situation of part (a).

Suppose now that $m' = 2$: that is, $m = 2^j$ for some integer j. By Remark 5.3, we have $m \mid (q+1, n)$, so in particular, $q \equiv -1 \bmod m$. For $j \geq 2$, observe that -1 is not a square modulo $m = 2^j$ and consequently q is not a square in this case. Hence, the Sylow 2-subgroup of $\mathcal{B}$ is generated by γ'. By Lemma 6.8, we therefore obtain $P_{m'} = N_{m'}\langle m_{\gamma_0'}\gamma_0'\rangle$ with $m_{\gamma_0'}\gamma_0'$ stabilising $(\mathbf{L}', \mathbf{P}')$. Thus, we are in the situation of part (b).

It remains to consider the case $m' = m = 2$. In particular, t is ϕ^*-stable and w_0 generates N_m/L_m. Assume first that $g^{-1}F^*(g) \in \mathbf{L}^* = \mathrm{C}^\circ_{\mathbf{G}^*}(s)$ so that s and t are $(\mathbf{G}^*)^F$-conjugate. We may thus safely replace s by t. We obtain $P_m = N_m\langle\phi\rangle$. Since $n_0 = n_0\gamma'\phi^r$ in $\mathbf{G}^F \rtimes \mathcal{B}$, we therefore obtain $P_m = \langle n_0\gamma', \phi\rangle$, and $n_0\gamma'$ stabilises $(\mathbf{L}_m, \mathbf{P}_m)$; see the proof of Lemma 6.8.

Assume finally that $g^{-1}F^*(g) \notin \mathbf{L}^*$. It follows that $\mathbf{L}$ is a 1-split Levi subgroup. For this case, it will be more convenient to work in the group $\mathbf{G}^{\tilde{F}}$, where $\tilde{F} := F_q\gamma = \mathrm{ad}(n_0)F$. Let $x \in \mathbf{G}$ be such that $n_0 = \tilde{F}(x)^{-1}x$ and $\tilde{\mathcal{B}} \subset \mathrm{Aut}(\mathbf{G}^{\tilde{F}})$ the subgroup generated by F_p. Since $n_0 = \tilde{F}(x)^{-1}x$ is F_p-stable, it follows that $xF_p(x)^{-1} \in \mathbf{G}^{\tilde{F}}$. We therefore obtain that the map

$$\mathbf{G}^F \rtimes \mathcal{B} \to \mathbf{G}^{\tilde{F}} \rtimes \tilde{\mathcal{B}}, \quad a \mapsto {}^x a,$$

is a well-defined isomorphism. Instead of considering the element $s \in (\mathbf{G}^*)^{F^*}$, we might therefore as well consider the element $t_m \in (\mathbf{G}^*)^{\tilde{F}^*}$. We observe that $F_p^*(t_m) = t_m$. In particular, $\mathbf{L}_m$ and $e_{t_m}^{\mathbf{G}^{\tilde{F}}}$ are ϕ-stable and $\phi^r = \tilde{F}\gamma$. Using the explicit isomorphism, it is now easy to conclude that situation (a) holds. □

7. Global equivalences

7.1. Extending modules

Let X be a normal subgroup of a finite group Y. If M is a ΛX-module, then in practice, it is often quite hard to decide whether M extends to a ΛY-module if $\ell \mid |Y : X|$ unless M is simple or projective. We will therefore often consider the following weaker notion:

Definition 7.1. Let X be a normal subgroup of a finite group Y such that the quotient group Y/X is solvable. If M is a ΛX-module, then we say that it *almost extends to* Y if M extends to a ΛH-module M', where H/X is a Hall ℓ'-subgroup of Y/X, such that the extension M' is Y-stable.

We will check almost extendability with the following remark.

Remark 7.2. In the situation of the previous definition, assume that M is Y-stable and $E := \mathrm{End}_{\Lambda X}(M)/J(\mathrm{End}_{\Lambda X}(M))$ is commutative. If the Hall ℓ'-subgroup H/X is normal in Y/X and M extends to H, then by [45, Corollary 2.6], the module M automatically almost extends to Y.

We will later use the following result about almost extendability.

Lemma 7.3. *Let X be a normal subgroup of a finite group Y such that Y/X is abelian. Assume that M is a ΛX-module such that $E := \mathrm{End}_{\Lambda X}(M)/J(\mathrm{End}_{\Lambda X}(M))$ is commutative and M almost extends to Y. If M' is an extension of M to a subgroup X' of Y such that $\ell \nmid |X' : X|$, then M' almost extends to Y as well.*

Proof. Let H/X be the Hall ℓ'-subgroup of Y/X. By assumption, H/X' is the Hall ℓ'-subgroup of Y/X'. By Remark 7.2, it is therefore enough to show that M' extends to H.

Assume that U is maximal among the subgroups of H containing X' with the property that M' extends to U. Denote by M'' an extension of M' to U. If M'' is stable under some element $x \in H \setminus U$, then by [35, Lemma 10.2.13], the module M'' extends to $\langle U, x\rangle$. Therefore, the stabiliser of M'' in H is U.

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Denote by $^-: Y \to Y/X$ the projection map. By the remark following [45, Theorem 2.5], the set of H-invariant extension of M to U is in bijection with $H^1(\overline{U}, E^\times)^{\overline{H}/\overline{U}}$. Since Y/X is abelian, the action of $\overline{H}/\overline{U}$ on this cohomology group is trivial. On the other hand, by [45, Proposition 2.4], the set of extensions of M to U is in bijection with $H^1(\overline{U}, E^\times)$. From this, it follows that every extension of M to U is H-stable. In particular, M'' is H-stable, and we conclude that $U = H$. □

7.2. Some auxiliary results

As before, we consider $\mathbf{G} = \mathrm{SL}_{n+1}(\overline{\mathbb{F}_p})$ and a fixed strictly quasi-isolated element $s \in (\mathbf{G}^*)^{F^*}$ of ℓ'-order. Our aim in this section is to show that the cohomology module $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}'}^{\mathbf{G}}, \Lambda)e_s^{L'}$ almost extends to a $\Lambda(G \times (L')^{\mathrm{opp}}\Delta \mathrm{N}_{\tilde{G}\mathcal{A}}(\mathbf{L}', e_s^{L'}))$-module, where $\mathcal{A}$ (defined below in Definition 7.6) is a subgroup of $\mathrm{Aut}(\tilde{\mathbf{G}}^F)$ whose image in the outer automorphism group is $\mathrm{Out}(\mathbf{G}^F)_{e_s^{\mathbf{G}^F}}$. Thanks to Lemma 6.8, many situations can be handled by the following lemma.

Lemma 7.4. *Assume that there exist $m_\phi, m_\gamma \in \mathbf{G}^F$ such that $\mathrm{N}_{\tilde{G}\mathcal{B}}(\mathbf{L}', e_s^{L'}) = \tilde{L}'\langle m_\phi\phi, m_\gamma\gamma\rangle$. If $m_\gamma\gamma$ stabilises $\mathbf{P}'$ and $\mathrm{N}_{\tilde{G}\mathcal{B}}(\mathbf{L}', e_s^{L'})/\tilde{L}'$ is abelian, then $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}'}^{\mathbf{G}}, \Lambda)e_s^{L'}$ almost extends to $G \times (L')^{\mathrm{opp}}\Delta(\mathrm{N}_{\tilde{G}\mathcal{B}}(\mathbf{L}', e_s^{L'}))$.*

Proof. Recall that for disconnected reductive groups, we define parabolic subgroups and Levi subgroups as in [37, Section 2.1]. Consider the disconnected reductive group $\hat{\mathbf{G}} := \tilde{\mathbf{G}} \rtimes \langle\gamma\rangle$. Since ϕ and F commute with γ, there exist unique extensions of ϕ and F to $\hat{\mathbf{G}}$ (denoted by the same letter) with $\phi(\gamma) = \gamma$ and $F(\gamma) = \gamma$, respectively. We observe that $\hat{\mathbf{P}}' = \mathrm{N}_{\hat{\mathbf{G}}}(\tilde{\mathbf{P}}')$ is a parabolic subgroup of $\hat{\mathbf{G}}$ with Levi subgroup $\hat{\mathbf{L}}' = \mathrm{N}_{\hat{\mathbf{G}}}(\tilde{\mathbf{L}}', \tilde{\mathbf{P}}) = \mathrm{N}_{\hat{\mathbf{P}}'}(\tilde{\mathbf{L}}')$. We abbreviate $\hat{L}' := (\hat{\mathbf{L}}')^F$. The Deligne–Lusztig variety $\mathbf{Y}_{\mathbf{U}'}^{\mathbf{G}}$ has a natural $G \times (L')^{\mathrm{opp}}\Delta(\hat{L}')$-action. Observe that $\hat{\mathbf{L}}'$ is $m_\phi\phi$-stable since $[m_\gamma\gamma, m_\phi\phi] \in \mathbf{L}'$. Consider the morphism $\phi' := \mathrm{ad}(m_\phi)\phi$ of $\hat{\mathbf{G}}$. It follows that $\phi' : \mathbf{Y}_{\mathbf{U}'}^{\mathbf{G}} \to \mathbf{Y}_{\phi'(\mathbf{U}')}^{\mathbf{G}}$ is a bijective morphism of $G \times (L')^{\mathrm{opp}}\Delta(\hat{L}')$-varieties. Since $m_\gamma\gamma$ stabilises $\mathbf{P}'$ by assumption, it follows that $m_\gamma\gamma \in \hat{\mathbf{L}}'$ and thus the quotient group $\hat{\mathbf{L}}'/\mathbf{L}'$ is generated by $m_\gamma\gamma$. We therefore obtain that $e_s^{L'}$ is $\hat{L}'$-stable. By [37, Lemma 3.1], we have $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}'}^{\mathbf{G}}, \Lambda)e_s^{L'} \cong H_c^{\dim}(\mathbf{Y}_{\phi'(\mathbf{U}')}^{\mathbf{G}}, \Lambda)e_s^{L'}$ as $G \times (L')^{\mathrm{opp}}\Delta\hat{L}'$-modules. From this, we conclude that the cohomology module $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}'}^{\mathbf{G}}, \Lambda)e_s^{L'}$ is $\Delta\langle m_\phi\phi\rangle$-stable. Hence it almost extends to $G \times (L')^{\mathrm{opp}}\Delta(\hat{L}'\langle m_\phi\phi\rangle)$. □

The following corollary is the local version of Lemma 7.4.

Corollary 7.5. *Let Q be an ℓ-subgroup of L'. Under the assumptions of Lemma 7.4, the module $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}'}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)\,\mathrm{br}_Q(e_s^{L'})$ almost extends to a $\Lambda(\mathrm{N}_G(Q) \times (\mathrm{N}_{L'}(Q))^{\mathrm{opp}}\Delta(\mathrm{N}_{\tilde{G}\mathcal{B}}(\mathbf{L}', Q, e_s^{L'}))$-module.*

Proof. We have an injective map $\mathrm{N}_{G\mathcal{B}}(\mathbf{L}', Q, e_s^{L'})/\mathrm{N}_{L'}(Q) \hookrightarrow \mathrm{N}_{G\mathcal{B}}(\mathbf{L}', e_s^{L'})/L'$. If the quotient group $\mathrm{N}_{\tilde{G}\mathcal{B}}(\mathbf{L}', Q, e_s^{L'})/\mathrm{N}_{L'}(Q)$ is cyclic, the result follows. We can therefore assume that there exist $l_\phi, l_\gamma \in \tilde{L}'$ such that $\mathrm{N}_{\tilde{G}\mathcal{B}}(\mathbf{L}', Q, e_s^{L'}) = \mathrm{N}_{\tilde{L}'}(Q)\langle l_\phi(m_\phi\phi)^i, l_\gamma m_\gamma\gamma\rangle$ for some i.

The Deligne–Lusztig variety $\mathbf{Y}_{\mathrm{C}_{\mathbf{U}'}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}$ has a natural $\mathrm{N}_G(Q) \times (\mathrm{N}_{L'}(Q))^{\mathrm{opp}}\Delta(\mathrm{N}_{\hat{\mathbf{L}}'^F}(Q))$-action. Consider the morphism ϕ' induced by the action of $l_\phi(m_\phi\phi)^i$ on $\hat{\mathbf{G}}$. It follows that $\phi' : \mathbf{Y}_{\mathrm{C}_{\phi'(\mathbf{U}')}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)} \to \mathbf{Y}_{\mathrm{C}_{\mathbf{U}'}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}$ is a bijective morphism of $\mathrm{N}_G(Q) \times (\mathrm{N}_{L'}(Q))^{\mathrm{opp}}\Delta(\mathrm{N}_{\hat{\mathbf{L}}'^F}(Q))$-varieties. Since $m_\gamma\gamma \in \hat{\mathbf{L}}'$, we deduce that the quotient group $\mathrm{N}_{\hat{\mathbf{L}}'}(Q)/\mathrm{N}_{\mathbf{L}'}(Q)$ is generated by $l_\gamma m_\gamma\gamma$. We have $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}'}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)\,\mathrm{br}_Q(e_s^{L'}) \cong H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\phi'(\mathbf{U}')}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)\,\mathrm{br}_Q(e_s^{L'})$ as $\mathrm{N}_G(Q) \times (\mathrm{N}_{L'}(Q))^{\mathrm{opp}}\Delta(\mathrm{N}_{\hat{\mathbf{L}}'^F}(Q))$-modules. From this, we conclude that the cohomology module $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}'}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)$ is $\Delta\langle\phi'\rangle$-stable as $\mathrm{N}_G(Q) \times (\mathrm{N}_{L'}(Q))^{\mathrm{opp}}\Delta(\mathrm{N}_{\hat{\mathbf{L}}'^F}(Q))$-module. It therefore almost extends to a $\Lambda(\mathrm{N}_G(Q) \times (\mathrm{N}_{L'}(Q))^{\mathrm{opp}}\Delta(\mathrm{N}_{\tilde{G}\mathcal{B}}(\mathbf{L}', Q, e_s^{L'}))$-module. □

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7.3. Global equivalences

In the following, we prove an analogue of Theorem 4.2 for strictly quasi-isolated blocks in groups of type A. To formulate this theorem, we need the following definition.

Definition 7.6. If we are in the situation of Lemma 6.12, then we define $\mathcal{A} := \langle \gamma_0, F_0 \rangle \subseteq \mathrm{Aut}(\tilde{\mathbf{G}}^F)$, where F_0 is as defined in the proof of Lemma 6.12. Otherwise, we define $\mathcal{A} := \mathcal{B}_{e_s^{\mathbf{G}^F}} \subseteq \mathrm{Aut}(\tilde{\mathbf{G}}^F)$. In both cases we abbreviate $\mathcal{N} := \mathrm{N}_{\tilde{\mathbf{G}}^F \mathcal{A}}(\mathbf{L}', e_s^{L'})$.

Firstly, observe that $\mathcal{A}$ is always abelian by Lemma 6.12. In addition, the Hall ℓ'-subgroup of $\mathcal{N}/\tilde{L}'$ is always a normal subgroup. This follows from the fact that N'/L' is of ℓ'-order (see [2, Corollary 2.9]) and $\mathrm{N}_{G\mathcal{A}}(\mathbf{L}', e_s^{L'})/N' \cong \mathcal{A}$ is abelian. We therefore denote by $\mathcal{N}_{\ell'}/\tilde{L}'$ the unique Hall ℓ'-subgroup of $\mathcal{N}/\tilde{L}'$.

Theorem 7.7. *The $\Lambda(G \times (L')^{\mathrm{opp}} \Delta(\tilde{L}'))$-module $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}'}^{\mathbf{G}}, \Lambda) e_s^{L'}$ almost extends to a $\Lambda(G \times (L')^{\mathrm{opp}} \Delta \mathrm{N}_{\tilde{G}\mathcal{A}}(\mathbf{L}', e_s^{L'}))$-module, where $\mathcal{A} \subseteq \mathrm{Aut}(\tilde{\mathbf{G}}^F)$ is the group of automorphisms defined in Definition 7.6.*

Proof. We first focus on the more involved case when $(\mathbf{G}, F)$ is untwisted. Moreover, let us first suppose that $\mathbf{L}'$ is not 1-split. Thanks to the proof of Lemma 6.11 and Lemma 6.13, we know that the quotient group $\mathcal{N}/\tilde{L}'$ is generated by $m_{\gamma_0}\gamma_0, m_\phi \phi$, which satisfy $[m_{\gamma_0}\gamma_0, m_\phi \phi] \in L'$. We can therefore use Lemma 7.4 to conclude that $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}'}^{\mathbf{G}}, \Lambda) e_s^{L'}$ almost extends.

Assume therefore now that $\mathbf{L}'$ is 1-split. We let $X := G \times (L')^{\mathrm{opp}} \Delta \tilde{L}'$ and $M := H_c^{\dim}(\mathbf{Y}_{\mathbf{U}'}^{\mathbf{G}}, \Lambda) e_s^{L'}$. According to [4, Theorem 7.5], the module $\mathrm{Ind}_X^{\tilde{\mathbf{G}}^F \times (\tilde{\mathbf{L}}'^F)^{\mathrm{opp}}}(M)$ is multiplicity free, so M is multiplicity free as well. Furthermore, the ΛX-module M is $\Delta(\mathrm{N}_{G\mathcal{A}}(\mathbf{L}', e_s^{L'}))$-stable. Therefore, [45, Proposition 1.13] and Remark 7.2 are applicable, and we conclude that it is enough to show that for every prime b with $b \neq \ell$, the module M extends to $X\Delta(P_b)$, where $P_b/\tilde{L}'$ is a Sylow b-group of $\mathrm{N}_{\tilde{G}\mathcal{A}}(\mathbf{L}', e_s^{L'})/\tilde{L}'$. If such a Sylow b-subgroup is cyclic, then the claim follows from [35, Lemma 10.2.13], and hence we can concentrate on all b such that the Sylow b-subgroup is noncyclic. Therefore, we can assume that $b \in \{2, m'\}$.

