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ON THE DISTRIBUTION OF ZEROS OF THE DERIVATIVE OF SELBERG'S ZETA FUNCTION ASSOCIATED TO FINITE VOLUME RIEMANN SURFACES

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Abstract. We study the distribution of zeros of the derivative of the Selberg zeta function associated to a noncompact, finite volume hyperbolic Riemann surface M. Actually, we study the zeros of $(Z_M H_M)'$, where Z_M is the Selberg zeta function and H_M is the Dirichlet series component of the scattering matrix, both associated to an arbitrary finite volume hyperbolic Riemann surface M. Our main results address finiteness of number of zeros of $(Z_M H_M)'$ in the half-plane $\mathrm{Re}(s) < 1/2$, an asymptotic count for the vertical distribution of zeros, and an asymptotic count for the horizontal distance of zeros. One realization of the spectral analysis of the Laplacian is the location of the zeros of Z_M, or, equivalently, the zeros of $Z_M H_M$. Our analysis yields an invariant A_M which appears in the vertical and weighted vertical distribution of zeros of $(Z_M H_M)'$, and we show that A_M has different values for surfaces associated to two topologically equivalent yet different arithmetically defined Fuchsian groups. We view this aspect of our main theorem as indicating the existence of further spectral phenomena which provides an additional refinement within the set of arithmetically defined Fuchsian groups.

§1. Introduction

1.1 Selberg zeta functions for compact Riemann surfaces

In [22], Luo initiated the study of the nontrivial zeros of the derivative Z'_M of the Selberg zeta function Z_M associated to a compact, hyperbolic Riemann surface M, proving analogues of results obtained by Spira [27] and Berndt [4] for the Riemann zeta function. Further refinements of results by Luo were established in [11] and [12]. As is standard in analytic number theory, the nontrivial zeros of Z_M, or Z'_M, are its zeros which do not arise from the poles of the multiplicative factor of the functional equation. In the case of a compact Riemann surface, the nontrivial zeros of Z_M and Z'_M are zeros different from negative integers. Let us summarize the three

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main results stemming from the aforementioned articles, which are location, vertical distribution, and an asymptotic count for the weighted vertical distribution of the nontrivial zeros of Z'_M.

In [22] it is shown that $Z'_M(s)$ has at most a finite number of nontrivial zeros in the half-plane $\mathrm{Re}(s) < 1/2$. This result was strengthened in [24] and [25] where it is proved that $Z'_M(s)$ has no nontrivial zeros in the half-plane $\mathrm{Re}(s) < 1/2$.

Let $\mathrm{vol}(M)$ denote the hyperbolic volume of M. Let $\ell_{M,0}$ be the length of the shortest closed geodesic on M. Let $m_{M,0}$ denote the number of inconjugate geodesics whose length is $\ell_{M,0}$. Let $N_{\mathrm{ver}}(T; Z'_M)$ be the number of nontrivial zeros of $Z'_M(s)$ where $s = \sigma + it$ with $\sigma \geqslant 1/2$ and $0 < t < T$, and let

$$N_{\mathrm{w}}(T; Z'_M) = \sum_{\substack{Z'_M(\sigma+it)=0 \\ 0<t<T, \sigma>1/2}} (\sigma - 1/2)$$

be the weighted vertical distribution with weights equal to distances of zeros to the critical line. Then, building on the results form [22], it is proved in [11] and [12] that

$$N_{\mathrm{ver}}(T; Z'_M) = \frac{\mathrm{vol}(M)}{4\pi} T^2 - \frac{\ell_{M,0}}{2\pi} T + o(T) \quad \text{as } T \to \infty, \tag{1}$$

and

$$N_{\mathrm{w}}(T; Z'_M) = \frac{T}{2\pi} \log T + \frac{T}{2\pi} \left(\frac{1}{2} \ell_{M,0} + \log \left(\frac{\mathrm{vol}(M)(1 - e^{-\ell_{M,0}})}{m_{M,0} \ell_{M,0}} \right) - 1 \right) + o(T) \quad \text{as } T \to \infty. \tag{2}$$

