It was thus suggested that the standard equation of the tangent or polar, viz.,

$$xx' + yy' + g(x + x') + f(y + y') + c = 0$$

could be written

$$(x - x')^2 + (y - y')^2 = x^2 + y^2 + 2gx + 2fy + c$$
$$+ x'^2 + y'^2 + 2gx' + 2fy' + c,$$

which reduces to (1) if (x', y') is on the circle. This form of the equation shows that the polar of a point P, (x', y') with respect to a circle is the locus of a point Q, (x, y), which moves so that PQ^2 is equal to the sum of the powers of P and Q with respect to the circle. This well-known property of the polar can be very easily proved, for, from the triangle PCQ (Fig. 2), we have

$$\begin{aligned} PQ^2 &= CP^2 + CQ^2 - 2CP \,.\, CP' \\ &= CP^2 + CQ^2 - 2r^2 \\ &= (CP^2 - r^2) + (CQ^2 - r^2) \\ &= \text{sum of the powers of P and Q.} \end{aligned}$$

Thus, reversing the above steps, we have an easy geometrical way of obtaining the equation of the polar.

R. J. T. BELL.

Two Theorems on Determinants, and their Application to the Proof of the Volume-Formula for Tetrahedra —*Theorem I.* If to each of the elements of any row of a determinant there be added the same fraction of the difference between the corresponding elements in two other rows, the value of the determinant is unaltered. This is a case of a well known elementary theorem.

Theorem II. If to each of the elements of two rows of a determinant there be added the same fraction of the difference between the corresponding elements in these rows, the value of the determinant is unaltered.

For

$$\begin{vmatrix} a_1 + \lambda(a_2 - a_1), & b_1 + \lambda(b_2 - b_1), & c_1 + \lambda(c_2 - c_1) \\ a_2 + \lambda(a_2 - a_1), & b_2 + \lambda(b_2 - b_1), & c_2 + \lambda(c_2 - c_1) \\ a_3 \quad , & b_3 \quad , & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + \lambda(a_2 - a_1), & \ldots\ldots, & \ldots\ldots \\ a_2 - a_1 \quad , & b_2 - b_1, & c_1 - c_1 \\ a_3 \quad , & b_3 \;, & c_3 \end{vmatrix}$$

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$$= \begin{vmatrix} a_1 \;, & b_1 \;, & c_1 \\ a_2 - a_1, & b_2 - b_1, & c_2 - c_1 \\ a_3 \;, & b_3 \;, & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

and a like proof obviously applies to a determinant of any order.

Now take the determinant $\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_0 & y_0 & z_0 & 1 \end{vmatrix}$ and suppose x_0, y_0, z_0, etc., the coordinates of four non-coplanar points P_0, P_1, P_2, P_3. Then the change referred to in Theorem I. will occur if *one* vertex is displaced parallel to an opposite side. The change referred to in Theorem II. will occur when any edge of the tetrahedron, remaining of constant length, is displaced any distance along its own line. Neither of these transformations alters the volume of the tetrahedron in magnitude or in sense.

We note further that by two displacements of the former kind, a vertex being displaced successively parallel to two sides of the opposite face, the vertex can be brought to any point in the plane through it parallel to the opposite face.

If then we are given a tetrahedron $P_1P_2P_3P_0$ and a set of three non-coplanar concurrent straight lines OX, OY, OZ, we can by a series of displacements of the kind described transform $P_1P_2P_3P_0$ into a tetrahedron having three concurrent edges along OX, OY, OZ.

For there must be one at least of the three axes not parallel to the plane $P_1P_2P_3$. Let OX be that axis.

Let the plane through P_0 parallel to $P_1P_2P_3$ meet OX in P_0'.

" " " " P_1 " " $P_0'P_2P_3$ " " " P_1'.

Let OP_1'' be equal to $P_0'P_1'$ in magnitude and have the same sense.

Let the plane through P_2 parallel to $OP_1''P_3$ meet OY in P_2'.

" " " " P_3 " " $OP_1''P_2'$ " OZ in P_3'.

Thus we have the tetrahedron $P_1''P_2'P_3'O$ equal in volume to the tetrahedron $P_1P_2P_3P_0$ and having the same value for the associated determinant.

But the determinant of the former is

$$\begin{vmatrix} x_1'' & 0 & 0 & 1 \\ 0 & y_2' & 0 & 1 \\ 0 & 0 & z_3' & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = + x_1'' y_2' z_3'$$

and its volume is obviously equal to $x_1'' y_2' z_3'$ times that of a "unit tetrahedron" ABCO, when OA is measured along OX, OB along OY, and OC along OZ, each of unit length.

Hence $\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_0 & y_0 & z_0 & 1 \end{vmatrix}$ gives the ratio of the volume of $P_1P_2P_3P_0$ to that of the same "unit tetrahedron" in magnitude and sense.

This reasoning can be applied also in two dimensions to the area of a triangle, and to the content of a simplex in n dimensions.

Remark.—The number of steps required in transforming the tetrahedron can be reduced to three, and Theorem II. is not required if we start with P_0 coincident with O.

R. F. MUIRHEAD.

A Note on Induction.—An application of the method of Induction to the proof of a general result or theorem should involve two steps: (i) the discovery of the result to be proved by the consideration of particular cases; (ii) a proof, by Mathematical Induction, or otherwise, that the result, found to be true in particular cases, is universally true.

The first step is often omitted, the pupil being supplied with the result and asked to show that it is universally true. A most valuable part of the work is thereby lost; to find a formula from the examination of particular cases demands careful observation and a power of generalising which should be cultivated. In fact, step (i) by itself, apart from step (ii), is an excellent exercise. It is, of course, in many cases, extremely difficult, if not impossible, for