Let us consider the case where $b = m'$ and $m' \neq 2$. In this case, Lemma 6.12 shows that $P_{m'} = N'\langle F_0' \rangle$, where $F_0' = F_0^i$ for some $i \mid r$. Since $N'/L' \cong C_{m'}$ and F_0' has m'-power order, we obtain that F_0' centralises N'/L'. By Lemma 2.1, we conclude that M' extends to $X\Delta(P_{m'})$.

It remains to consider the case $b = 2$. Assume first that $m' \neq 2$. By conjugating P_2 with an element of $\mathcal{N}$, we can assume that $P_2 = L'\langle F_0^i, \gamma_0 \rangle$. We can therefore use Lemma 7.4 to conclude that M' extends to $X\Delta(P_2)$. Finally, assume that $m' = 2$. This implies that the quotient group $N'/L' \cong C_2$ gets centralised by $\mathcal{A}$. We obtain $P_2 = N'\langle \gamma_0, F_0' \rangle$, where $F_0' = F_0^i$ for some i dividing r. In particular, there exists k such that $(F_0')^k = F$. We can therefore use Lemma 2.1 to conclude that M extends to a $X\Delta(P_2)$-module.

Now, in the case where $(\mathbf{G}, F)$ is twisted, we see that the Sylow 2-subgroup of $\mathcal{A}$ is cyclic. Hence, with the arguments from above, it suffices to see that the module M extends to an $X\Delta(P_{m'})$-module. However, the structure of the Sylow m'-subgroup $P_{m'}$ was already considered in Lemma 6.14. Using the same methods as in the untwisted case also shows the result in this case. □

We let $\hat{M}$ be an extension of $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}'}^{\mathbf{G}}, \Lambda) e_s^{L'}$ to $\Lambda(G \times (L')^{\mathrm{opp}} \Delta \mathcal{N}_{\ell'})$ that is $\Delta\mathcal{N}$-stable. Furthermore, we denote by M' the restriction of $\hat{M}$ to $G \times (N')^{\mathrm{opp}} \Delta \tilde{N}'$.

Corollary 7.8. *Suppose the assumptions and notation as above. Then the bimodule M' induces a Morita equivalence between $\Lambda N' e_s^{L'}$ and $\Lambda G e_s^G$, which lifts to a Morita equivalence between $\Lambda \mathcal{N}_{\ell'} e_s^{L'}$ and $\Lambda \tilde{G}^F \mathcal{A}_{\ell'} e_s^G$ given by $\mathrm{Ind}_{G \times (L')^{\mathrm{opp}} \Delta \mathcal{N}_{\ell'}}^{\tilde{G}\mathcal{A}_{\ell'} \times (\mathcal{N}_{\ell'})^{\mathrm{opp}}}(\hat{M})$.*

Proof. The first statement is a consequence of (the proof of) [4, Theorem 7.5]. The second statement then follows from [28, Theorem 3.4]. □

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8. Local equivalences

Let s be our fixed strictly quasi-isolated element $s \in (\mathbf{G}^*)^{F^*}$ of ℓ'-order, where $\mathbf{G}$ is simple, simply connected of type A. In the previous section, we have shown that the bimodule $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}'}^{\mathbf{G}}, \Lambda)e_s^{L'}$ almost extends to a $\Lambda(G \times (L')^{\mathrm{opp}}\Delta \mathrm{N}_{\tilde{G}\mathcal{A}}(\mathbf{L}', e_s^{L'}))$-module. Our aim is now to prove a corresponding result for local subgroups.

For the proof of Lemma 8.2, we need the following simple group-theoretic fact.

Lemma 8.1. *Let X be a subgroup of a finite group Y and P a Sylow b-subgroup of Y. If $[Y, Y] \subseteq X$, then $P \cap X$ is a Sylow b-subgroup of X.*

Proof. The hypothesis implies that $[P, X] \subseteq X$. From this, we deduce that $PX = XP$, so PX is a subgroup of Y containing P. It follows that $|P|_b = |PX|_b = \frac{|P|_b|X|_b}{|P\cap X|_b}$, so $|P \cap X|_b = |X|_b$. □

Lemma 8.2. *Let Q be an ℓ-subgroup of L'. If $(\mathbf{G}, F)$ is untwisted, suppose that either $\mathrm{N}_{\tilde{N}'}(Q)\tilde{L}' = \tilde{N}'$ or $\mathcal{N}/\tilde{L}'$ is abelian. Then the bimodule $H_c^{\dim}(\mathbf{Y}_{\mathbf{C}_{\mathbf{U}'}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)\,\mathrm{br}_Q(e_s^{L'})$ almost extends to an $\mathrm{N}_G(Q) \times \mathrm{N}_{N'}(Q)^{\mathrm{opp}}\Delta\,\mathrm{N}_{\mathcal{N}}(Q)$-module.*

Proof. Let $X_Q := \mathrm{N}_G(Q) \times \mathrm{N}_{L'}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\tilde{L}'}(Q)$, and consider the $\Lambda[X_Q]$-module $M_Q := H_c^{\dim}(\mathbf{Y}_{\mathbf{C}_{\mathbf{U}'}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)\,\mathrm{br}_Q(e_s^{L'})$. By Lemma 1.2, the module M_Q is multiplicity free. Let $\tilde{P}_b$ be (the preimage) a Sylow b-subgroup of $\mathrm{N}_{\mathcal{N}}(Q)/\mathrm{N}_{\tilde{L}'}(Q)$. We conclude by Lemma 7.2 and [45, Proposition 1.13] that it is enough to show that for every prime b with $\tilde{P}_b$ noncyclic, the module M_Q almost extends to $X_Q\Delta(\tilde{P}_b)$. Arguing as in the proof of Theorem 7.7, we can also assume that $b \in \{2, m'\}$ and $b = m'$ when $(\mathbf{G}, F)$ is twisted. In the proof of Theorem 7.7, we have shown that $H_c^{\dim}(\mathbf{Y}_{\mathbf{U}'}^{\mathbf{G}}, \Lambda)e_s^{L'}$ almost extends to a $G \times (L')^{\mathrm{opp}}\Delta(P_b)$-module, where $P_b/\tilde{L}'$ is a Sylow b-group of $\mathcal{N}/\tilde{L}'$. We claim that $P_b \cap \mathrm{N}_{\mathcal{N}}(Q)$ is (the preimage) of a Sylow b-subgroup of $\mathrm{N}_{\mathcal{N}}(Q)/\mathrm{N}_{\tilde{L}'}(Q)$.

For $b = m'$, observe that the Sylow m'-subgroup of $P_{m'}$ is normal in $\mathcal{N}$. Observe that $\mathrm{N}_{\mathcal{N}}(Q)/\mathrm{N}_{\tilde{L}'}(Q)$ embeds as a subgroup of $\mathcal{N}/\tilde{L}'$. In particular, $\tilde{P}_{m'} = P_{m'} \cap \mathrm{N}_{\mathcal{N}}(Q)$ is (the preimage) of a Sylow m'-subgroup of $\mathrm{N}_{\mathcal{N}}(Q)/\mathrm{N}_{\tilde{L}'}(Q)$. Now suppose that $(\mathbf{G}, F)$ is untwisted and $b = 2$. Since $\mathcal{A}$ is abelian, we have $[\mathcal{N}, \mathcal{N}] \subseteq \tilde{N}'$. It follows from Lemma 8.1 and our assumption on Q that $(P_b \cap \mathrm{N}_{\mathcal{N}}(Q))/\mathrm{N}_{\tilde{L}'}(Q)$ is a Sylow b-subgroup of $\mathrm{N}_{\mathcal{N}}(Q)/\mathrm{N}_{\tilde{L}'}(Q)$.

Using the corresponding local results in Lemma 7.5, Lemma 2.2 and Lemma 3.6, now show that $H_c^{\dim}(\mathbf{Y}_{\mathbf{C}_{\mathbf{U}'}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)\mathrm{br}_Q(e_s^{L'})$ almost extends to an $X_Q\Delta\mathrm{N}_{P_b}(Q)$-module. As explained before, this implies that $H_c^{\dim}(\mathbf{Y}_{\mathbf{C}_{\mathbf{U}'}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)\,\mathrm{br}_Q(e_s^{L'})$ almost extends to $\mathrm{N}_G(Q) \times \mathrm{N}_{N'}(Q)^{\mathrm{opp}}\Delta\,\mathrm{N}_{\mathcal{N}}(Q)$. □

Suppose now that $\ell \nmid |H^1(F, \mathrm{Z}(\mathbf{G}))|$. As in Lemma 1.3, let $\mathcal{C}'$ be a complex of $\Lambda(G \times (N')^{\mathrm{opp}}\Delta\tilde{N})$-modules such that $H^d(\mathcal{C}') \cong M'$ and $\mathcal{C}'$ induces a splendid Rickard equivalence between $\Lambda G e_s^G$ and $\Lambda N' e_s^{L'}$. According to the proof of [34, Theorem 5.2], there exists a unique complex $\mathcal{C}'_Q$ of ℓ-permutation $\Lambda(\mathrm{C}_G(Q) \times \mathrm{C}_{N'}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\tilde{N}'}(Q))$-modules that lifts the complex $\mathrm{Br}_{\Delta Q}(\mathcal{C}')$ of ℓ-permutation modules from k to Λ. For simplicity, we denote

$$M'_Q := \mathrm{Ind}_{\mathrm{C}_G(Q)\times\mathrm{C}_{N'}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\tilde{N}'}(Q)}^{\mathrm{N}_G(Q)\times\mathrm{N}_{N'}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\tilde{N}'}(Q)}(H^{d_Q}(\mathcal{C}'_Q))\mathrm{br}_Q(e_s^{L'}).$$

As in the proof of Lemma 1.2, we see that M'_Q is an extension of $H_c^{d_Q}(\mathbf{Y}_{\mathbf{C}_{\mathbf{U}}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)}, \Lambda)\,\mathrm{br}_Q(e_s^{L'})$.

Lemma 8.3. *Suppose that $\ell \nmid |H^1(F, \mathrm{Z}(\mathbf{G}))|$ and $(\mathbf{G}, F)$ is untwisted. Then the bimodule M'_Q almost extends to $\mathrm{N}_G(Q) \times \mathrm{N}_{N'}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\mathcal{N}}(Q)$.*

Proof. We first assume that $\mathbf{L}'$ is 1-split. Then $M \cong \Lambda[\mathbf{G}^F/\mathbf{U}'^F]e_s^{L'}$ is an ℓ-permutation module and $\mathcal{C} \cong M$. It follows that $\mathcal{C}' \cong M'$ and thus $H^0(\mathrm{Br}_{\Delta Q}(\mathcal{C}')) \cong \mathrm{Br}_{\Delta Q}(M')$. The bimodule M' extends to a $G \times (L')^{\mathrm{opp}}\Delta\mathcal{N}_{\ell'}$-module $\hat{M}$, which is $\mathcal{N}$-stable. Thus, $\mathrm{Br}_{\Delta Q}(\hat{M})$ is a $k(\mathrm{C}_G(Q) \times (\mathrm{C}_{L'}(Q))^{\mathrm{opp}}\Delta\,\mathrm{N}_{\mathcal{N}_{\ell'}}(Q))$-module extending $\mathrm{Br}_{\Delta Q}(M')$. It follows that $\mathrm{Br}_{\Delta Q}(\hat{M})$ is an ℓ-permutation module as well. Thus, there exists a unique ℓ-permutation module $\hat{M}_Q$, which lifts $\mathrm{Br}_{\Delta Q}(\hat{M})$ to Λ; see [1, Corollary 3.11.4]. Since

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this lift is unique and $\mathrm{Br}_{\Delta Q}(\hat{M})$ is $\mathrm{N}_{\mathcal{N}}(Q)$-stable, it follows that $\hat{M}_Q$ is $\mathrm{N}_{\mathcal{N}}(Q)$-stable as well. In other words, $H^0(\mathcal{C}'_Q)\mathrm{br}_Q(e_s^{L'})$ almost extends to $\mathrm{C}_G(Q)\times\mathrm{C}_{N'}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\mathcal{N}}(Q)$. The claim follows by applying the induction functor.

Now let us assume that $\mathbf{L}'$ is not 1-split. By Lemma 8.2, the bimodule $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}'}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)},\Lambda)\mathrm{br}_Q(e_s^{L'})$ almost extends to $\mathrm{N}_G(Q)\times\mathrm{N}_{L'}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\mathcal{N}}(Q)$. On the other hand, the bimodule M'_Q is an extension of $H_c^{\dim}(\mathbf{Y}_{\mathrm{C}_{\mathbf{U}'}(Q)}^{\mathrm{N}_{\mathbf{G}}(Q)},\Lambda)\mathrm{br}_Q(e_s^{L'})$. The quotient group $\mathcal{N}/\tilde{L}'$ is abelian by Lemma 6.11. Therefore, Lemma 7.3 shows that $H^{d_Q}(\mathcal{C}'_Q)\mathrm{br}_Q(e_s^{L'})$ almost extends as well to a $\Lambda(\mathrm{N}_G(Q)\times\mathrm{N}_{L'}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\mathcal{N}}(Q))$-module. □

According to Lemma 8.3, there exists an extension $\hat{M}_Q$ of M'_Q to $\Lambda(\mathrm{N}_G(Q)\times\mathrm{N}_N(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\mathcal{N}_{\ell'}}(Q))$. With this notation, the following is immediate.

Corollary 8.4. *Suppose the assumptions and notation as above. Then the bimodule M'_QC_Q induces a Morita equivalence between $\Lambda\mathrm{N}_{N'}(Q)C_Q$ and $\Lambda\mathrm{N}_G(Q)B_Q$, which lifts to the Morita equivalence $\hat{M}_Q$ between $\Lambda\mathrm{N}_{\mathcal{N}_{\ell'}}(Q,C_Q)C_Q$ and $\Lambda\mathrm{N}_{G\mathcal{A}_{\ell'}}(Q,B_Q)B_Q$.*

Proof. Note that $M'_Q\cong\mathrm{Ind}_{\mathrm{C}_G(Q)\times\mathrm{C}_{N'}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{N'}(Q)}^{\mathrm{N}_G(Q)\times\mathrm{N}_{\tilde{N}'}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\tilde{N}'}(Q)}(H^{d_Q}(\mathcal{C}'_Q))\mathrm{br}_Q(e_s^{L'})$. By the proof of Lemma 1.2 and [37, Proposition 1.12], the bimodule M'_QC_Q induces a Morita equivalence between $\Lambda\mathrm{N}_{N'}(Q)C_Q$ and $\Lambda\mathrm{N}_G(Q)B_Q$. The second claim now follows from this and [28, Theorem 3.4]. □

9. The first reduction

Let b be a block of ΛGe_s^G, where $s\in(\mathbf{G}^*)^{F^*}$ is the fixed strictly-quasi isolated element in $\mathbf{G}=\mathrm{SL}_{n+1}(\overline{\mathbb{F}_p})$ from before. By [37, Theorem 1.3], there exists a defect group D contained in N; and since $\ell\nmid|N/L|$, we have $D\subset L$. In what follows, we let Q be a fixed characteristic subgroup of D. For a given character χ, we use $\mathrm{bl}(\chi)$ to denote the ℓ-block χ belongs to. For the language of character triples and the definition of the order relation $\geq_b$ on character triples, we refer the reader to [38, Section 1.1]. We will use the following criterion due to Cabanes–Späth to check the inductive Alperin–McKay condition.

Theorem 9.1. *Let $\chi\in\mathrm{Irr}(G,b)$ and $\chi'\in\mathrm{Irr}(\mathrm{N}_G(Q),B_Q)$ such that the following holds:*

(i) *We have $(\tilde{G}\mathcal{B})_\chi=\tilde{G}_\chi\mathcal{B}_\chi$, and χ extends to $(G\mathcal{B})_\chi$.*
(ii) *We have $(\mathrm{N}_{\tilde{G}}(Q)\mathrm{N}_{G\mathcal{B}}(Q))_{\chi'}=\mathrm{N}_{\tilde{G}}(Q)_{\chi'}\mathrm{N}_{G\mathcal{B}}(Q)_{\chi'}$, and χ' extends to $\mathrm{N}_{G\mathcal{B}}(Q)_{\chi'}$.*
(iii) *$(\tilde{G}\mathcal{B})_\chi=G(\mathrm{N}_{\tilde{G}}(Q)\mathrm{N}_{G\mathcal{B}}(Q))_{\chi'}$.*
(iv) *There exist $\tilde{\chi}\in\mathrm{Irr}(\tilde{G}\mid\chi)$ and $\tilde{\chi}'\in\mathrm{Irr}(\mathrm{N}_{\tilde{G}}(Q)\mid\chi')$ such that the following holds:*
   - *For all $m\in\mathrm{N}_{G\mathcal{B}}(Q)_{\chi'}$, there exists $\nu\in\mathrm{Irr}(\tilde{G}/G)$ with $\tilde{\chi}^m=\nu\tilde{\chi}$ and $\tilde{\chi}'^m=\mathrm{Res}^{\tilde{G}}_{\mathrm{N}_{\tilde{G}}(Q)}(\nu)\tilde{\chi}'$.*
   - *The characters $\tilde{\chi}$ and $\tilde{\chi}'$ cover the same underlying central character of $\mathrm{Z}(\tilde{G})$.*
(v) *The Clifford correspondents $\tilde{\chi}_0\in\mathrm{Irr}(\tilde{G}_\chi\mid\chi)$ and $\tilde{\chi}'_0\in\mathrm{Irr}(\mathrm{N}_{\tilde{G}}(Q)_{\chi'}\mid\chi')$ of $\tilde{\chi}$ and $\tilde{\chi}'$, respectively, satisfy $\mathrm{bl}(\tilde{\chi}_0)=\mathrm{bl}(\tilde{\chi}'_0)^{\tilde{G}_\chi}$.*

Let $\mathrm{Z}:=\mathrm{Ker}(\chi)\cap\mathrm{Z}(G)$. Then

$$((\tilde{G}\mathcal{B})_\chi/Z,G/Z,\overline{\chi})\geq_b((\mathrm{N}_{\tilde{G}}(Q)\mathrm{N}_{G\mathcal{B}}(Q))_{\chi'}/Z,\mathrm{N}_G(Q)/Z,\overline{\chi'}),$$

where $\overline{\chi}$ and $\overline{\chi'}$ are the characters that inflate to χ, respectively χ'.