The study of the zeros of Z'_M is of particular interest because of the connection with spectral analysis. Recall that if s is a nontrivial zero of $Z_M(s)$, then $\lambda = s(1-s)$ is an eigenvalue of an L^2-eigenfunction of the hyperbolic Laplacian which acts on the space of smooth functions on M. Common zeros of Z_M and Z'_M are, in fact, zeros of $Z_M(s)$ with multiplicity greater than one. Such zeros of Z_M correspond to multi-dimensional eigenspaces of the Laplacian. As shown in [22, p. 1143], all zeros of $Z'_M(s)$ on the line $\mathrm{Re}(s) = 1/2$, except possibly at $s = 1/2$, correspond to multiple zeros of Z_M. The problem of obtaining nontrivial bounds for the dimension of eigenspaces of the Laplacian is very difficult; see [17, p. 160]. Thus, it is possible that refined information regarding (1) possibly could shed light on this important, outstanding question.

1.2 Noncompact Riemann surfaces

Let $\mathbb{H}$ denote the hyperbolic upper half-plane. Let $\Gamma \subseteq \mathrm{PSL}(2, \mathbb{R})$ be any Fuchsian group of the first kind acting by fractional linear transformations on $\mathbb{H}$, and let M be the quotient space $\Gamma\backslash\mathbb{H}$.

One realization of the spectral analysis of the Laplacian on the surface M is the location of nontrivial zeros of the associated Selberg zeta function Z_M, defined for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ by the Euler product

$$Z_M(s) = \prod_{n=0}^{\infty} \prod_{P_0 \in \mathcal{H}(\Gamma)} \left(1 - e^{-(s+n)\ell_{P_0}}\right) = \prod_{n=0}^{\infty} \prod_{P_0 \in \mathcal{H}(\Gamma)} \left(1 - N(P_0)^{-(s+n)}\right). \tag{3}$$

Here $\mathcal{H}(\Gamma)$ denotes a complete set of representatives of inconjugate, primitive hyperbolic elements of Γ, P_0 is a primitive hyperbolic element, ℓ_{P_0} is the hyperbolic length of the geodesic path in the homotopy class determined by P_0 and the norm $N(P_0)$ is equal to $\exp(\ell_{P_0})$.

The function Z_M possesses a meromorphic continuation to the whole complex plane and satisfies the functional equation $Z_M(s)\phi_M(s) = \eta_M(s)Z_M(1-s)$, where $\phi_M(s)$ denotes the determinant of the scattering matrix $\Phi_M(s)$,

$$\begin{aligned} \frac{\eta_M'}{\eta_M}(s) &= \mathrm{vol}(M)(s-1/2)\tan(\pi(s-1/2)) - \pi \\ &\quad \cdot \sum_{\substack{\{R\} \\ 0<\theta(R)<\pi}} \frac{1}{M_R \sin\theta} \frac{\cos(2\theta-\pi)(s-1/2)}{\cos\pi(s-1/2)} \\ &\quad + 2n_1 \log 2 + n_1\left(\frac{\Gamma'}{\Gamma}(1/2+s) + \frac{\Gamma'}{\Gamma}(3/2-s)\right) = \frac{\eta_M'}{\eta_M}(1-s) \end{aligned} \tag{4}$$

and where $\{R\}$ denotes a complete, finite set of inconjugate elliptic elements of Γ so that $0 < \theta(R) < \pi$ is the uniquely determined real number such that R is conjugate to the matrix

$$\begin{pmatrix} \cos\theta(R) & -\sin\theta(R) \\ \sin\theta(R) & \cos\theta(R) \end{pmatrix}.$$

The scattering determinant $\phi_M(s)$ has a decomposition into a product of a general Dirichlet series and Gamma functions. Specifically, we can write

$$\phi_M(s) = \pi^{n_1/2}\left(\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\right)^{n_1} \sum_{n=1}^{\infty} \frac{d(n)}{\mathfrak{g}_n^{2s}}$$

where n_1 is the number of cusps of M, and $\{d(n)\}$ and $\{\mathfrak{g}_n\}$ are sequences of real numbers with

$$0 < \mathfrak{g}_1 < \cdots < \mathfrak{g}_n < \mathfrak{g}_{n+1} < \cdots ;$$

given in terms of Kloosterman sums (see [17, p. 160]). Let us write $\phi_M(s) = K_M(s) \cdot H_M(s)$ where

$$K_M(s) = \pi^{n_1/2} \left(\frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \right)^{n_1} e^{c_1 s + c_2}, \quad \text{with}$$
$$c_1 = -2 \log \mathfrak{g}_1 \qquad \text{and} \qquad c_2 = \log d(1), \tag{5}$$

and

$$H_M(s) = 1 + \sum_{n=2}^{\infty} \frac{a(n)}{r_n^{2s}} \quad \text{with } r_n = \mathfrak{g}_n/\mathfrak{g}_1 > 1 \qquad \text{and} \qquad a(n) = d(n)/d(1). \tag{6}$$