Proof. This is a consequence of [38, Theorem 2.1] and [38, Lemma 2.2]. □

Note that all conditions in Theorem 9.1 except condition (v) only depend on the character theory of G and $\tilde{G}$ (together with its associated groups).

Let M' be the bimodule constructed in Corollary 7.8, which induces a Morita equivalence between ΛGe_s^G and $\Lambda N'e_s^{L'}$. We let c be the block of $\Lambda N'e_s^{L'}$ corresponding to b under this equivalence. Recall the group $\mathcal{A}$ from Definition 7.6 and that $\mathcal{N}=\mathrm{N}_{\tilde{G}\mathcal{A}}(\mathbf{L}',e_s^{L'})$. We denote by $\mathcal{N}_b$ the stabiliser of the block b in $\mathcal{N}$.

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With this notation, we are now ready to state the following theorem, which can be seen as an analogue of [38, Theorem 2.12].

Theorem 9.2. *Let* $\mathbf{G}$ *be a simple, simply connected group of type A and b a strictly quasi-isolated* ℓ*-block of* $\Lambda G e_s^G$*. Assume that* $\ell \nmid 2|H^1(F, \mathrm{Z}(\mathbf{G}))|$*, and suppose that the following hold:*

(i) *There exists an* $\mathrm{Irr}(\tilde{N}'/N') \rtimes \mathrm{N}_{\mathcal{N}}(Q)$*-equivariant bijection* $\tilde{\varphi} : \mathrm{Irr}(\tilde{N}' \mid \mathrm{Irr}_0(c)) \to \mathrm{Irr}(\mathrm{N}_{\tilde{N}'}(Q) \mid \mathrm{Irr}_0(C_Q))$ *such that it maps characters covering the character* $\nu \in \mathrm{Irr}(\mathrm{Z}(\tilde{G}))$ *to a character covering* ν*.*

(ii) *There exists an* $\mathrm{N}_{\mathcal{N}}(Q, C_Q)$*-equivariant bijection* $\varphi : \mathrm{Irr}_0(N', c) \to \mathrm{Irr}_0(\mathrm{N}_{N'}(Q), C_Q)$ *that satisfies the following two conditions:*
- *If* $\chi \in \mathrm{Irr}_0(N', c)$ *extends to a subgroup* H *of* $\mathcal{N}_b$*, then* $\varphi(\chi)$ *extends to* $\mathrm{N}_H(Q)$*.*
- $\tilde{\varphi}(\mathrm{Irr}(\tilde{N}' \mid \chi)) = \mathrm{Irr}(\mathrm{N}_{\tilde{N}'}(Q) \mid \varphi(\chi))$ *for all* $\chi \in \mathrm{Irr}_0(c)$*.*

(iii) *For every* $\theta \in \mathrm{Irr}_0(c)$ *and* $\tilde{\theta} \in \mathrm{Irr}(\tilde{N}' \mid \theta)$*, the following holds: If* $\theta_0 \in \mathrm{Irr}(\tilde{N}'_\theta \mid \theta)$ *is the Clifford correspondent of* $\tilde{\theta} \in \mathrm{Irr}(\tilde{N}')$*, then* $\mathrm{bl}(\theta_0) = \mathrm{bl}(\theta'_0)^{\tilde{N}'_\theta}$*, where* $\theta'_0 \in \mathrm{Irr}(\mathrm{N}_{\tilde{N}'}(Q)_{\varphi(\theta)} \mid \varphi(\theta))$ *is the Clifford correspondent of* $\tilde{\varphi}(\tilde{\theta})$*.*

Then the block b is AM-good.

Proof. According to Corollary 7.8, the bimodule M' induces an $\mathcal{N}$-equivariant bijection

$$R : \mathrm{Irr}_0(N', e_s^{L'}) \to \mathrm{Irr}_0(G, e_s^G).$$

Let M'_Q be the bimodule constructed before the proof of Lemma 8.3. By Corollary 8.4, the bimodule M'_Q induces an $\mathrm{N}_{\mathcal{N}}(Q, C_Q)$-equivariant bijection

$$R_Q : \mathrm{Irr}_0(\mathrm{N}_{N'}(Q), C_Q) \to \mathrm{Irr}_0(\mathrm{N}_G(Q), B_Q).$$

We define $\Psi := R_Q \circ \varphi \circ R^{-1} : \mathrm{Irr}_0(G, b) \to \mathrm{Irr}_0(\mathrm{N}_G(Q), B_Q)$, which is by construction $\mathrm{N}_{\tilde{G}\mathcal{A}}(Q, B_Q)$-equivariant.

As in the proof of [38, Theorem 2.12], we can assume that the character χ satisfies condition (i) of Theorem 9.1. We denote $\chi' := \Psi(\chi)$ and show that the characters χ and χ' satisfy the conditions of Theorem 9.1.

Since the bijection $\Psi : \mathrm{Irr}_0(G, b) \to \mathrm{Irr}_0(\mathrm{N}_G(Q), B_Q)$ is $\mathrm{N}_{\tilde{G}\mathcal{A}}(Q, B_Q)$-equivariant, we deduce that condition (iii) in Theorem 9.1 is satisfied and we have

$$\mathrm{N}_{\tilde{G}\mathcal{B}}(Q)_{\chi'} = \mathrm{N}_{\tilde{G}}(Q)_{\chi'} \mathrm{N}_{G\mathcal{B}}(Q)_{\chi'}.$$

The following lemma finishes the verification of condition (ii) in Theorem 9.1:

Lemma 9.3. *The character* χ' *extends to its inertia group in* $\mathrm{N}_{G\mathcal{B}}(Q)$*.*

Proof. We have $\mathrm{N}_{G\mathcal{B}}(Q)_{\chi'}/\mathrm{N}_G(Q) \cong \mathcal{B}_\chi$. Since the Sylow r-subgroups of $\mathcal{B}_\chi$ for $r \neq 2$ are cyclic, it suffices to show that χ' extends to $\mathrm{N}_{G\mathcal{B}_2}(Q)$, where $\mathcal{B}_2$ is the Sylow 2-subgroup of $\mathcal{B}_\chi$. Moreover, $\mathcal{B}$ is cyclic unless $(\mathbf{G}, F)$ is untwisted.

We may assume that $\mathcal{B}_2 = \langle \phi^i, \gamma \rangle$ for some i since $\mathcal{B}_2$ is cyclic otherwise. By definition of F_0 in the proof of Lemma 6.12, there exists some $g \in G^\gamma$ such that $F_0 = \mathrm{ad}(g)\phi$. Recall that the character χ extends to $G\mathcal{B}_2$. It follows that χ has a ϕ^i-stable extension to $G\langle\gamma\rangle$. Consequently, this extension is F_0^i-stable, and the character χ thus extends to $G\langle F_0^i, \gamma\rangle$. Let H be the subgroup with $N \leq H \leq \mathcal{N}$ corresponding to $G\langle F_0^i, \gamma\rangle$ under the isomorphism $\mathcal{N}/N \cong \tilde{G}\mathcal{A}/G$. By Corollary 7.8, the character $R^{-1}(\chi)$ extends to H. By assumption (ii), the character $\varphi(R^{-1}(\chi))$ therefore extends to its inertia group in $\mathrm{N}_H(Q)$. Hence, by Corollary 8.4, the character χ' has an extension to $\mathrm{N}_{G\langle F_0^i, \gamma\rangle}(Q)$. We deduce that χ' has an extension to $\mathrm{N}_{G\langle\gamma\rangle}(Q)$, which is $\mathrm{N}_{G\langle\phi^i,\gamma\rangle}(Q)$-stable. It follows that χ' extends to $\mathrm{N}_{G\mathcal{B}_2}(Q)$. □

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Fix a character $\tilde{\chi} \in \mathrm{Irr}(\tilde{G} \mid \chi)$. We define $\tilde{\chi}' := \tilde{R}_Q \circ \tilde{\varphi} \circ \tilde{R}^{-1}(\tilde{\chi})$, where $\tilde{R}$ and $\tilde{R}_Q$ are defined as in 1.8. By the remarks following Lemma 1.4 and Lemma 1.5 together with assumption (ii), we deduce that $\tilde{\chi}' \in \mathrm{Irr}(\mathrm{N}_{\tilde{N}'}(Q) \mid \Psi(\chi))$.

Recall the description of $\tilde{R}$ and $\tilde{R}_Q$ in terms of the Lusztig induction functors given in Lemma 1.4 and Lemma 1.5. Using [38, Lemma 2.9] and assumption (i), we can deduce that the first part of condition (iv) in Theorem 9.1 is satisfied. Moreover, [37, Lemma 2.10] and assumption (i) imply that the second part of condition (iv) in Theorem 9.1 is satisfied.

We now verify condition (v) in Theorem 9.1. Let $\tilde{\chi}_0 \in \mathrm{Irr}(\tilde{G}_\chi \mid \chi)$ be the Clifford correspondent of $\tilde{\chi}$. Moreover, let $\tilde{\chi}_0' \in \mathrm{Irr}(\mathrm{N}_{\tilde{G}}(Q)_{\chi'} \mid \chi')$ be the Clifford correspondent of $\tilde{\chi}'$. Recall that there exists a complex $\mathcal{C}'$ of $\Lambda(G \times (N')^{\mathrm{opp}}\Delta\tilde{N}')$-modules such that $H^d(\mathcal{C}') \cong M'$ and $\mathcal{C}'$ induces a splendid Rickard equivalence between $\Lambda G e_s^G$ and $\Lambda N' e_s^{L'}$; see Lemma 1.3.

Lemma 9.4. *The characters* $\tilde{\chi}_0 \in \mathrm{Irr}(\tilde{G}_\chi \mid \chi)$ *and* $\tilde{\chi}_0' \in \mathrm{Irr}(\mathrm{N}_{\tilde{G}}(Q)_{\chi'} \mid \chi')$ *satisfy* $\mathrm{bl}(\tilde{\chi}_0')^{\tilde{G}_\chi} = \mathrm{bl}(\tilde{\chi}_0)$.

Proof. Define $J := \tilde{G}_\chi$, and let J_0 be the subgroup of $\tilde{N}'/N'$ corresponding to J under the natural isomorphism $\tilde{N}'/N' \cong \tilde{G}/G$.

Consider $\mathcal{C} := G\Gamma_c(\mathbf{Y}_{\mathbf{U}'}^{\mathbf{G}}, \Lambda)e_s^{L'}$ as a complex of $\Lambda(G \times (L')^{\mathrm{opp}}\Delta(\tilde{L}'))$-modules, and define $\tilde{\mathcal{C}} := \mathrm{Ind}_{G\times(L')^{\mathrm{opp}}\Delta(\tilde{L}')}^{\tilde{G}\times(\tilde{L}')^{\mathrm{opp}}}(\mathcal{C})$. We have $\tilde{\mathcal{C}} \cong G\Gamma_c(\mathbf{Y}_{\mathbf{U}'}^{\tilde{\mathbf{G}}}, \Lambda)e_s^{L'}$ by [5, Proposition 1.1].

As in 1.8, we let $e \in \mathrm{Z}(\Lambda\tilde{L}')$ be the central idempotent such that $\sum_{n\in N'/L'} {}^n e = e_s^{L'}$. It follows similar to arguments given in the proof of [4, Theorem 7.5] that the complex $\tilde{\mathcal{C}}' := \tilde{\mathcal{C}}e \otimes_{\Lambda\tilde{L}'} \Lambda\tilde{N}'$ induces a splendid Rickard equivalence between $\Lambda\tilde{N}' e_s^{L'}$ and $\Lambda\tilde{G}e_s^G$. Moreover, we have $\mathrm{Ind}_{G\times(N')^{\mathrm{opp}}\Delta(\tilde{L}')}^{\tilde{G}\times(\tilde{N}')^{\mathrm{opp}}}(\mathcal{C}') \cong \tilde{\mathcal{C}}'$. The cohomology of $\mathcal{C}'$ is concentrated in degree $d := \dim(\mathbf{Y}_{\mathbf{U}}^{\mathbf{G}})$ and $H^d(\tilde{\mathcal{C}}_0') \cong \mathrm{Ind}_{(G\times(N')^{\mathrm{opp}})\Delta J_0}^{J\times J_0^{\mathrm{opp}}} H^d(\mathcal{C}')$. By [28, Theorem 3.4], the bimodule $H^d(\tilde{\mathcal{C}}_0')$ induces a Morita equivalence between $\mathcal{O}J_0c$ and $\mathcal{O}Jb$. We denote by

$$R_0 : \mathrm{Irr}(J_0, c) \to \mathrm{Irr}(J, b)$$

the associated bijection between irreducible characters. Using [37, Lemma 1.9], we see that the complex $\tilde{\mathcal{C}}_0' := \mathrm{Ind}_{(G\times N'^{\mathrm{opp}})\Delta(J_0)}^{J\times J_0^{\mathrm{opp}}}(\mathcal{C}')c$ induces a splendid Rickard equivalence between $\mathcal{O}Jb$ and $\mathcal{O}J_0c$. Denote

$$(M_Q')_0 := \mathrm{Ind}_{\mathrm{N}_G(Q)\times(\mathrm{N}_{L'}(Q))^{\mathrm{opp}}\Delta\,\mathrm{N}_{J_0}(Q)}^{\mathrm{N}_J(Q)\times(\mathrm{N}_{J_0}(Q))^{\mathrm{opp}}}(\mathrm{Res}_{\mathrm{N}_G(Q)\times(\mathrm{N}_{L'}(Q))^{\mathrm{opp}}\Delta\,\mathrm{N}_{J_0}(Q)}^{\mathrm{N}_G(Q)\times(\mathrm{N}_L(Q))^{\mathrm{opp}}\Delta\,\mathrm{N}_{\tilde{N}'}(Q)}(M_Q'))$$

so that the bimodule $(M_Q')_0$ induces a Morita equivalence between $\mathcal{O}\mathrm{N}_{J_0}(Q)C_Q$ and $\mathcal{O}\mathrm{N}_J(Q)B_Q$. We denote the associated character bijection by

$$(R_0)_Q : \mathrm{Irr}(\mathrm{N}_{J_0}(Q), C_Q) \to \mathrm{Irr}(\mathrm{N}_J(Q), B_Q).$$

By construction, $\tilde{\chi}' \in \mathrm{Irr}(\mathrm{N}_{\tilde{G}}(Q) \mid \chi')$. Let $\theta := R^{-1}(\chi)$ and $\tilde{\theta} := \tilde{R}^{-1}(\tilde{\chi})$. We obtain that $(R_0)_Q^{-1}(\tilde{\chi}_0') \in \mathrm{Irr}(\mathrm{N}_{J_0}(Q) \mid \varphi(\theta))$ is the Clifford correspondent of $\tilde{\varphi}(\tilde{\theta})$. Consequently, we have

$$\mathrm{bl}((R_0)_Q^{-1}(\tilde{\chi}_0'))^{J_0} = \mathrm{bl}(R_0^{-1}(\tilde{\chi}_0))$$

by assumption (iii). By the definition of M_Q' after Lemma 8.3, we have

$$M_Q' \otimes_\Lambda k \cong \mathrm{Ind}_{\mathrm{C}_G(Q)\times\mathrm{C}_{N'}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{N'}(Q)}^{\mathrm{N}_G(Q)\times\mathrm{N}_{N'}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{\tilde{N}'}(Q)} H^{d_Q}(\mathrm{Br}_{\Delta Q}(C'))\mathrm{br}_Q(e_s^{L'})$$

and therefore

$$(M_Q')_0 \otimes_\Lambda k \cong \mathrm{Ind}_{\mathrm{C}_G(Q)\times\mathrm{C}_{N'}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{J_0}(Q)}^{\mathrm{N}_G(Q)\times\mathrm{N}_{N'}(Q)^{\mathrm{opp}}\Delta\mathrm{N}_{J_0}(Q)} H^{d_Q}(\mathrm{Br}_{\Delta Q}(\mathcal{C}_0'))\mathrm{br}_Q(e_s^{L'}).$$

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Therefore, by [37, Remark 1.21], the equality $\mathrm{bl}((R_0)_Q^{-1}(\tilde{\chi}_0'))^{J_0} = \mathrm{bl}(R_0^{-1}(\tilde{\chi}_0))$ implies that $\mathrm{bl}(\tilde{\chi}_0')^J = \mathrm{bl}(\tilde{\chi}_0)$. □

Theorem 9.1 applies, and we obtain

$$((\tilde{G}\mathcal{B})_\chi/Z, G/Z, \overline{\chi}) \geq_b (\mathrm{N}_{\tilde{G}\mathcal{B}}(Q)_{\chi'}/Z, \mathrm{N}_G(Q), \overline{\chi'}),$$

where $Z := \mathrm{Z}(G) \cap \mathrm{Ker}(\chi)$. Using [37, Theorem 1.10], we deduce that the bijection $\Psi : \mathrm{Irr}_0(G, b) \to \mathrm{Irr}_0(\mathrm{N}_G(Q), B_Q)$ is a strong iAM-bijection in the sense of [38, Definition 1.9], which shows that the block b is AM-good; see [38, Definition 1.11]. □

Remark 9.5. The statement of the previous theorem remains true for the groups occurring in Theorem 4.2 if we take $\mathbf{L}'^*$ to be the minimal Levi subgroup containing $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$ and replace $\mathcal{A}$ by the group $\langle F_0 \rangle$. To see this, just replace Corollary 7.8 and Corollary 8.4 by Theorem 4.2 (and its local version, which can be proved as in Corollary 8.4).

Remark 9.6. In the proof of Theorem 9.2, we have shown that there exist a strong iAM-bijection $\Psi : \mathrm{Irr}_0(G, b) \to \mathrm{Irr}_0(\mathrm{N}_G(Q), B_Q)$ for the characteristic subgroup Q of a defect group D of the block b. In Section 12.2, it will be important to keep track of the subgroup Q. If such a strong iAM-bijection exists, we will say that the block b is AM-good relative to the subgroup Q.

10. The reduction for linear primes and twisted groups

10.1. Clifford theory of blocks

In the last section, we needed to assume in Theorem 9.2 that $\ell \nmid 2|H^1(F, \mathrm{Z}(\mathbf{G}))| = 2(n+1, q-\varepsilon)$ and that $(\mathbf{G}, F)$ is untwisted. Therefore, in this section, we will consider the remaining cases.

Let Y be a finite group and X be a normal subgroup of Y. For an ℓ-block e of X, we denote by $Y[e]$ the group of elements in Y, which stabilise e and act as inner automorphisms on the block algebra ΛXe. This group is called the *Dade ramification group of e in Y*; see for instance [30].

Lemma 10.1. *Let* $\mathbf{G}$ *be a simple, simply connected group of type A, and let b be a (nonnecessarily quasi-isolated)* ℓ*-block of G. If* $\ell \mid (q-\varepsilon)$*, then block induction yields a bijective map* $\mathrm{Bl}(\tilde{G}[b] \mid b) \to \mathrm{Bl}(\tilde{G} \mid b)$.

Proof. Suppose that b is a block of $\Lambda G e_s^G$ for a semisimple element $s \in (\mathbf{G}^*)^{F^*}$ of ℓ'-order. Consider a regular embedding $\iota : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ with dual map $\iota^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$. Fix an element $\tilde{s}$ of ℓ'-order of $(\tilde{\mathbf{G}}^*)^F$ with $\iota^*(\tilde{s}) = s$. Let $\tilde{\chi} \in \mathcal{E}(\tilde{G}, \tilde{s})$ be the unique semisimple character in its Lusztig series; see [3, Theorem 15.10]. Since $\Lambda\tilde{G}e_{\tilde{s}}^{\tilde{G}}$ is a block, there exists a constituent $\chi \in \mathrm{Irr}(G \mid \tilde{\chi})$, which lies in the block b. We have $\mathrm{Irr}(\tilde{G} \mid \chi) = \{\hat{z} \otimes \tilde{\chi} \mid z \in \mathrm{Z}(\tilde{G})\}$ and such a character $\hat{z} \otimes \tilde{\chi}$ lies in the Lusztig series $\mathcal{E}(\tilde{G}, \tilde{s}z)$. Note that $\mathcal{E}_\ell(\tilde{G}, \tilde{s}) = \mathrm{Irr}(\tilde{G}, \tilde{b})$, where $\tilde{b}$ is the block of $\tilde{\chi}$. Hence, $\hat{z} \otimes \tilde{\chi}$ lies in $\tilde{b}$ if and only if the ℓ'-part $(\tilde{s}z)_{\ell'}$ of $\tilde{s}z$ is $\tilde{\mathbf{G}}^*$-conjugate to $\tilde{s}$. This is equivalent to $\tilde{s}z_{\ell'}$ being $\tilde{\mathbf{G}}^*$-conjugate to $\tilde{s}$. In this case we obtain that $\tilde{\chi} = \hat{z}_{\ell'} \otimes \tilde{\chi}$ since $\tilde{\chi}$ is the unique semisimple character in $\mathcal{E}(\tilde{G}, \tilde{s})$. Thus, $\tilde{b} \otimes \hat{z} = \tilde{b}$ if and only if $\chi = \hat{z}_{\ell'} \otimes \chi$. From this, we deduce by Clifford theory that $|\mathrm{Bl}(\tilde{G} \mid b)| = |\tilde{G}_\chi : G|_{\ell'}$.

On the other hand, by [30, Proposition 3.9], we deduce that $|\mathrm{Bl}(\tilde{G}[b] \mid b)| \leq |\tilde{G}[b] : G|_{\ell'}$. According to [30, Theorem 3.5] and the Fong–Reynolds reduction block induction yields a surjective map $\mathrm{Bl}(\tilde{G}[b] \mid b) \to \mathrm{Bl}(\tilde{G} \mid b)$. This shows that $|\tilde{G}[b] : G|_{\ell'} \geq |\tilde{G}_\chi : G|_{\ell'}$. On the other hand, by [30, Theorem 4.1], we deduce that $|\tilde{G}[b] : G|_{\ell'} \leq |\tilde{G}_\chi : G|_{\ell'}$. Therefore, all inequalities must actually be equalities. This shows the statement. □

With a little more effort, the assumption on ℓ from the previous lemma can be lifted.

Lemma 10.2. *Let* $\mathbf{G}$ *be a simple, simply connected group of type A, and let b be a (nonnecessarily quasi-isolated)* ℓ*-block of G. Then block induction yields a bijective map* $\mathrm{Bl}(\tilde{G}[b] \mid b) \to \mathrm{Bl}(\tilde{G} \mid b)$.

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Proof. Our aim is to explicitly construct a character $\chi \in \mathrm{Irr}(G,b)$ with $|\mathrm{Bl}(\tilde{G} \mid b)| = |\tilde{G}_\chi : G|_{\ell'}$. As in Lemma 10.1 fix an element $\tilde{s}$ of ℓ'-order of $(\tilde{\mathbf{G}}^*)^{F^*}$ with $\iota^*(\tilde{s}) = s$. Let $\tilde{b}$ be a block of $e_{\tilde{s}}^{\tilde{G}}$ covering b. We have a Morita equivalence between $\mathcal{O}\tilde{L}e_{\tilde{s}}^{\tilde{L}}$ and $\mathcal{O}\tilde{G}e_{\tilde{s}}^{\tilde{G}}$ given by Deligne–Lusztig induction and we let $\tilde{c}$ be the block corresponding to $\tilde{b}$ under it. Let $z \in \mathrm{Z}(\tilde{G}^*)$ of ℓ'-order such that $e_{\tilde{s}}^{\tilde{G}} = \hat{z} \otimes e_{\tilde{s}}^{\tilde{G}}$. By duality, there exists some element $n^* \in (\iota^*)^{-1}(\mathrm{C}_{\mathbf{G}^*}(s)) = \tilde{N}^*$ such that ${}^{n^*}\tilde{s} = \tilde{s}z$. We let $n \in \tilde{N}$ be the element corresponding to n^* under the canonical isomorphism $\tilde{N}/\tilde{L} \cong \tilde{N}^*/\tilde{L}^*$ so that ${}^n e_{\tilde{s}}^{\tilde{L}} = e_{\tilde{s}\tilde{z}}^{\tilde{L}}$. Since $\hat{z}R_{\tilde{L}}^{\tilde{G}} = R_{\tilde{L}}^{\tilde{G}}\hat{z}$, it follows that the block ${}^{n^{-1}}(\tilde{c} \otimes \hat{z})$ corresponds to $\tilde{b} \otimes \hat{z}$ under the Bonnafé–Rouquier Morita equivalence. In particular, $\tilde{b} \otimes \hat{z} = \tilde{b}$ if and only if ${}^{n^{-1}}(\tilde{c} \otimes \hat{z}) = \tilde{c}$. Let $\tilde{c}_0 = \tilde{c} \otimes \hat{s}^{-1}$ be the corresponding unipotent block of $\tilde{L}$. Then the latter equality is equivalent to ${}^n\tilde{c}_0 = \tilde{c}_0$.

Now let $\tilde{\chi} \in \mathrm{Irr}(\tilde{G}, \tilde{b}) \cap \mathcal{E}(\tilde{G}, \tilde{s})$ and $\tilde{\psi} \in \mathcal{E}(\tilde{L}, 1)$ the unipotent character with $\tilde{\chi} = R_{\tilde{L}}^{\tilde{G}}(\hat{s}\tilde{\psi})$. A similar calculation as above now shows that $\tilde{\chi} \otimes \hat{z} = \tilde{\chi}$ if and only if ${}^n\tilde{\psi} = \tilde{\psi}$. Now, write $\tilde{L}$ as $\tilde{L} = \tilde{H}_1 \times \cdots \times \tilde{H}_r$, where the $\tilde{H}_i$ are the fixed points of the minimal F-stable components of $\tilde{\mathbf{L}}$. Then we can write $\tilde{c}_0 = c_1 \otimes \cdots \otimes c_r$ and $\tilde{\psi} = \psi_1 \times \cdots \times \psi_r$. We can choose a character $\tilde{\chi}$ such that the corresponding character $\tilde{\psi}$ satisfies $\psi_i = \psi_j$ whenever $c_i = c_j$. The group $\tilde{N}/\tilde{L}$ acts by permuting the rational components of $\tilde{\mathbf{L}}$. By considering each orbit under this action individually, we can assume that $\tilde{N}/\tilde{L}$ acts transitively. We let $\tilde{N}_0$ be the stabiliser of this permutation action. Observe that all unipotent characters of $\mathcal{E}(\tilde{H}_i, 1)$ are $\mathrm{Aut}(\tilde{H}_i)$-stable by [25, Theorem 2.5]. From this, it follows that every character in $\mathcal{E}(\tilde{L}, 1)$ is $\tilde{N}_0$-stable. We deduce that $\tilde{N}_{\tilde{\psi}} = \tilde{N}_{\tilde{c}_0}$. This shows that $\tilde{b} \otimes \hat{z} = \tilde{b}$ if and only if $\tilde{\chi}\hat{z} = \tilde{\chi}$. Hence, $|\mathrm{Bl}(\tilde{G} \mid b)| = |\tilde{G}_\chi : G|_{\ell'}$ by Clifford theory. The arguments in the second paragraph of Lemma 10.1 now show the result. □

Lemma 10.3. *Assume the assumptions of Lemma 10.2 and let $\psi \in \mathrm{Irr}(G,b)$. Then every block of $\tilde{G}_\psi$ covering b is $\tilde{G}_b$-stable.*

Proof. Denote $H := \tilde{G}_\psi\tilde{G}[b]$. Since $H[b] = \tilde{G}[b]$, we know by [30, Theorem 3.5] that block induction yields a surjective map $\mathrm{Bl}(\tilde{G}[b] \mid b) \to \mathrm{Bl}(H \mid b)$. Moreover, by [31, Corollary 6.2] block induction induces a map $\mathrm{Bl}(H \mid b) \to \mathrm{Bl}(\tilde{G} \mid b)$. By transitivity of block induction the composition of these maps is the bijection $\mathrm{Bl}(\tilde{G}[b] \mid b) \to \mathrm{Bl}(\tilde{G} \mid b)$ from Lemma 10.2. Thus, the map $\mathrm{Bl}(H \mid b) \to \mathrm{Bl}(\tilde{G} \mid b)$ is necessarily bijective as well.

Again, by [31, Corollary 6.2] block induction yields a surjective map $\mathrm{Bl}(\tilde{G}_\psi \mid b) \to \mathrm{Bl}(H \mid b)$. We show that it is actually bijective as well. Every block of $\mathrm{Bl}(\tilde{G}_\psi \mid b)$ contains a character ψ_0 extending ψ. Two such extensions differ by multiplication by a linear character $\lambda \in \mathrm{Irr}(\tilde{G}_\psi/G)$. Moreover, if λ has ℓ-power order then ψ_0 and $\psi_0\lambda$ lie in the same ℓ-block. From this, we deduce that $\ell \nmid |\mathrm{Bl}(\tilde{G}_\psi \mid b)|$.

Since blocks in $\mathrm{Bl}(\tilde{G}_\psi \mid b)$ differ only by multiplication with a linear character in $\mathrm{Irr}(\tilde{G}_\psi/G)$, every block $\tilde{b} \in \mathrm{Bl}(\tilde{G}_\psi \mid b)$ has the same stabiliser $H_{\tilde{b}}$ in H. Hence the fibres of the surjection $\mathrm{Bl}(\tilde{G}_\psi \mid b) \to \mathrm{Bl}(H \mid b)$ have cardinality $|H : H_{\tilde{b}}|$, so

$$|H : H_{\tilde{b}}|\,|\mathrm{Bl}(H \mid b)| = |\mathrm{Bl}(\tilde{G}_\psi \mid b)|$$

and therefore $\ell \nmid |H : H_{\tilde{b}}|$. Since $H/H_{\tilde{b}}$ is an ℓ-group by [30, Theorem 4.1], it follows that $H = H_{\tilde{b}}$. Therefore, block induction yields a bijective map $\mathrm{Bl}(\tilde{G}_\psi \mid b) \to \mathrm{Bl}(\tilde{G} \mid b)$. This proves the statement. □

10.2. The first reduction for twisted groups and linear primes

As in Section 9, we let b be an ℓ-block of $\Lambda G e_s^G$, where $s \in (\mathbf{G}^*)^{F^*}$ is a strictly quasi-isolated element of ℓ'-order. Recall the Levi subgroup $\mathbf{L}'$ of $\mathbf{G}$ defined in Lemma 6.2.

Lemma 10.4. *With the notation as above, there exists a block c_1 of $\Lambda L' e_s^{L'}$, unique up to N'-conjugation, such that $bH_c^{\dim}(\mathbf{Y}_{\mathbf{U}'}^{\mathbf{G}}, \Lambda)c_1 \neq 0$.*

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Proof. By Theorem 1.1, there exists a bimodule M_0 inducing a Morita equivalence between $\Lambda G e_s^G$ and $\Lambda N' e_s^{L'}$. We let c be the block of $\Lambda N' e_s^{L'}$ corresponding to b under the Morita equivalence induced by M_0. The $\Lambda(G \times (N')^{\mathrm{opp}})$-bimodule $bM_0c \cong bM_0$ is thus indecomposable. We have $\mathrm{Res}^{G\times(N')^{\mathrm{opp}}}_{G\times(L')^{\mathrm{opp}}}(M_0) \cong bH_c^{\dim}(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}'}, \Lambda)$. Observe that bM_0 is a right $\Lambda N'c$-module. Thus, if c_1 is a block of L' such that $bH_c^{\dim}(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}'}, \Lambda)c_1 \neq 0$, then c_1 has to lie below c. This determines c_1 up to N'-conjugation. □

We keep the notation of the proof of the previous lemma, and we fix a block c_1 of L' below c.

Definition 10.5. If $(\mathbf{G}, F)$ is untwisted and ℓ is a nonlinear prime: that is, $\ell \nmid (q-1)$, then we denote $T := N'$ and $\hat{c} := c$. Otherwise, we let T be the largest subgroup of N' containing L' such that there exists a unique block $\hat{c}$ covering c_1. Additionally, we set $\tilde{T} := T\tilde{L}'$, and we denote by $\hat{C}_Q$ the Brauer correspondent of $\hat{c}$ in $\mathrm{N}_T(Q)$.

The advantage of working with the block $\hat{c}$ (in the second case of the definition) instead of the block c is that the definition of the former does not depend on the particular choice of the extension of the bimodule $H^{\dim}(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}'}, \Lambda)e_s^{L'}$ to $G \times (N')^{\mathrm{opp}}$.

There are also some further simplifications specific to the situation when $\ell \mid (q-\varepsilon)$. The block b of $\Lambda G e_s^G$ is Morita equivalent to a block of $\Lambda N e_1^L$. This block of N is covered by the principal block $\Lambda L e_1^L$ of L. Thus, any Sylow ℓ-subgroup D of L is a defect group of b. A Frattini argument therefore shows that $\mathrm{N}_{\tilde{N}'}(Q)\tilde{L}' = \tilde{N}'$, where Q, as in the previous section, denotes a characteristic subgroup of D. In particular, the subgroup Q of L satisfies the group-theoretic assumptions of Lemma 8.2. With this, we now obtain a version of Theorem 9.2 for the cases excluded in this theorem.

Theorem 10.6. *Let* $\mathbf{G}$ *be a simple, simply connected group of type A and* b *a strictly quasi-isolated* ℓ*-block of* $\Lambda G e_s^G$*. Assume that* $\ell \mid (q-\varepsilon)$ *or that* $(\mathbf{G}, F)$ *is twisted and suppose that the following holds.*

(i) *There exists an* $\mathrm{Irr}(\tilde{N}'/N') \rtimes \mathrm{N}_{\mathcal{N}}(Q)$*-equivariant bijection* $\tilde{\varphi} : \mathrm{Irr}(\tilde{N}' \mid \mathrm{Irr}_0(T, \hat{c})) \to \mathrm{Irr}(\mathrm{N}_{\tilde{N}'}(Q) \mid \mathrm{Irr}_0(\mathrm{N}_T(Q), \hat{C}_Q))$ *such that it maps characters covering the character* $\nu \in \mathrm{Irr}(\mathrm{Z}(\tilde{G}))$ *to a character covering* ν*.*

(ii) *There exists an* $\mathrm{N}_{\mathcal{N}}(Q, \hat{C}_Q)$*-equivariant bijection* $\varphi : \mathrm{Irr}_0(T, \hat{c}) \to \mathrm{Irr}_0(\mathrm{N}_T(Q), \hat{C}_Q)$*, which satisfies the following two conditions:*
   - *If* $\chi \in \mathrm{Irr}_0(T, \hat{c})$ *extends to a subgroup* H *of* $\mathcal{N}_b$*, then* $\varphi(\chi)$ *extends to* $\mathrm{N}_H(Q)$*.*
   - $\tilde{\varphi}(\mathrm{Irr}(\tilde{N}' \mid \chi)) = \mathrm{Irr}(\mathrm{N}_{\tilde{N}'}(Q) \mid \varphi(\chi))$ *for all* $\chi \in \mathrm{Irr}_0(T, \hat{c})$*.*

(iii) *For every* $\tilde{\theta} \in \mathrm{Irr}(\tilde{N}' \mid \mathrm{Irr}_0(T, \hat{c}))$*, we have* $\mathrm{bl}(\tilde{\theta}) = \mathrm{bl}(\tilde{\varphi}(\tilde{\theta}))^{\tilde{N}'}$*.*

Then the block b *is AM-good.*

Proof. According to the proof of Theorem 7.7, there exists a $K[G \times (L')^{\mathrm{opp}} \Delta \mathcal{N}]$-module $\hat{M}$ extending $H_c^{\dim}(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}'}, K)e_s^{L'}$. Denote by M' the restriction of $\hat{M}$ to $G \times (N')^{\mathrm{opp}} \Delta \tilde{N}'$.