The Dirichlet series expansion for $H_M(s)$ converges for all $\text{Re}(s) > 1$. We call the function H_M the *Dirichlet series portion of the scattering determinant* ϕ_M. In general, the function H_M can be expressed as the determinant of a matrix whose entries are general Kloosterman sums; see [17, Theorem 3.4]. The constants $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are explained in terms of the left lower entries of the matrices appearing in the double coset decomposition of Γ. Therefore, the constants $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are precisely connected to the Fuchsian group Γ and $H_M(s)$ is a Dirichlet series carrying the information related to parabolic subgroups of Γ.

By nontrivial zeros of $Z_M(s)$ we mean all nonreal zeros and real zeros at points $s \in [0, 1]$ such that $s(1 - s)$ is equal to an eigenvalue of the Laplacian that is less than or equal to $1/4$. The nontrivial zeros are related to the spectrum of the Laplacian in the sense that, according to [14, Theorem 5.3], the nontrivial zeros of Z_M are located at points of the form $1/2 \pm ir_n$ where $1/4 + r_n^2$ is a discrete eigenvalue of the Laplacian and at points $1 - \rho$ in the half-plane $\text{Re}(s) < 1/2$ which are poles of the determinant ϕ_M of the scattering matrix.

We may conclude that the function $Z_M H_M$ is “spectrally equivalent” to Z_M in the following sense: The function $Z_M H_M$ can be represented as a general Dirichlet series converging in the half-plane $\text{Re}(s) > 1$ and carrying information about the underlying group Γ; it possesses a meromorphic

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continuation to the entire complex plane satisfying the functional equation

$$(Z_M H_M)(s) = \eta_M(s) K_M^{-1}(s) Z_M(1-s); \tag{7}$$

and its nontrivial zeros are at points $s = 1/2 \pm i r_n$, where $1/4 + r_n^2$ is a discrete eigenvalue of the Laplacian and at points $s = \rho$ in the half-plane $\mathrm{Re}(s) > 1/2$ which are zeros of the determinant ϕ_M of the scattering matrix. Based on this argument, we may, loosely speaking, say that $Z_M H_M$ carries the same amount of spectral information as Z_M does. Besides that, the function $Z_M H_M$ has no nontrivial zeros in the half-plane $\mathrm{Re}(s) < 1/2$.

The question of studying the zeros of Z'_M when M is not compact by applying methods presented in this paper begins with one possible technical difficulty stemming from the fact that the function Z_M has an infinite number of zeros in the half-plane $\mathrm{Re}(s) < 1/2$, each one of which would produce a negative weight in the weighted counting function N_{w}. On the other hand, the function $Z_M H_M$ has no nontrivial zeros in the half-plane $\mathrm{Re}(s) < 1/2$ and carries the same spectral information as Z_M does. Therefore, as a result we shall study the zeros of $(Z_M H_M)'$.

As we shall see below, the choice of $Z_M H_M$ instead of Z_M is further justified by the fact that, according to the statement (a) of the Main Theorem, the only zeros of $(Z_M H_M)'$ on the critical line (with imaginary part greater than some constant depending upon the group) are the multiple zeros of Z_M; therefore, the study of zeros of $(Z_M H_M)'$ is related to the problem of obtaining bounds for the dimension of eigenspaces of discrete eigenvalues of the Laplacian on M.

In conclusion, we followed the guide provided by the technical issues we faced and chose to study the zeros of $(Z_M H_M)'$. To be specific, we viewed the positivity issues described above as important, thus we focused our attention on the zeros of $(Z_M H_M)'$. Nonetheless, the problem of studying the zeros of Z'_M is both well-posed and remains open. It is quite possible that a successful study of the zeros of Z'_M when combined with the results of the present paper would yield interesting results. We leave such a study to a motivated reader.