We claim that the bimodule $b\,\mathrm{Res}^{G\times(N')^{\mathrm{opp}}\Delta\tilde{N}'}_{G\times T^{\mathrm{opp}}}(M')\hat{c}$ induces an $\mathcal{N}_b$-equivariant bijection $R : \mathrm{Irr}(T, \hat{c}) \to \mathrm{Irr}(G, b)$. For $T = N'$, we observe that $\hat{c}$ is the unique block of N' covering the block c_1 of L'. By Lemma 10.4, we deduce that $bM' = bM'\hat{c}$. Observe that $\tilde{L}'$ acts transitively on the set $\mathrm{Irr}(N' \mid \zeta)$ for every $\zeta \in \mathrm{Irr}(L', e_s^{L'})$ by [13, Lemma 5.8(b)] and M' is $\Delta(\tilde{L}')$-stable. Hence, Lemma 1.2 together with N'/L' being cyclic implies that $bM'\hat{c}$ is multiplicity free as $K(G \times (N')^{\mathrm{opp}})$-module. Now Lemma 10.4 together with Theorem 1.1 imply that $|\mathrm{Irr}(G, b)| = |\mathrm{Irr}(N', \hat{c})|$, and thus the $\Delta(\tilde{N}')_b$-stable bimodule $bM'\hat{c}$ necessarily induces a bijection between these two sets.

Assume therefore now that $T = L'$. Then c_1 is necessarily N'-stable and there are exactly $|N'/L'|$ different blocks covering c_1. Consequently, by [30, Proposition 3.9] and [30, Theorem 4.1] in this case the block $\Lambda N'c$ is isomorphic to $\Lambda L'c_1$ via restriction. We have $b\,\mathrm{Res}^{G\times(N')^{\mathrm{opp}}\Delta\tilde{N}'}_{G\times(L')^{\mathrm{opp}}}(M')\hat{c} \cong bH^{\dim}(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}'}, K)\hat{c}$. Hence, Theorem 1.1 shows that $bH^{\dim}(\mathbf{Y}^{\mathbf{G}}_{\mathbf{U}'}, \Lambda)\hat{c}$ induces a Morita equivalence between $\Lambda L'\hat{c}$, and $\Lambda G b$. The claim thus also follows in this case.

We now construct a local bijection.As explained before the statement of this theorem, the subgroup Q satisfies the group theoretic requirement in Lemma 8.2. The proof of said lemma therefore shows

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that the bimodule $H_c^{\dim}(\mathbf{Y}_{\mathbf{C}_{\mathbf{U}'}(Q)}^{\mathbf{N}_{\mathbf{G}}(Q)}, K)\,\mathrm{br}_Q(e_s^{L'})$ extends to an $\mathrm{N}_G(Q)\times \mathrm{N}_{N'}(Q)^{\mathrm{opp}}\Delta\,\mathrm{N}_{\mathcal{N}}(Q)$-module. Let $\hat{M}_Q$ be such an extension and denote by M'_Q its restriction to $\mathrm{N}_G(Q)\times \mathrm{N}_{N'}(Q)^{\mathrm{opp}}\Delta\,\mathrm{N}_{\tilde{N}'}(Q)$.

Arguing as in the global case we obtain that the bimodule $B_Q\,\mathrm{Res}^{\mathrm{N}_G(Q)\times \mathrm{N}_{N'}(Q)^{\mathrm{opp}}\Delta\,\mathrm{N}_{\tilde{N}'}(Q)}_{\mathrm{N}_G(Q)\times \mathrm{N}_T(Q)^{\mathrm{opp}}}(M'_Q)\hat{C}_Q$ induces an $\mathrm{N}_{\mathcal{N}}(Q,B_Q)$-equivariant bijection $R_Q:\mathrm{Irr}_0(\mathrm{N}_T(Q),\hat{C}_Q)\to\mathrm{Irr}_0(\mathrm{N}_G(Q),B_Q)$. As in Theorem 9.2, we define $\Psi:=R_Q\circ\varphi\circ R^{-1}:\mathrm{Irr}_0(G,b)\to\mathrm{Irr}_0(\mathrm{N}_G(Q),B_Q)$, which is by construction $\mathrm{N}_{\tilde{G}\mathcal{A}}(Q,B_Q)$-equivariant.

We fix a character $\chi\in\mathrm{Irr}_0(G,b)$ and denote $\chi':=\Psi(\chi)$. As in Theorem 9.2, we want to show that the characters χ and χ' satisfy the conditions in Theorem 9.1. Moreover, as in the proof of Theorem 9.2, we fix a character $\tilde{\chi}\in\mathrm{Irr}(\tilde{G}\mid\chi)$, and define $\tilde{\chi}':=\tilde{R}_Q\circ\tilde{\varphi}\circ\tilde{R}^{-1}(\tilde{\chi})$, where $\tilde{R}$ and $\tilde{R}_Q$ are again defined as in 1.8. One easily sees that conditions (i)-(iv) of Theorem 9.1 can now be verified exactly in the same way as in Theorem 9.2. However, to show condition (v) we crucially used Lemma 8.3. Nevertheless according to Lemma 1.5 and assumption (iii) we see that $\mathrm{bl}(\tilde{\chi}')^{\tilde{G}}=\mathrm{bl}(\tilde{\chi})$. According to Lemma 10.3, the Clifford correspondents $\tilde{\chi}_0\in\mathrm{Irr}(\tilde{G}_\chi\mid\chi)$ and $\tilde{\chi}'_0\in\mathrm{Irr}(\mathrm{N}_{\tilde{G}}(Q)_{\chi'}\mid\chi')$ of $\tilde{\chi}$ and $\tilde{\chi}'$, respectively, therefore satisfy $\mathrm{bl}(\tilde{\chi}_0)=\mathrm{bl}(\tilde{\chi}'_0)^{\tilde{G}_\chi}$. In other words also condition (v) is satisfied. □

11. Results on defect groups of blocks of groups of Lie type

In this section, we prove some properties of defect groups of groups of Lie type. These results will be needed in the proofs of Section 12.2.

11.1. Defect groups of groups of type A

Following the terminology in [21, Section 3.4], we say that an ℓ-group D is *Cabanes* if it has a unique maximal abelian normal subgroup Q. Moreover, in this case we say that Q is the *Cabanes subgroup* of D. We recall [10, Lemma 4.16]:

Theorem 11.1. *Let* $\mathbf{H}$ *be a connected reductive group defined over* $\overline{\mathbb{F}}_p$ *such that* $\ell\geq 5$ *and* $\ell\geq 7$ *if* $\mathbf{H}$ *has a component of type* E_8. *Then the defect group of any* ℓ*-block of* $\mathbf{H}^F$ *is Cabanes.*

We keep the notation from the previous section. In particular, $\mathbf{G}$ is simple, simply connected of type A, and b denotes a strictly quasi-isolated block of $\Lambda G e_s^G$, and we let $\mathbf{L}'$ be the proper Levi subgroup of $\mathbf{G}$ constructed before. Moreover, c denotes the block of $\Lambda N' e_s^{L'}$, which corresponds to b under the Morita equivalence given by the bimodule M' defined after the proof of Theorem 7.7.

We fix a defect group D of the block c. If $\ell\geq 5$, we define Q to be the Cabanes subgroup of D and if $\ell<5$, we let $Q:=D$. Let $\tilde{b}$ be a block of $\tilde{G}$ covering b with defect group $\tilde{D}$ satisfying $\tilde{D}\cap G=D$. If $\ell\geq 5$, let $\tilde{Q}$ be the Cabanes subgroup of $\tilde{D}$ and otherwise define $\tilde{Q}:=\tilde{D}$.

Lemma 11.2. *Suppose that either* $\ell\geq 5$ *or* $\ell=2$ *and* D *is abelian. With the notation as above, we have* $\mathrm{C}_{\tilde{G}}(Q)=\mathrm{C}_{\tilde{G}}(\tilde{Q})$ *and* $\mathrm{N}_{\tilde{G}}(Q)=\mathrm{N}_{\tilde{G}}(\tilde{Q})$.

Proof. We may assume that $\ell\mid(n+1,q-\varepsilon)=|\tilde{\mathbf{G}}^F:\mathrm{Z}(\tilde{\mathbf{G}}^F)\mathbf{G}^F|$ since otherwise we have $\tilde{Q}=Q\mathrm{Z}(\tilde{\mathbf{G}}^F)_\ell$, and the statement follows.

If s has order $n+1$, then we have $\ell\nmid(n+1)$. As explained at the beginning of the proof, this implies the statement in this case. Hence, by [17, Lemma 5.2] (see also Lemma 14.1, below), the statement of the lemma holds for $\ell=2$ and D abelian. We can therefore assume that $\ell\geq 5$.

Suppose first that $s=1$. Let $\mathbf{S}$ be the diagonal torus in $\mathbf{G}$. The Cabanes subgroup of a Sylow ℓ-subgroup of $\mathbf{G}^F$ is given by $\mathbf{S}_\ell^F$. Since $\ell\neq 2$, we have $\mathrm{C}_{\mathbf{G}}(\mathbf{S}_\ell^F)=\mathbf{S}$; see [11, Proposition 22.6]. Similarly, $\tilde{\mathbf{S}}_\ell^F$ is the Cabanes subgroup of a Sylow ℓ-subgroup of $\tilde{\mathbf{G}}^F$ and $\mathrm{C}_{\mathbf{G}}(\tilde{\mathbf{S}}_\ell^F)=\mathbf{S}$. From this, the claim of the lemma follows.

Now, let $1\neq s$ and $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$ dual to the Levi subgroup $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$ of $\mathbf{G}^*$. Note that $\Lambda\mathbf{N}^F e_1^{\mathbf{L}^F}$ is Morita equivalent to $\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}$; see [4, Example 7.10]. Any block of $\Lambda\mathbf{N}^F e_1^{\mathbf{L}^F}$ has the same

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defect group as a block of $\mathbf{L}^F$ that is covered by it. It is well known (see [9, Theorem 13]) that since $\ell \mid (q-\varepsilon)$, the algebra $\Lambda\mathbf{L}^F e_1^{\mathbf{L}^F}$ consists of only one block: the principal block of $\mathbf{L}^F$.

Therefore, Q is the Cabanes subgroup of a Sylow ℓ-subgroup of $\mathbf{L}^F$. By Corollary 5.4, we have $\tilde{\mathbf{L}} \cong (\mathrm{GL}_e)^d \times \cdots \times (\mathrm{GL}_e)^d$ with the Frobenius endomorphism F transitively permuting the d copies of each $(\mathrm{GL}_e)^d$ such that $\tilde{\mathbf{L}}^F \cong \mathrm{GL}_e((\varepsilon q)^d) \times \cdots \times \mathrm{GL}_e((\varepsilon q)^d)$ with $e > 1$. Denote $Q_0 := Q \cap [\mathbf{L},\mathbf{L}]^F$. We conclude $[\mathbf{L},\mathbf{L}]^F \cong \mathrm{SL}_e((\varepsilon q)^d) \times \cdots \times \mathrm{SL}_e((\varepsilon q)^d)$. Here, d divides $|A(s)|$, which is of order prime to ℓ; see Corollary 5.4. Let $\mathbf{S}$ be the diagonal torus in SL_e. The subgroup $\mathbf{S}_\ell^{F^d}$ is a Cabanes subgroup of a Sylow ℓ-subgroup of $\mathrm{SL}_e((\varepsilon q)^d)$. Since $\ell \nmid d$ and $\ell \mid (q-\varepsilon)$, it follows that $((\varepsilon q)^d - 1)_\ell = (\varepsilon q - 1)_\ell$. We deduce that $|\Delta_d(\mathbf{S})_\ell^F| = |\mathbf{S}_\ell^{F^d}|$, where $\Delta_d : \mathrm{GL}_e \to (\mathrm{GL}_e)^d$ denotes the d-fold diagonal embedding. By the proof of [10, Lemma 4.16], we can therefore assume that

$$Q_0 = \Delta_d(\mathbf{S})_\ell^F \times \cdots \times \Delta_d(\mathbf{S})_\ell^F.$$

Hence $\mathrm{C}_{\tilde{\mathbf{G}}}(Q_0) \subseteq \mathrm{GL}_{ed} \times \cdots \times \mathrm{GL}_{ed}$. Since $\ell \neq 2$, we have $\mathrm{C}_{\mathrm{GL}_e}(\mathbf{S}_\ell^{F^d}) = \mathbf{S}$; see [11, Proposition 22.6]. Therefore, $\mathrm{C}_{\tilde{\mathbf{G}}}(Q_0) \cong (\mathrm{GL}_d)^e \times \cdots \times (\mathrm{GL}_d)^e$. A similar calculation shows that this coincides with $\mathrm{C}_{\tilde{\mathbf{G}}}(\tilde{Q})$. This implies that $\mathrm{C}_{\mathbf{G}}(\tilde{Q}) = \mathrm{C}_{\mathbf{G}}(Q) = \mathrm{C}_{\mathbf{G}}(Q_0)$.

Observe that $\mathbf{M} := \mathrm{C}_{\tilde{\mathbf{G}}}(Q)$ is a Levi subgroup of $\tilde{\mathbf{G}}$. It follows that $\mathrm{N}_{\tilde{\mathbf{G}}}(Q)$ and $\mathrm{N}_{\tilde{\mathbf{G}}}(\tilde{Q})$ are both contained in $\mathrm{N}_{\tilde{\mathbf{G}}}(\mathbf{M})$. On the other hand, $\mathrm{N}_{\tilde{\mathbf{G}}}(\mathbf{M})/\mathbf{M} \cong S_{(n+1/d)}$ given by permuting the $n + 1/d$ components of $\mathbf{M} = (\mathrm{GL}_d)^e \times \cdots \times (\mathrm{GL}_d)^e$. By the description of Q_0, it is clear that these automorphisms stabilise Q_0. This shows that $\mathrm{N}_{\tilde{\mathbf{G}}}(Q) = \mathrm{N}_{\tilde{\mathbf{G}}}(\tilde{Q}) = \mathrm{N}_{\tilde{\mathbf{G}}}(\mathbf{M})$. □

Remark 11.3. Suppose that D is the Sylow 2-subgroup of G. Then $\mathrm{C}_{\tilde{G}}(Q) = \mathrm{C}_{\tilde{G}}(\tilde{Q})$ unless $\mathbf{G}$ is of type A_1 and $q \equiv \pm 3 \bmod 8$; see [23, Theorem 1].

We denote $L_0 := [\mathbf{L}', \mathbf{L}']^F$, and we fix a block c_0 of L_0 below c such that $D_0 := D \cap L_0$ is a defect group of c_0. Additionally, we set $Q_0 := Q \cap L_0$.

The induction step in the proof of Theorem 12.4 below requires the following property of Cabanes subgroups.

Lemma 11.4. *Suppose that $\ell \neq 2$ and that $D/\mathrm{Z}(G)_\ell$ is abelian if $\ell = 2$. With the notation as introduced above, we have $\mathrm{C}_{\tilde{L}'}(Q_0) = \mathrm{C}_{\tilde{L}'}(Q)$ and $\mathrm{N}_{\tilde{L}'}(Q_0) = \mathrm{N}_{\tilde{L}'}(Q)$.*

Proof. Let us first consider the special case where $\ell = 2$ and D is nonabelian but $D/\mathrm{Z}(G)_\ell$ is abelian. According to the proof of [17, Lemma 5.2] and [17, Proposition 3.4(a)], we observe that $D \subset [\mathbf{L},\mathbf{L}]^F$. Since $\mathbf{L} \subset \mathbf{L}'$ by construction, we have $D \subset [\mathbf{L}', \mathbf{L}']^F$, so $D_0 = D$. Hence, the statement holds trivially in this case.We can therefore assume that the assumptions of Lemma 11.2 are satisfied.

We only prove the first part of the statement since the same arguments apply when we replace the centraliser subgroups everywhere by their corresponding normaliser subgroups. Consider the composition $[\mathbf{L}', \mathbf{L}'] \hookrightarrow \mathbf{L}' \hookrightarrow \tilde{\mathbf{L}}'$. We let $\tilde{c}$ be a block of $\tilde{N}'$ covering c. By [31, Theorem 9.26], there exists a defect group $\tilde{D}$ of $\tilde{c}$ such that $\tilde{D} \cap N' = \tilde{D} \cap L' = D$. It follows that the Cabanes subgroup $\tilde{Q}$ of $\tilde{D}$ satisfies $\tilde{Q} \cap L' = Q$. We have $\mathrm{C}_{\tilde{L}'}(\tilde{Q}) \subseteq \mathrm{C}_{\tilde{L}'}(Q) \subseteq \mathrm{C}_{\tilde{L}'}(Q_0)$, so it's enough to show that $\mathrm{C}_{\tilde{L}'}(\tilde{Q}) = \mathrm{C}_{\tilde{L}'}(Q_0)$.

As in the proof of [38, Proposition 3.8], it follows that $[\mathbf{L}', \mathbf{L}'] = \mathbf{H}_1 \times \cdots \times \mathbf{H}_r$,, where the $\mathbf{H}_i$ are simple algebraic groups of simply connected type. The action of the Frobenius endomorphism F induces a permutation π on the set of simple components of $[\mathbf{L}', \mathbf{L}']$. We let $\pi = \pi_1 \cdots \pi_t$ be the decomposition of this permutation into disjoint cycles. For $i = 1, \ldots, t$ choose $x_i \in \Pi_i$ in the support Π_i of the permutation π_i, and let $n_i = |\Pi_i|$ be the length of the cycle π_i. We then have

$$L_0 := [\mathbf{L}', \mathbf{L}']^F \cong \mathbf{H}_{x_1}^{F^{n_1}} \times \cdots \times \mathbf{H}_{x_t}^{F^{n_t}}.$$

Similarly, we can decompose $\tilde{\mathbf{L}}'$ as $\tilde{\mathbf{L}}' = \tilde{\mathbf{H}}_1 \times \cdots \times \tilde{\mathbf{H}}_r$, where $\tilde{\mathbf{H}}_i \cap [\mathbf{L}', \mathbf{L}'] = \mathbf{H}_i$. Therefore, $\tilde{\mathbf{L}}'^F \cong \tilde{\mathbf{H}}_{x_1}^{F^{n_1}} \times \cdots \times \tilde{\mathbf{H}}_{x_t}^{F^{n_t}}$.

Denote $\tilde{H}_i := \tilde{\mathbf{H}}_{x_i}^{F^{n_i}}$ and $H_i := \mathbf{H}_{x_i}^{F^{n_i}}$. The block c_0 is strictly quasi-isolated and decomposes as a direct product $c_0 = c_1 \otimes \cdots \otimes c_t$ of blocks that are strictly quasi-isolated in H_i; see the proof of [38,

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Proposition 3.8]. It follows that $Q_0 = Q_1 \times \cdots \times Q_t$. Similarly, let $\tilde{c}'$ be a block of $\tilde{L}'$ below $\tilde{c}$ that covers c_0. We have a decomposition $\tilde{c}' = \tilde{c}_1 \otimes \cdots \otimes \tilde{c}_t$, where $\tilde{c}_i$ is a block of $\tilde{H}_i$ covering c_i, and we obtain a decomposition $\tilde{Q} = \tilde{Q}_1 \times \cdots \times \tilde{Q}_t$ with $\tilde{Q}_i \cap H_i = Q_i$. It is therefore enough to show that $\mathrm{C}_{\tilde{H}_i}(Q_i) = \mathrm{C}_{\tilde{H}_i}(\tilde{Q}_i)$. This follows now from Lemma 11.2. □

12. Reduction to isolated blocks

12.1. Defect groups for groups not of type A

Let $\mathbf{G}$ be again simple, simply connected, but from now on, not necessarily of type A. Suppose that $\ell \geq 5$ and $\ell \geq 7$ if $\mathbf{G}$ is of type E_8. Fix an ℓ-block b of $\Lambda G e_s^G$. We let $\mathbf{L}^*$ be the minimal Levi subgroup containing $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$. Let c be a block of $\Lambda N e_s^L$ with defect group D corresponding to b under the Bonnafé–Dat–Rouquier Morita equivalence. Let c_0 be a block of $L_0 = [\mathbf{L}, \mathbf{L}]^F$ lying below c with defect group D_0 satisfying $D_0 = D \cap L_0$. Again we let Q be the Cabanes subgroup of D so that $Q_0 := Q \cap D_0$ is the Cabanes subgroup of D_0.

Proposition 12.1. *With the notation and assumptions as above, we have* $\mathrm{N}_{\tilde{L}}(Q_0) = \mathrm{N}_{\tilde{L}}(Q)$.

Proof. We let $\tilde{c}$ be a block of $\tilde{L}$ covering c with defect group $\tilde{D}$ satisfying $\tilde{D} \cap L = D$. Let $\tilde{Q}$ be the Cabanes subgroup of $\tilde{D}$ such that $\tilde{Q} \cap L = Q$. As in the proof of Lemma 11.4, it suffices to show that $\mathrm{N}_{\tilde{L}}(\tilde{Q}) = \mathrm{N}_{\tilde{L}}(Q_0)$.

Since $\mathbf{L}_0 = [\mathbf{L}, \mathbf{L}]$ is simply connected, we can write L_0 as

$$L_0 \cong \mathrm{SL}_{n_1}(\pm q^{d_1}) \times \cdots \times \mathrm{SL}_{n_r}(\pm q^{d_r}) \times H,$$

where H is a finite group obtained by taking fixed points under a Frobenius endomorphism of a simple, simply connected group $\mathbf{H}$ (of the same Lie type as $\mathbf{G}$). By individually considering each minimal F-stable component of $\mathbf{L}_0$, it is now easy to construct a regular embedding $\mathbf{L}_0 \hookrightarrow \mathbf{L}^+$ such that

$$L^+ \cong \mathrm{GL}_{n_1}(\pm q^{d_1}) \times \cdots \times \mathrm{GL}_{n_r}(\pm q^{d_r}) \times \tilde{H}.$$

Observe that $\mathbf{L}_0 \hookrightarrow \tilde{\mathbf{L}}$ is a second regular embedding. Using [11, Problem 15.2], we deduce that there exist regular embeddings $\tilde{\mathbf{L}} \hookrightarrow \tilde{\mathbf{L}}^+$ and $\mathbf{L}^+ \hookrightarrow \tilde{\mathbf{L}}^+$ such that the so-obtained square is commutative. Note that the centre of $\tilde{\mathbf{L}}$ and $\mathbf{L}^+$ is already connected. Therefore, it follows that $\tilde{L}^+ = \tilde{L}\,\mathrm{Z}(\tilde{L}^+)$ and $\tilde{L}^+ = L^+\,\mathrm{Z}(\tilde{L}^+)$ by Lang's theorem.

We fix a block $\tilde{c}^+$ of $\tilde{L}^+$ covering $\tilde{c}$ and observe that it has a defect group whose Cabanes group is $\tilde{Q}^+ := \tilde{Q}\,\mathrm{Z}(\tilde{L}^+)_\ell$. Furthermore, there exists a block c^+ covering the block c_0 such that c^+ has a defect group whose Cabanes group is $Q^+ := \tilde{Q}^+ \cap L^+$. We obtain $\mathrm{N}_{\tilde{L}^+}(\tilde{Q}) = \mathrm{N}_{\tilde{L}^+}(\tilde{Q}^+) = \mathrm{N}_{\tilde{L}^+}(Q^+)$ since all Cabanes groups differ only by a subgroup central in $\tilde{L}^+$. Therefore, it suffices to show that $\mathrm{N}_{\tilde{L}^+}(Q^+) = \mathrm{N}_{\tilde{L}^+}(Q_0)$. Since $\tilde{L}^+ = L^+\,\mathrm{Z}(\tilde{L}^+)$, this is equivalent to $\mathrm{N}_{L^+}(Q^+) = \mathrm{N}_{L^+}(Q_0)$.

We are now essentially in the situation of Lemma 11.4. The block c^+ is a block of L^+ covering the strictly quasi-isolated block c_0 of L_0. Again we argue componentwise as in Lemma 11.4. This deals with all rational components H of L_0 of type A. If the group H is not of type A, then $\ell \nmid |\mathrm{Z}(H)|$ by [11, Table 13.11]. Therefore, any ℓ-subgroup of $\tilde{H}$ is contained in $H\mathrm{Z}(\tilde{H})$, and the claim also follows in this case. □

Remark 12.2. The proof of the previous proposition also shows that [38, Hypothesis 3.3'] is satisfied for all groups $\mathbf{G}$.

12.2. Reduction to isolated blocks

We first consider certain blocks of simple groups with exceptional Schur multiplier.

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Lemma 12.3. *Let $\ell \geq 5$ and S be a simple group of Lie type A, B or C defined over a field of characteristic $\neq \ell$ with exceptional Schur multiplier. Then the Sylow ℓ-subgroups of the universal covering group of S are cyclic. In particular, the inductive Alperin–McKay condition holds for all ℓ-blocks of the universal covering group of S.*

Proof. An examination of the groups S with exceptional Schur multiplier shows that $|S|_\ell \in \{1, \ell\}$ and $\ell \nmid |M(S)|$, where $M(S)$ is the Schur multiplier of S. Therefore, the Sylow ℓ-subgroups of the universal covering group of S are cyclic. Therefore, the inductive Alperin–McKay condition holds for all ℓ-blocks of S by the work of Koshitani–Späth; see [24]. □

We can now show that ℓ-blocks of groups of type A are AM-good provided that $\ell \geq 5$. The proof is very similar to the proof of [38, Theorem 3.14] using all the new ingredients proved up to here. For what follows, recall from Remark 9.6 that a block b of a finite group G is called AM-good relative to a subgroup Q if there exists a strong iAM-bijection $\Psi : \mathrm{Irr}_0(G, b) \to \mathrm{Irr}_0(\mathrm{N}_G(Q), B_Q)$.

Theorem 12.4. *Let $\ell \geq 5$, and assume that all isolated ℓ-blocks of quasi-simple groups of type A defined over a field of characteristic $\neq \ell$ are AM-good relative to the Cabanes subgroup of their defect group. Then all ℓ-blocks of quasi-simple groups of type A are AM-good.*

Proof. Note that for $\ell \geq 5$, all blocks of simple groups of type A with exceptional Schur multiplier are AM-good (with respect to the prime ℓ) by Lemma 12.3. By [38, Theorem 3.14] and Remark 12.2, it suffices to show that the strictly quasi-isolated blocks of $\mathbf{G}^F$, where $\mathbf{G}$ is of type A, are AM-good relative to the Cabanes subgroup of their defect group. We show this statement by induction on the rank of $\mathbf{G}$.

Assume that b is a strictly quasi-isolated block of $\mathbf{G}^F$ that is not isolated. Recall that we have a decomposition $L_0 := H_1 \times \cdots \times H_t$, where the finite groups H_i are either quasi-simple or solvable. In the former case, our induction hypothesis implies that the blocks c_i are AM-good with respect to the Cabanes subgroup of their defect group. By the proof of [38, Proposition 3.8], we therefore obtain an iAM-bijection $\varphi_0 : \mathrm{Irr}_0(L_0, c_0) \to \mathrm{Irr}_0(\mathrm{N}_{L_0}(Q_0), (C_0)_{Q_0})$. Here, Q_0 is the Cabanes subgroup of the defect group D_0 of c_0 and $(C_0)_{Q_0} := \mathrm{br}_{Q_0}(c_0)$.

Arguing as in the proof of [38, Lemma 3.9], we obtain a bijection

$$\varphi_0 : \mathrm{Irr}_0(L_0, c_0) \to \mathrm{Irr}_0(\mathrm{N}_{L_0}(Q_0), (C_0)_{Q_0})$$

that satisfies

$$(\mathcal{N}_\chi, L_0, \chi) \geq_b (\mathrm{N}_{\mathcal{N}}(Q_0)_{\varphi_0(\chi)}, \mathrm{N}_{L_0}(Q_0), \varphi_0(\chi))$$

for every character $\chi \in \mathrm{Irr}_0(L_0, c_0)$.

Recall the subgroup T and the block $\hat{c}$ from Definition 10.5. Let c_1 be a block of L' below $\hat{c}$ with defect group D. Denote by T' the stabiliser of c_1 in N'. Lemma 11.4 shows that $\mathrm{N}_L(Q) = \mathrm{N}_L(Q_0)$, where Q is the Cabanes subgroup of D. Since T' stabilises c_1, it follows that $\mathrm{N}_{T'}(Q)/\mathrm{N}_{L'}(Q) \cong T'/L'$. From these two facts, we deduce that $\mathrm{N}_{T'}(Q) = \mathrm{N}_{T'}(Q_0)$.

Suppose first that $T = T'$. Since $\mathrm{N}_T(Q) = \mathrm{N}_T(Q_0)$, we can argue as in the proof of [38, Lemma 1.12] to obtain an $\mathrm{N}_{\mathcal{N}}(Q, \hat{C}_Q)$-equivariant bijection $\varphi' : \mathrm{Irr}_0(T, \hat{c}) \to \mathrm{Irr}_0(\mathrm{N}_T(Q), \hat{C}_Q)$ such that

$$(\mathcal{N}_\chi, T, \chi) \geq_b (\mathrm{N}_{\mathcal{N}}(Q_0)_{\varphi(\chi)}, \mathrm{N}_T(Q), \varphi(\chi))$$

holds for every character $\chi \in \mathrm{Irr}_0(T, \hat{c})$. Using the equality $\mathrm{N}_T(Q) = \mathrm{N}_T(Q_0)$, we can argue as in the proof of [38, Lemma 3.12] and obtain $\mathrm{N}_{\mathcal{N}}(Q_0)_{\varphi(\chi)} = \mathrm{N}_{\mathcal{N}}(Q)_{\varphi(\chi)}$. We thus obtain

$$(\mathcal{N}_\chi, T, \chi) \geq_b (\mathrm{N}_{\mathcal{N}}(Q)_{\varphi(\chi)}, \mathrm{N}_T(Q), \varphi(\chi)).$$

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Suppose now that $T \neq T'$: that is, $T' = L'$ and $T = N'$. By arguing as above, we obtain an $\mathrm{N}_{\mathcal{N}}(Q, (C_1)_Q)$-equivariant bijection $\varphi' : \mathrm{Irr}_0(L', c_1) \to \mathrm{Irr}_0(\mathrm{N}_{L'}(Q), (C_1)_Q)$ such that

$$(\mathcal{N}_{\chi'}, L', \chi') \geq_b (\mathrm{N}_{\mathcal{N}}(Q)_{\varphi'(\chi')}, \mathrm{N}_{L'}(Q), \varphi(\chi'))$$

holds for every character $\chi' \in \mathrm{Irr}_0(L', c_1)$. Clifford theory yields an $\mathrm{N}_{\mathcal{N}}(Q, \hat{C}_Q)$-equivariant bijection $\varphi : \mathrm{Irr}_0(N', \hat{c}) \to \mathrm{Irr}_0(\mathrm{N}_{N'}(Q), \hat{C}_Q)$. Moreover, by applying [32, Corollary 3.14] to the character $\chi := \mathrm{Ind}_{L'}^{N'}(\chi')$, we get

$$(\mathcal{N}_{\chi}, N', \chi) \geq_b (\mathrm{N}_{\mathcal{N}}(Q)_{\varphi(\chi)}, \mathrm{N}_{N'}(Q), \varphi(\chi)).$$

The proof of [38, Lemma 3.13] now shows the existence of a bijection $\tilde{\varphi} : \mathrm{Irr}(\tilde{N}' \mid \mathrm{Irr}_0(\hat{c})) \to \mathrm{Irr}(\mathrm{N}_{\tilde{N}'}(Q) \mid \mathrm{Irr}_0(\hat{C}_Q))$ such that $\tilde{\varphi}$ together with the bijection $\varphi : \mathrm{Irr}_0(T, \hat{c}) \to \mathrm{Irr}_0(\mathrm{N}_T(Q), \hat{C}_Q)$ satisfies assumptions (i)–(iii) of Theorem 9.2 and Theorem 10.6, respectively.

We can therefore apply Theorem 9.2 (respectively Theorem 10.6) and obtain that the block b is AM-good with respect to the Cabanes subgroup Q. □

Corollary 12.5. *The inductive Alperin–McKay condition holds for all ℓ-blocks of quasi-simple groups of type A, whenever $\ell \geq 5$ is a nondefining prime.*

Proof. By Theorem 12.4, it is enough to show that the isolated (that means unipotent) ℓ-blocks of type A are AM-good relative to the Cabanes subgroup of their defect group. Let d denote the order of q modulo ℓ.

Suppose first that $\ell \nmid (q - \varepsilon)$. Consider a unipotent block b of $\mathbf{G}^F$. We fix a block $\tilde{b}$ of $\tilde{\mathbf{G}}^F$ covering b. We observe that $\tilde{b}$ has the same defect group as b. There exists a d-cuspidal pair $(\mathbf{K}, \zeta)$ of $(\tilde{\mathbf{G}}, F)$ associated to $\tilde{b}$. By the proof of [11, Theorem 22.9], it follows that $Q := \mathrm{Z}(\mathbf{K})_\ell^F$ is the Cabanes subgroup of a defect group D of b, and we have $\mathbf{K} = \mathrm{C}_{\tilde{\mathbf{G}}}(Q)$. Since Q is characteristic in $\mathbf{K}$, it follows that $\mathrm{N}_{\tilde{\mathbf{G}}}(\mathbf{K}) = \mathrm{N}_{\tilde{\mathbf{G}}}(Q)$. Note that unipotent blocks satisfy the requirements of [8, Corollary 6.1]. Therefore, [8, Corollary 6.1] shows that b is AM-good relative to $\mathrm{N}_G(\mathbf{K}) = \mathrm{N}_G(Q)$.

Assume now that $\ell \mid (q - \varepsilon)$. It is well known (see [11, Example 22.10] and [11, Remark 22.11]) that in this case, $\tilde{\mathbf{G}}^F$ and also therefore $\mathbf{G}^F$ has only one unipotent ℓ-block. This is the principal block of $\mathbf{G}^F$ and therefore has maximal defect group. In particular, this block is AM-good relative to $\mathrm{N}_G(\mathbf{S})$, where $\mathbf{S}$ is the centraliser of a Sylow Φ_d-torus of $\mathbf{G}$, by the main theorem of [12]. It is thus sufficient to show that $\mathrm{N}_{\mathbf{G}}(Q) = \mathrm{N}_{\mathbf{G}}(\mathbf{S})$. Since $\ell \mid (q - \varepsilon)$, we know that $\mathbf{S}$ is the diagonal torus of $\mathbf{G}$. Again by the proof of [11, Theorem 22.9], it follows that $Q = \mathbf{S}_\ell^F$ is the Cabanes subgroup of a defect group of b and $\mathrm{C}_{\mathbf{G}}(Q) = \mathbf{S} = \mathrm{C}_{\mathbf{G}}(\mathbf{S})$. This implies $\mathrm{N}_{\mathbf{G}}(Q) = \mathrm{N}_{\mathbf{G}}(\mathbf{S})$, which proves the claim. □

Theorem 12.6. *Let $\ell \geq 5$, and let X be one of the symbols B or C. Assume that all isolated ℓ-blocks of a quasi-simple group of type X are AM-good relative to the Cabanes subgroup of the defect group. Then all ℓ-blocks of quasi-simple groups of type X are AM-good.*

Proof. Assume first that $n = 2$ and q is even. Then $\mathrm{Z}(\mathbf{G})$ is trivial, so the isolated ℓ-blocks of $\mathbf{G}^F$ are precisely the quasi-isolated ℓ-blocks. The statement is then a consequence of the main theorem of [38]. We also observe that for $\ell \geq 5$, all blocks of simple groups of type X with nonexceptional Schur multiplier are AM-good by Lemma 12.3. We can therefore assume that we are in none of these exceptional cases.