1.3 The main result

The function H'_M/H_M admits the general Dirichlet series expansion

$$\frac{H'_M}{H_M}(s) = \sum_{i=1}^{\infty} \frac{b(q_i)}{q_i^s}, \tag{8}$$

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where the series on the right converges absolutely and uniformly for $\mathrm{Re}(s) \geqslant \sigma_0 + \epsilon > \sigma_0 \gg 0$, and where $\{q_i\}$ is a nondecreasing sequence of positive real numbers consisting of all finite products of numbers $r_n^2 > 1$. Obviously, $q_2 > q_1 = \inf q_i = (\mathfrak{g}_2/\mathfrak{g}_1)^2$. Furthermore,

$$b(q_1) = -a(2)\log q_1 = -2(d(2)/d(1))\log(\mathfrak{g}_2/\mathfrak{g}_1).$$

Let $\ell_{M,0}$ be the length of a shortest closed geodesic, or systole, on M. With our notation from above, let

$$A_M = \min\left\{e^{\ell_{M,0}}, (\mathfrak{g}_2/\mathfrak{g}_1)^2\right\}. \tag{9}$$

Here, we have dropped the subscript M on $(\mathfrak{g}_2/\mathfrak{g}_1)^2$ in order to ease the notation; however, it is clear that $(\mathfrak{g}_2/\mathfrak{g}_1)^2$ depends on M. Let $m_{M,0}$ denote the number of inconjugate closed geodesics on M with length $\ell_{M,0}$. If $e^{\ell_{M,0}} \neq (\mathfrak{g}_2/\mathfrak{g}_1)^2$, let

$$a_M = \begin{Bmatrix} \dfrac{m_{M,0}\ell_{M,0}}{1-e^{-\ell_{M,0}}}; & \text{if } e^{\ell_{M,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2 \\ b((\mathfrak{g}_2/\mathfrak{g}_1)^2); & \text{if } e^{\ell_{M,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2 \end{Bmatrix}. \tag{10}$$

If $e^{\ell_{M,0}} = (\mathfrak{g}_2/\mathfrak{g}_1)^2$, let

$$a_M = \frac{m_{M,0}\ell_{M,0}}{1-e^{-\ell_{M,0}}} + b((\mathfrak{g}_2/\mathfrak{g}_1)^2). \tag{11}$$

Observe that a_M is the sum of the two terms which appear in the two cases in (10), not the arithmetic average as one would expect from elementary Fourier analysis.

With all this, the main result of this article is the following.

MAIN THEOREM. *Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ be any Fuchsian group of the first kind acting by fractional linear transformations on $\mathbb{H}$, and let M be the quotient space $\Gamma\backslash\mathbb{H}$. Let $Z_M(s)$ be the associated Selberg zeta function, and $H_M(s)$ be the Dirichlet series portion of the determinant of the associated scattering matrix.*

(a) *There are a finite number of nontrivial zeros of $(Z_M H_M)'(s)$ in the half-plane $\mathrm{Re}(s) < 1/2$. In addition, there exist some $t_0 > 0$ such that any zero of $(Z_M H_M)'(s)$ on the line $\mathrm{Re}(s) = 1/2$ with property $|\,\mathrm{Im}(s)| > t_0$ arises from a multiple zero of $Z_M(s)$.*

(b) *Let us define the vertical counting function*

$$N_{\mathrm{ver}}(T;(Z_M H_M)') = \#\{\rho=\sigma+it \mid (Z_M H_M)'(\rho)=0 \text{ with } 0<t<T\}.$$

Then

$$N_{\mathrm{ver}}(T;(Z_M H_M)') = \frac{\mathrm{vol}(M)}{4\pi}T^2 - \frac{T}{2\pi}(\log A_M + 2n_1\log 2 + 2\log \mathfrak{g}_1) + o(T), \quad \text{as } T\to\infty.$$

In particular, if M is co-compact, then (1) holds true.

(c) *Let us define the weighted vertical counting function*

$$N_{\mathrm{w}}(T;(Z_M H_M)') = \sum_{\substack{(Z_M H_M)'(\sigma+it)=0 \\ 0<t<T \text{ and } \sigma>1/2}} (\sigma - 1/2).$$

Then

$$N_{\mathrm{w}}(T;(Z_M H_M)') = \left(\frac{n_1}{2}+1\right)\frac{T\log T}{2\pi} + \frac{T}{2\pi}\left(\log\frac{\mathrm{vol}(M)A_M^{1/2}}{|a_M|} - 1\right) + \frac{T}{2\pi}\left(\log\left(\frac{\mathfrak{g}_1}{\pi^{n_1/2}|\,d(1)|}\right) - \frac{n_1}{2}\right) + o(T), \quad \text{as } T\to\infty.$$

In particular, if M is co-compact, then (2) holds true.