Fix an ℓ-block b of $\Lambda\mathbf{G}^F e_s^{\mathbf{G}^F}$ (not necessarily quasi-isolated), where $\mathbf{G}$ is of type X and $s \in (\mathbf{G}^*)^{F^*}$ is semisimple of ℓ'-order.

In contrast to the proof of Theorem 12.4, we don't need to argue by induction. We let $\mathbf{L}^*$ be the minimal Levi subgroup of $\mathbf{G}^*$ containing $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$ and let $\mathbf{L}$ be the Levi subgroup dual to $\mathbf{L}^*$. We let $\mathcal{A} := \langle F_0 \rangle$ as in Theorem 4.2. Let c be the block of $\Lambda\mathbf{N}^F e_s^{\mathbf{L}^F}$ corresponding to b under the Bonnafé–Dat–Rouquier equivalence from Theorem 4.2, and let c_0 be a block of $L_0 := [\mathbf{L}, \mathbf{L}]^F$ below c. We conclude that the block c_0 is then an isolated block of L_0. As in the proof of Theorem 12.4, we obtain a decomposition $L_0 = H_1 \times \cdots \times H_t$ into groups that are either quasi-simple of type A or X or solvable

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and a corresponding decomposition $c_0 = c_1 \otimes \cdots \otimes c_t$ of c_0 into block c_i, which are isolated in H_i. Our assumption together with the proof of Corollary 12.5 implies that these blocks are AM-good relative to the Cabanes subgroup of their defect group. Following the proof of Theorem 12.4 (using Remark 9.5 instead of Theorem 9.2, and using Proposition 12.1 instead of Lemma 11.4), we deduce that the block b is AM-good relative to the ℓ-subgroup Q. □

13. A variant of Späth's reduction theorem

In her paper [42], Späth shows that the Alperin–McKay conjecture holds for every finite group if and only if the inductive Alperin–McKay condition holds for every simple group. Our aim here is to modify her proof to get a similar statement involving preferably only blocks with abelian defect. Unfortunately, in her proof, it is necessary to consider central extensions of groups, and a block might have an abelian defect group, whereas a block of a central extension dominating it might not. Therefore, we need to consider blocks whose defect group lies in a slightly larger class of groups.

For the following definition, recall that if G is a finite group, its upper central series is defined recursively as $\mathrm{Z}_0(G) := 1$ and $\mathrm{Z}_i(G)$ is the unique subgroup of G containing $\mathrm{Z}_{i-1}(G)$ such that $\mathrm{Z}_i(G)/\mathrm{Z}_{i-1}(G) = \mathrm{Z}(G/\mathrm{Z}_{i-1}(G))$.

Definition 13.1. We say that a subgroup D of a finite group G is *almost abelian in G* if there exists an i such that $D\,\mathrm{Z}_i(G)/\mathrm{Z}_i(G)$ is abelian.

Observe that the property of D being almost abelian can depend on the ambient group G. Moreover, if one considers the hypercentre $\mathrm{Z}_\infty(G)$ of G (i.e., the union of all $\mathrm{Z}_i(G)$ for $i \geq 0$), then D is almost abelian in G if and only $D\,\mathrm{Z}_\infty(G)/\mathrm{Z}_\infty(G)$ is abelian.

Lemma 13.2. *Suppose that D is almost abelian in G:*

(a) *Let H be a subgroup of G and $E \leq H \cap D$. Then E is almost abelian in H.*
(b) *For $j = 1, 2$, the groups D_j are almost abelian in G_j if and only if $D_1 \times D_2$ is almost abelian in $G_1 \times G_2$.*
(c) *A subgroup E of G is almost abelian in G if and only if $E\,\mathrm{Z}(G)/\mathrm{Z}(G)$ is almost abelian in $G/\mathrm{Z}(G)$.*

Proof. Let us first prove part (a). By assumption, there is an i such that $D\,\mathrm{Z}_i(G)/\mathrm{Z}_i(G)$ is abelian. By induction, one easily shows that $\mathrm{Z}_i(G) \cap H \lhd \mathrm{Z}_i(H)$. From this, we deduce that $\frac{(D\cap H)\,\mathrm{Z}_i(H)}{\mathrm{Z}_i(H)}$ is abelian. Therefore, $D \cap H$ is almost abelian in H. Consequently, $E \leq D \cap H$ is almost abelian as well.

For part (b), we observe that $\mathrm{Z}_i(G_1 \times G_2) = \mathrm{Z}_i(G_1) \times \mathrm{Z}_i(G_2)$. Thus

$$\frac{(D_1 \times D_2)\,\mathrm{Z}_i(G_1 \times G_2)}{\mathrm{Z}_i(G_1 \times G_2)} \cong \frac{D_1\,\mathrm{Z}_i(G)}{\mathrm{Z}_i(G)} \times \frac{D_2\,\mathrm{Z}_i(G)}{\mathrm{Z}_i(G)},$$

and the claim follows from this.

For part (c), one first shows by induction that $\mathrm{Z}_{i-1}(G/\mathrm{Z}(G)) = \mathrm{Z}_i(G)/Z(G)$ for all i. Hence, $D\,\mathrm{Z}(G)/\mathrm{Z}(G)$ is almost abelian in $G/\mathrm{Z}(G)$ if and only if there exists an i such that

$$\frac{\mathrm{Z}_i(G/\mathrm{Z}(G))D\,\mathrm{Z}(G)/\mathrm{Z}(G)}{\mathrm{Z}_i(G/\mathrm{Z}(G))} \cong \frac{\mathrm{Z}_{i+1}(G)D}{\mathrm{Z}_{i+1}(G)}$$

is abelian. From this, is follows that $D\,\mathrm{Z}(G)/\mathrm{Z}(G)$ is almost abelian in $G/\mathrm{Z}(G)$ if and only if D is almost abelian in G. □

The aim of this section is to prove the following variant of [42, Theorem C]. We closely follow the proof of [12, Proposition 2.5].

Proposition 13.3. *Let X be a finite group and ℓ a prime. Assume that for every nonabelian simple subquotient S of X with $\ell \mid |S|$, the following holds: Every ℓ-block of the universal covering group H*

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of S with almost abelian defect group satisfies the iAM-condition. Then the Alperin–McKay conjecture holds for any ℓ-block of X with an almost abelian defect.

For a finite group X, let $F^*(X)$ be its generalised Fitting subgroup.

Proposition 13.4. *Let X be a finite group and b an ℓ-block of X with an almost abelian defect. Suppose that the Alperin–McKay conjecture is true for any ℓ-block with almost abelian defect group of any group H with $|H : \mathrm{Z}(H)| < |X : \mathrm{Z}(X)|$ and such that H is isomorphic to a subquotient of $X/\mathrm{Z}(X)$. Then one of the following holds:*

(i) *The Alperin–McKay conjecture holds for b.*
(ii) *For any noncentral normal subgroup K of X, we have $X = K\,\mathrm{N}_X(D)F^*(X)$.*

Proof. Let $K \lhd X$ be a noncentral normal subgroup of X. Replacing K by $K\,\mathrm{Z}(X)$ if necessary, we can assume that $\mathrm{Z}(X) \leq K$. By assumption, the Alperin–McKay conjecture is true for every block with almost abelian defect of any central extension of X/K. An analysis of the proof of [29, Theorem 6] then shows that the proof of said theorem can be adapted, and we obtain $|\operatorname{Irr}_0(B)| = |\operatorname{Irr}_0(b)|$, where $B \in \mathrm{Bl}(K\,\mathrm{N}_X(D))$ is the unique block with $B^X = b$. If $K\,\mathrm{N}_X(D)$ is a proper subgroup of X, then our assumption implies that the Alperin–McKay conjecture holds for the block B of $K\,\mathrm{N}_X(D)$. Therefore, $|\operatorname{Irr}_0(B)| = |\operatorname{Irr}_0(B_D)|$, where B_D is the Brauer correspondent of b. This would imply that the Alperin–McKay conjecture holds for b. Since the generalised Fitting subgroup $F^*(X)$ is such a noncentral normal subgroup of X, this argument shows in particular that $X = F^*(X)\,\mathrm{N}_X(D)$. □

Proof of Proposition 13.3. The proof of the statement is by induction on $|X : \mathrm{Z}(X)|$. We let b be an ℓ-block of X with almost abelian defect group D. We let B_D be its Brauer correspondent in $\mathrm{N}_X(D)$.

According to Proposition 13.4, we may assume that X and b satisfy the statement in Proposition 13.4(ii). Consequently, any normal ℓ-subgroup of X is central. As in [42, Section 6 and 7], we distinguish two cases.

Assume that there exists a normal noncentral subgroup $K \lhd X$ with $K \leq F^*(X)$. If $K \cap D \leq \mathrm{Z}(X)$, then $|\operatorname{Irr}_0(b)| = |\operatorname{Irr}_0(B_D)|$, according to [42, Proposition 6.6].

Assume otherwise that the Fitting subgroup is central in X and $E(X)$, the group of components of X, is noncentral. Inside $E(X)$, we can take a normal subgroup $K \lhd X$ such that $K = [K, K]$ and $K/\mathrm{Z}(K) \cong S^r$ for a nonabelian simple group S and an integer $r \geq 1$. The proof of [29, Proposition 9(ii)] shows that the block b covers a unique block b_0 of K. Note that $D_0 := D \cap K$ is a defect group of b_0, which is almost abelian by Lemma 13.2(a). Let $\tilde{G} = G^r$ be the universal covering group of $K/\mathrm{Z}(K) \cong S^r$, where G is the universal covering group of S. Let $\pi : \tilde{G} \to S^r$ be the associated quotient map. Let $\tilde{b}_0$ be a block of $\tilde{G}$ dominating b_0. Again Lemma 13.2 ensures that $\tilde{b}_0$ has an almost abelian defect group $\tilde{D}$. Following the proof of [42, Theorem 7.9], we obtain a bijection $\tilde{\Omega} : \operatorname{Irr}(\tilde{G}, \tilde{b}_0) \to \operatorname{Irr}(\tilde{M}, \tilde{B}_0)$. Here, $\tilde{M}$ is a suitably defined subgroup of $\tilde{G}$ containing $\mathrm{N}_{\tilde{G}}(\tilde{D})$ and $\tilde{B}_0^{\tilde{G}} = \tilde{b}_0$. From this, the proof of [42, Theorem 7.9] then yields a bijection $\hat{\Omega} : \operatorname{Irr}(K, b_0) \to \operatorname{Irr}(M\,\mathrm{N}_K(D_0), B_0)$ having the properties of the bijection in the statement of [42, Theorem 7.9]. Here, $M := \pi(\tilde{M})$ and B_0 is the unique block of $M\,\mathrm{N}_K(D_0)$ with $B_0^K = b_0$. Using the counting argument in the proof of [42, Theorem C], we can then deduce that $|\operatorname{Irr}_0(X, b)| = |\operatorname{Irr}_0(M\,\mathrm{N}_X(D_0), B)|$.

Since $|M\,\mathrm{N}_X(D_0)/\mathrm{Z}(X)| < |X/\mathrm{Z}(X)|$, we can apply the induction hypothesis, and we conclude that $|\operatorname{Irr}_0(M\,\mathrm{N}_X(D_0), B)| = |\operatorname{Irr}_0(\mathrm{N}_X(D), B_D)|$, where B_D is the Brauer correspondent of b. □

14. The Alperin–McKay conjecture for 2-blocks with abelian defect group

14.1. Classification of 2-blocks with abelian defect group

We start by recalling the following classification of quasi-isolated 2-blocks of finite groups of Lie type (see [17, Lemma 5.2]):

Lemma 14.1. *Assume that p is odd and* **G** *is simple, simply connected. Let b be a quasi-isolated 2-block of G with semi-simple label $s \in \mathbf{G}^*$.*

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(a) *Suppose that b has abelian defect groups. Then one of the following holds:*
  (i) *$\mathbf{G}$ is of type A_n, n is even and $C^\circ_{\mathbf{G}^*}(s)$ is a torus.*
  (ii) *$\mathbf{G}$ is of type G_2, F_4, E_6 or E_8, $s = 1$ and b is of defect 0.*

(b) *Suppose that b has nonabelian defect groups, but for some central 2-subgroup Z of $\mathbf{G}^F$, the image $\overline{b}$ in G/Z has abelian defect group. Then Z is cyclic of order 2 and one of the following holds:*
  (i) *$\mathbf{G}$ is of type A_n, $n \equiv 1 \bmod 4$ and the defect groups of $\overline{b}$ are $C_2 \times C_2$.*
  (ii) *$\mathbf{G}$ is of type E_7 and the defect groups of $\overline{b}$ are $C_2 \times C_2$.*

In the following remark, we collect some additional information from the proof of [17, Lemma 5.2].

Remark 14.2.

(a) If we assume additionally that the blocks in part (b)(i) are isolated (i.e., unipotent), then the proof of [17, Lemma 5.2] shows that $\mathbf{G}^F \cong A_1(q)$ and $q \equiv \pm 3 \bmod 8$. Moreover, their defect groups are isomorphic to the quaternion group Q_8.

(b) The blocks in (b)(ii) occur only if $4 \mid\mid (q-1)$. These blocks are unipotent, and their defect groups are isomorphic to the dihedral group D_8. They correspond to lines 3 and 7 of the table on page 354 of [18].

Our aim is now to show the iAM-condition for all isolated blocks of positive defect occurring in the classification of Lemma 14.1.

14.2. On a certain 2-block of $E_7(q)$

In this subsection, we consider the block b of $E_7(q)$ occurring in part (b)(ii) of the classification of Lemma 14.1. The author is very grateful to Gunter Malle for pointing out the proof of the following proposition to him.

Proposition 14.3. *Let b be one of the blocks of $G = E_7(q)$ occurring in Lemma 14.1(b)(ii). Then we have $|\operatorname{Irr}_0(b)| = 4$ and $|\operatorname{Irr}(b)| = 5$. Furthermore, the height zero characters of b have $\mathrm{Z}(G)$ in their kernel and are as follows:*

(i) *Two unipotent characters χ_1, χ_2.*
(ii) *Two nonunipotent characters χ_3, χ_4 that are conjugate under the diagonal automorphism of G.*

Proof. Let q be a prime power such that $4 \mid\mid (q-1)$. Set $G = E_7(q)_{sc}$ and $S = G/Z(G)$, the simple group of type $E_7(q)$. We consider the unipotent 2-block b of G parametrised by the 1-cuspidal pair $(E_6, E_6[\theta])$ (and its Galois conjugate block $(E_6, E_6[\theta^2])$ for which all arguments apply similarly).

Firstly, we note that according to the description in [18, 3.2], the defect group of b is isomorphic to the dihedral group D_8. By [39, Theorem 8.1], we can therefore deduce that $|\operatorname{Irr}_0(b)| = 4$ and $|\operatorname{Irr}(b)| = 5$. Moreover, the unique block $\overline{b}$ of S dominated by b has defect group $C_2 \times C_2$. Hence, by [39, Theorem 8.1], we know that $4 = |\operatorname{Irr}(\overline{b})| = |\operatorname{Irr}_0(\overline{b})|$; thus the unique character in $\operatorname{Irr}(b)$ with positive height is nontrivial on $\mathrm{Z}(G)$, and it is unique with this property among the irreducible characters of b. We will now describe the character of $\operatorname{Irr}(b)$ in more detail.

According to [18, Theorem B], the ordinary characters in b are described as follows: let $t \in G^* = E_7(q)_{\mathrm{ad}}$ be a (semisimple) 2-element such that $C^\circ_{\mathbf{G}^*}(t)$ has a Levi subgroup of type E_6. Then b contains those elements of $\mathcal{E}(G, t)$ that under Jordan decomposition correspond to characters in the 1-Harish-Chandra series of $\mathrm{C}^\circ_{\mathbf{G}^*}(t)$ above $(E_6, E_6[\theta])$, and moreover, these are the only characters in b.

In our particular case, the only centraliser in $\mathbf{G}^*$ that can possibly contain a Levi subgroup of type E_6 is, apart from $\mathbf{G}^*$ itself, also of type E_6. Now the centraliser of $E_6(q)$ in $E_7(q)$ is $K := E_6(q)(q-1)$, so its centre has order $(q-1)$. Given that $4 \mid\mid (q-1)$, there are hence only four 2-elements t in $\mathrm{Z}(K)$: the identity, an involution and two elements of order 4. Clearly the centraliser of the identity is all of G^*. Now $N(K) = K.2$ (by a calculation in the Weyl group), so for the involution $t \in \mathrm{Z}(K)$, we have $\mathrm{C}_{G^*}(t) = K.2$ (that is, the centraliser of t in $\mathbf{G}^*$ is disconnected). Thus there are two characters, say χ_3 and χ_4, of b in that geometric Lusztig series $\tilde{\mathcal{E}}(G, s)$ fused by the diagonal automorphism of G. On

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the other hand, $N(K)$ acts nontrivially on $\mathrm{Z}(K)$ (again by a calculation in the Weyl group), so the two elements t of order 4 in $\mathrm{Z}(K)$ are G^*-conjugate and moreover $\mathrm{C}_{G^*}(t) = K$. So here, $\mathcal{E}(G, t)$ contains one character χ_5 in b, which is left invariant under the diagonal automorphism.

From these calculations, it necessarily follows from $|\operatorname{Irr}(b)| = 5$ that $\mathcal{E}(G, 1)$ contains two characters χ_1 and χ_2 of b (in fact, these are the irreducible characters lying in the Harish-Chandra series of $E_6(\theta)$). These characters have $\mathrm{Z}(G)$ in their kernel. The characters χ_3 and χ_4 are $\tilde{G}$-conjugate and thus have the same underlying character of the centre. From this, we deduce that χ_5 has to be the unique character of b, which is nontrivial on $\mathrm{Z}(G)$. From this, our arguments above show that $\operatorname{Irr}_0(G, b) = \{\chi_1, \chi_2, \chi_3, \chi_4\}$.

An entirely similar argument applies when $4||(q + 1)$; here all groups $E_6(q)$ have to be replaced by ${}^2E_6(q)$ and all $(q - 1)$ by $(q + 1)$. □

To prove the iAM-condition for the block b, we also need to compute the action of group automorphisms on the height zero characters of its Brauer correspondent. The information given in [18] does not seem sufficient for this. Instead, we will try to obtain all the necessary local information from the invariants of the block b.

The following proposition is a consequence of [39, Proposition 10.26].

Proposition 14.4. *Let H be a group isomorphic $C_2 \times C_2$ or D_8, and for $n \geq 1$, consider a group extension*

$$1 \to H \to D \to C_{2^n} \to 1.$$

Then the invariants for every block of a finite group with defect group D are known.

Proof. We must show that the exceptions listed in [39, Proposition 10.26] do not occur if H is as in the statement of our proposition. The first assumption on the coupling ω in [39, Proposition 10.26] can be easily verified by examining the automorphism group structure of $C_2 \times C_2$ and D_8, respectively. We are therefore left to show that D is not isomorphic to $C_{2^m} \wr C_2$ for all $m \geq 3$. For $H \cong C_2 \times C_2$, this follows from the proof of [39, Proposition 10.26].

Assume therefore that $H \cong D_8$ and $H \lhd (C_{2^m} \times C_{2^m}) \rtimes C_2$. There exists an element $a \in H$ that is not contained in the base group $C_{2^m} \times C_{2^m}$. This element must act on the base group by interchanging both components. Let $g \in C_{2^m}$ be a generator. Then it follows that $a^{(g,1)} = (g, g^{-1})a$, so $(g, g^{-1}) \in D_8$. This implies that $m \leq 2$. □

Lemma 14.5. *Let b be a block of a finite group G with defect group D. Suppose that B is a block of a subgroup M of G containing $\mathrm{N}_G(D)$ with $B^G = b$. Let $A \subseteq \operatorname{Aut}(G)$ be a finite cyclic subgroup stabilising M. Assume that the Alperin–McKay conjecture holds for every block of $G \rtimes A$ and $M \rtimes A$ covering b and B, respectively.*

(a) *If A is a simple cyclic group or an ℓ-group, then the number of A-invariant characters in $\operatorname{Irr}_0(b)$ is equal to the number of A-invariant characters of $\operatorname{Irr}_0(B)$.*
(b) *We have $\operatorname{Irr}_0(b) = \operatorname{Irr}_0(b)^A$ if and only if $\operatorname{Irr}_0(B) = \operatorname{Irr}_0(B)^A$.*

Proof. This follows from the proof of [42, Lemma 8.1]. □

Proposition 14.6. *Let b be one of the blocks of $G = E_7(q)$ occurring in Lemma 14.1(b)(ii). Then b is AM-good relative to its defect group.*

Proof. Let $\mathbf{G}_{\mathrm{ad}}$ be the adjoint quotient of $\mathbf{G}$. There exists a Frobenius endomorphism F on $\mathbf{G}_{\mathrm{ad}}$ that commutes with the quotient map $\pi : \mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$. Then π induces an injective map $\pi : S \to G_{\mathrm{ad}}$, where $G_{\mathrm{ad}} := \mathbf{G}_{\mathrm{ad}}^F$ and G_{ad} induces all diagonal automorphisms on S. More precisely, we have $\tilde{G}/\mathrm{Z}(\tilde{G}) \cong G_{\mathrm{ad}}$.

Let D be a defect group of b and B be its Brauer correspondent in $\mathrm{N}_G(D)$. We first construct an $\operatorname{Aut}(G)_{b,D}$-equivariant bijection $\operatorname{Irr}_0(G, b) \to \operatorname{Irr}_0(\mathrm{N}_G(D), B)$. Let $\delta : G \to G$ be a diagonal automorphism induced by the action of an element of G_{ad}, and let $F_0 : G \to G$ be a generator of the group of field automorphisms of G that together generate $\operatorname{Out}(G)$. We first observe that the characters χ_1, χ_2 are $\langle \delta, F_0 \rangle$-stable by [25, Theorem 2.5]. Moreover, Proposition 14.3 together with [38, Theorem 2.11]

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implies that $\langle\delta, F_0\rangle_{\chi_i} = \langle F_0\rangle$ for $i = 3, 4$. In particular, the block b is $\langle\delta, F_0\rangle$-stable, and thus we can assume (by possibly replacing these automorphisms by a G-conjugate) that D is $\langle\delta, F_0\rangle$-stable.

We denote by c either block b or B. Moreover, we let $\overline{c}$ be the unique block of $S := G/Z(G)$, respectively $N_G(D)/Z(G)$, which is dominated by c. As in the proof of Proposition 14.3, we find that $|\operatorname{Irr}_0(c)| = 4$ and $|\operatorname{Irr}(\overline{c})| = |\operatorname{Irr}_0(\overline{c})| = 4$. Therefore, the quotient map induces a bijection $\operatorname{Irr}_0(c) \to \operatorname{Irr}(\overline{c})$. Note that the defect group $\overline{D} := D/Z(G)$ of the block c has a defect group isomorphic to $C_2 \times C_2$. According to Proposition 14.4, we can therefore apply Lemma 14.5 to the automorphisms δ and F_0. Using Proposition 14.3, we find that every character of $\operatorname{Irr}_0(B)$ is F_0-stable. Moreover, there exist two characters $\psi_1, \psi_2 \in \operatorname{Irr}_0(B)$ that are δ-stable, and the other two characters $\psi_3, \psi_4 \in \operatorname{Irr}_0(B)$ are δ-conjugate. We therefore obtain that the bijection $\operatorname{Irr}_0(b) \to \operatorname{Irr}(B)$, $\chi_i \mapsto \psi_i$, is $\langle\delta, F_0\rangle$-equivariant.

We now show that the characters of $\operatorname{Irr}_0(\overline{b})$ and $\operatorname{Irr}_0(\overline{B})$ extend to their inertia groups in $G_{\mathrm{ad}}\langle F_0\rangle$ and $N_{G_{\mathrm{ad}}\langle F_0\rangle}(\overline{D})$, respectively. Since χ_1, χ_2 are unipotent characters, it follows that they extend to $G_{\mathrm{ad}}\langle F_0\rangle$; see, for example, [25, Theorem 2.4]. For $i = 3, 4$ the stabiliser quotient $(G_{\mathrm{ad}}\langle F_0\rangle)_{\chi_i}/S$ of χ_i is cyclic, so χ_i extends to its inertia group in $G_{\mathrm{ad}}\langle F_0\rangle$. Similarly, for $i = 3, 4$ the local character ψ_i also extends to its inertia group in $N_{G_{\mathrm{ad}}\langle F_0\rangle}(\overline{D})$. It is therefore left to show that ψ_1 and ψ_2 extend to $G_{\mathrm{ad}}\langle F_0\rangle$ as well. We let b_2 be a block of $G_{\mathrm{ad}}\langle F_0\rangle$ covering $\overline{b}$. Let D_2 be a defect group of b_2 such that $D_2 \cap S = \overline{D}$. We observe that $D_1 := D_2 \cap G_{\mathrm{ad}}$ is a defect group of the unique block b_1 of G_{ad} covering $\overline{b}$. Since Brauer's height zero conjecture holds for blocks with defect group of order 8 (see, e.g., [39, Theorem 13.1] and [39, Theorem 8.1]), it follows that D_1 is nonabelian. Since $|\operatorname{Irr}(b_1)| = 5$, we must have $D_1 \cong D_8$; see for instance [39, Theorem 8.1]. From this, it follows that D_2 is a cyclic extension of D_8. Let B_1 be the Harris–Knörr correspondent of b_1 in $N_{G_{\mathrm{ad}}}(\overline{D})$. Again, Proposition 14.4 ensures that Lemma 14.5 is applicable to the automorphism F_0. Since every character of $\operatorname{Irr}_0(b_1)$ is F_0-stable, the same is true for every character of $\operatorname{Irr}_0(B_1)$. Since $\operatorname{Irr}_0(B_1)$ consists of the four extensions of ψ_1 and ψ_2 to G_{ad}, it follows that ψ_1 and ψ_2 must necessarily extend to $G_{\mathrm{ad}}\langle F_0\rangle$.

It is now easy to show that the above information is enough to verify the inductive conditions. For this, we observe that we can choose extensions $\hat{\chi}_i \in \operatorname{Irr}((G_{\mathrm{ad}}\langle F_0\rangle)_{\chi_i})$ of χ_i and extensions $\hat{\psi}_i \in \operatorname{Irr}((N_{G_{\mathrm{ad}}\langle F_0\rangle}(\overline{D}))_{\psi_i})$ of ψ_i lying in Harris–Knörr corresponding blocks. It follows from [43, Lemma 2.15] and [43, Proposition 4.4] that the inductive Alperin–McKay condition (in the version of [43, Definition 4.2]) is satisfied for the block b of G. □

14.3. The inductive Alperin–McKay condition for $\mathrm{SL}_2(\mathbb{F}_q)$

Proposition 14.7. *The iAM-condition holds for the principal 2-block of* $\mathbf{G}^F = \mathrm{SL}_2(\mathbb{F}_q)$ *relative to its defect group.*

Proof. By [6], we can assume that $\mathbf{G}^F$ has nonexceptional Schur multiplier. We follow the proof (and notation) of Proposition 14.6. Using [11, Theorem 21.14], it's easy to see that the principal block b is the unique 2-block of maximal defect. Hence, $\operatorname{Irr}_0(b) = \operatorname{Irr}_{2'}(G)$ and $\operatorname{Irr}_0(B) = \operatorname{Irr}_{2'}(N_G(D))$, where B is the Brauer correspondent of b.

We place ourselves in the situation of [19, Section 15], where it was shown that $S = G/Z(G)$ is McKay-good. More precisely, it was shown that there exist an intermediate subgroup $N_G(D) \subset H$ and an $\mathrm{Aut}(G)_H$-equivariant bijection $\operatorname{Irr}_{2'}(G) \to \operatorname{Irr}_{2'}(H)$. Modifying their bijection, we obtain an $\mathrm{Aut}(G)_D$-equivariant bijection $\operatorname{Irr}_0(G, b) \to \operatorname{Irr}_0(N_G(D), B)$. This bijection preserves central characters since every $2'$-character of G, respectively $N_G(D)$, has the 2-group $Z(G)$ in its kernel. One then checks that all characters of $\operatorname{Irr}_0(S, \overline{b})$ extend to their inertia group in $G_{\mathrm{ad}}\langle F_0\rangle$ (the stabiliser is either cyclic or the characters are unipotent). Similarly, one also checks that the characters of $\operatorname{Irr}_0(N_S(\overline{D}), \overline{B})$ extend to their inertia group: If D is self-normalising, then every character in $\operatorname{Irr}_0(N_S(\overline{D}), \overline{B})$ is linear, and one can check the claim by explicit computations. Otherwise, both characters of $\operatorname{Irr}_0(N_G(D), B)$ with noncyclic inertia group in $\mathrm{Out}(G)$ are unipotent characters of $N_G(D) \cong \mathrm{SL}_2(3)$, and their extension to the inertia group follows again from explicit computations. Now, using similar arguments as in Proposition 14.6 shows that the block b is AM-good. □

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14.4. Groups with exceptional Schur multiplier

After having checked the iAM-condition for all quasi-isolated 2-blocks of groups of Lie type with almost abelian defect group in nondefining characteristic, we are left to show the iAM-condition for the remaining simple groups and blocks under consideration.

Proposition 14.8. *Suppose that S is a finite simple nonabelian group that is not a group of Lie type in odd characteristic with generic Schur multiplier. Then the inductive Alperin–McKay condition holds for all 2-blocks with almost abelian defect group of the universal covering group X of S.*

Proof. Observe first that the iAM-condition holds for all simple sporadic groups by [6]. Assume that S is a simple group of Lie type in characteristic different from 2 with exceptional Schur multiplier, and let X be its universal covering group. Then S is one of the following groups: $A_1(9)$, ${}^2A_3(3)$, $B_3(3)$ or $G_2(3)$. According to [6], both $A_1(9) \cong A_6$ and $G_2(3)$ are AM-good. Observe that the blocks of the universal covering groups of ${}^2A_3(3)$ and $B_3(3)$ with maximal defect have a defect group that is not almost abelian. Recall that every defect group of X contains $\mathrm{Z}(X)_\ell$. An inspection of the blocks of the universal covering group of ${}^2A_3(3)$ using [7] now shows that all 2-blocks with almost abelian defect groups have central defect. Therefore, there is nothing to check in this case. For the universal covering group of $B_3(3)$, the same arguments easily rule out all blocks except the blocks denoted by 2, 3, 8 and 9 in [7].

We claim that the blocks 2, 8 and 9 of X have a defect group that is not almost abelian. For this, let b be any of these blocks. Consider a central extension $X \to X'$ whose kernel is of order 2. According to [33, Theorem 9.10], we obtain a bijection $b \to \overline{b}$ between blocks of X and X'. The block of X' corresponding to b under this bijection has, according to [33, Theorem 9.10], a defect group of order 8 and still two Brauer characters. The groups C_8 and $C_2 \times C_4$ admit no automorphisms of odd order. By the remarks following [39, Theorem 1.30], we therefore deduce that these groups arise as defect groups of nilpotent blocks only. Moreover, any block with defect group $C_2 \times C_2 \times C_2$ can't have exactly two Brauer characters; see [39, Theorem 13.1]. Since the block $\overline{b}$ is not nilpotent (it has 2 Brauer characters), its defect group is therefore not abelian. Since X' is a 3-cover of S, we deduce that the block b cannot have an almost abelian defect group.

It therefore remains to consider the block labeled 3 in [7]. This block can, however, be considered a block of the 2-cover of S – that is, as a block of $\mathbf{G}^F$ – and can therefore be treated as a block with nonexceptional covering group. Using the reduction theorem in [38], we can thus conclude that this block is AM-good.

For alternating groups, the inductive AM condition is known to hold by the main result of [14] and [42, Corollary 8.3]. By [26], the iAM-condition holds for Suzuki and Ree groups.

Finally, let us assume that S is a simple group of Lie type defined over a field of characteristic 2. Let $\mathbf{G}^F$ be such that $\mathbf{G}^F/\mathrm{Z}(\mathbf{G}^F) \cong S$. As before, let X be the universal covering group of S. Then there exists a surjective homomorphism $X \to \mathbf{G}^F$ whose kernel is the Sylow 2-group Z of $\mathrm{Z}(\hat{G})$. According to [33, Theorem 9.10], we obtain a bijection $B \to \overline{B}$ between blocks of $\hat{G}$ and $\mathbf{G}^F$ that maps blocks with almost abelian defect to each other.

It is known (see [11, Theorem 6.18]) that the only 2-blocks of $\mathbf{G}^F$ are blocks of maximal defect and blocks of height zero. Therefore, we must consider the cases where a Sylow 2-subgroup of S is abelian. However, this is precisely the case when $S \cong A_1(\pm 2^f)$. Again by the work of Breuer [6], we know that S is AM-good whenever S has an exceptional Schur multiplier. Using the properties of the bijection constructed in [41] in conjunction with [12, Theorem 4.1] shows that the principal block of S is AM-good. □

Theorem 14.9. *Suppose that S is a finite simple group of Lie type in characteristic unequal to 2 with nonexceptional Schur multiplier. Then the inductive Alperin–McKay condition holds for all 2-blocks with almost abelian defect group of the universal covering group X of S.*

Proof. Let $\mathbf{G}$ be a simple, simply connected algebraic group and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism such that $S = \mathbf{G}^F/\mathrm{Z}(\mathbf{G}^F)$. We fix a 2-block b of $\mathbf{G}^F$ with an almost abelian defect group. We want to show that b is AM-good relative to its defect group.

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Using the proof of [38, Theorem 3.14], we see that the statement of [38, Theorem 3.14] can be adapted to our situation as follows: If [38, Hypothesis 3.3] holds for all blocks with almost abelian defect groups, then the block b is AM-good. In other words, we can assume that the 2-block b of $\mathbf{G}^F$ is strictly quasi-isolated. According to Lemma 14.1, all these blocks are unipotent unless $\mathbf{G}$ is of type A. Suppose therefore that $\mathbf{G}$ is of type A and $s \neq 1$. The assumptions of Lemma 11.2 apply. We can therefore use the proof of Theorem 12.4 to show that we can also assume in this case that the block is unipotent. It therefore suffices to check that the iAM-condition holds for the unipotent blocks occurring in Lemma 14.1. Using Remark 14.2, we see that this was checked in Proposition 14.6 and Proposition 14.7. □

Theorem 14.10. *The Alperin–McKay conjecture holds for all* 2*-blocks of almost abelian defect.*

Proof. According to Proposition 13.3, to show the theorem, it is sufficient to prove the iAM-condition for all ℓ-blocks with almost abelian defect of the universal covering group of a simple finite group. The iAM-condition in these cases has been verified in Theorem 14.9 and Proposition 14.8. □

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