As stated in the Main Theorem, the above asymptotic formulas specialize in the case M is compact to give the main results in [11, 12, 22, 24] and [25]. More precisely, in [24] and [25] it is proved that Z'_M in the compact case possesses *no* nonreal zeros in the half-plane $\mathrm{Re}(s) < 1/2$, a statement which we believe to hold true for $(Z_M H_M)'$.

Similar results for the zeros of higher derivatives of $Z_M H_M$ are presented in a later section. In addition, corollaries of the main theorem, analogous to results from [21], are derived.

Since the proof of the Main Theorem is rather technical, let us present here a summary of the ideas involved in its proof.

Part (a) of the Main Theorem is proved in two parts. First, we employ the functional equation for the Selberg zeta function together with a bound for the growth of the logarithmic derivative $D_M(s) = Z'_M(s)/Z_M(s)$ in the right half of the critical strip in order to deduce that $\mathrm{Re}(Z_M H_M)'(s) \neq 0$ for sufficiently large $\mathrm{Im}(s)$ in the half-plane $\mathrm{Re}(s) < 1/2$. However, this method has a critical line as its boundary. Therefore, in order to show that all (but eventually finitely many) multiple zeros of Z_M on the critical line are also zeros of $(Z_M H_M)'$, we employ the Hadamard product representation of the completed Selberg zeta function, which was proved in [7], and conduct careful analysis of the imaginary part of the logarithmic derivative of $(Z_M H_M)'$.

Parts (b) and (c) of the Main Theorem are proved by an application of Littlewood's theorem [29, p. 132] to the function

$$X_M(s) := \frac{A_M^s}{a_M}(Z_M H_M)'(s) \tag{12}$$

followed by a careful technical analysis of the integrals obtained. There are two main difficulties appearing in the noncompact case. The first one is to control the growth of $D_M(s)$ inside the critical strip, which is resolved by an application of Theorem 5 below. What remains is the second technical point, which is to study the growth of $\arg X_M(\sigma + iT)$, for large T and $\sigma \in (a, \sigma_0)$, where $a \in (0, 1/2)$ is an arbitrary constant. In order to address this problem, we prove a Phragmen–Lindelöf type bound for $(Z_M H_M)(s)$ inside the strip $-\sigma_2 \leqslant -1 \leqslant \mathrm{Re}(s) \leqslant \sigma_0$ and the Lindelöf type bound for $(Z_M H_M)(s)$ for $\mathrm{Re}(s)$ close to $1/2$. These bounds are necessary in order to apply the generalized Backlund equivalent for the Lindelöf hypothesis (see § 2.5) which will yield a sharp bound for $(Z_M H_M)'(s)$ near the critical line. The resulting estimate enables one to apply Jensen's theorem and deduce that $\arg X_M(\sigma + iT) = o(T)$, as $T \to \infty$.

1.4 Properties of the invariant A_M

Aspects of the spectral analysis of the Laplacian acting on smooth functions on a hyperbolic Riemann surface can be measured by studying the zeros of the Selberg zeta function. As discussed above, one equivalently can study the zeros of $Z_M H_M$. Therefore, by slight extension, the zeros of $(Z_M H_M)'$ provide another measure of the spectral analysis of the Laplacian. In this regard, the quantity $(\mathfrak{g}_2/\mathfrak{g}_1)^2$ is a new spectral invariant. In addition, our Main Theorem asserts that for any given surface, the spectral analysis depends on the comparison of $e^{\ell_{M,0}}$ and $(\mathfrak{g}_2/\mathfrak{g}_1)^2$.

In § 7, we consider various arithmetic groups and compare $e^{\ell_{M,0}}$ to $(\mathfrak{g}_2/\mathfrak{g}_1)^2$. More precisely, we prove the following proposition.

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PROPOSITION 1.

(i) *For all surfaces* $M = \Gamma \setminus \mathbb{H}$, *where* Γ *is a congruence subgroup or principal congruence subgroup of the group* $\mathrm{PSL}(2,\mathbb{Z})$ *we have that* $A_M = e^{\ell_{M,0}}$.

(ii) *For the surface* M_5 *corresponding to the arithmetic group* $\Gamma_0^+(5)$ *we have that* $A_{M_5} = (\mathfrak{g}_2/\mathfrak{g}_1)^2 = ((1+\sqrt{5})/2)^2$. *For the surface* M_6 *corresponding to the arithmetic group* $\Gamma_0^+(6)$, *which has the same signature as* $\Gamma_0^+(5)$, *we have that* $A_{M_6} = e^{\ell_{M_6,0}} = 2$.

(iii) *There exists a surface* M *where* $e^{\ell_{M,0}} = (\mathfrak{g}_2/\mathfrak{g}_1)^2$.

With respect to statements (i) and (ii) of the above proposition we find it very interesting that, in the sense of our Main Theorem, not all arithmetic surfaces, even those with the same topological signature, have the same behavior.

Also in § 7, in order to prove statement (iii) of Proposition 1 we argue that if one considers a degenerating family of hyperbolic Riemann surfaces within the moduli space of surfaces of fixed topological type, one eventually has the inequality $e^{\ell_{M,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$ near the boundary. As a result, if one begins with congruence group and degenerates the corresponding surface, one will ultimately encounter a surface where $e^{\ell_{M,0}} = (\mathfrak{g}_2/\mathfrak{g}_1)^2$. More generally, however, it seems as if moduli space can be separated into sets defined by the sign of $e^{\ell_{M,0}} - (\mathfrak{g}_2/\mathfrak{g}_1)^2$ where most, but not all, arithmetic surfaces are in the component where $e^{\ell_{M,0}} - (\mathfrak{g}_2/\mathfrak{g}_1)^2 > 0$, and the Deligne–Mumford boundary lies in the component where $e^{\ell_{M,0}} - (\mathfrak{g}_2/\mathfrak{g}_1)^2 < 0$.

We could not explicitly construct a surface where $e^{\ell_{M,0}} - (\mathfrak{g}_2/\mathfrak{g}_1)^2 = 0$, even though we prove that such surfaces exist.

1.5 A comparison of counting functions

In [14, Theorem 2.22], D. Hejhal establishes the asymptotic behavior of the weighted vertical distribution of zeros of ϕ_M within the critical strip. In our notation, the zeros of ϕ_M within the critical strip coincide with the zeros of the Dirichlet series H_M, so then [14, Theorem 2.22] establishes the asymptotic behavior of the weighted vertical counting function $N_{\mathrm{w}}(T; H_M)$.

Let M be any finite volume hyperbolic Riemann surface. We claim there exists a co-compact hyperbolic Riemann surface $\widetilde{M}$ such that $\mathrm{vol}(M) = \mathrm{vol}(\widetilde{M})$, $\ell_{M,0} = \ell_{\widetilde{M},0}$ and $m_{M,0} = m_{\widetilde{M},0}$, which we argue as follows. In the case when the number n_1 of cusps of the surface M is even, we choose the surface $\widetilde{M}_1$ to be any co-compact surface with genus $g_{\widetilde{M}} = g_M + n_1/2$ and

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the same structure of elliptic points as M, hence $\mathrm{vol}(M)=\mathrm{vol}(\widetilde{M}_1)$. If the number of cusps of the surface M is odd, we choose the surface $\widetilde{M}_1$ to be any co-compact surface with genus $g_{\widetilde{M}}=g_M+(n_1-1)/2$ such that it has the same structure of elliptic points as M, plus one additional elliptic point of order 2. By the Gauss–Bonnet formula, $\mathrm{vol}(M)=\mathrm{vol}(\widetilde{M}_1)$. We then deform the surface $\widetilde{M}_1$ in moduli space so that its shortest geodesic has the length equal to $\ell_{M,0}$ and the number of inconjugate geodesics of length $\ell_{M,0}$ is $m_{M,0}$.

In § 9.2, we show that one can compare [14, Theorem 2.22] with generalization of the part (c) of the Main Theorem to higher derivatives in order to establish a simple asymptotic relation between $N_{\mathrm{w}}(T;(Z_M H_M)^{(k)})$, $N_{\mathrm{w}}(T;Z_{\widetilde{M}}^{(k)})$ for $k\geqslant 1$ and $N_{\mathrm{w}}(T;H_M)$. Namely, we prove the following theorem.

THEOREM 2. *Let M be a finite volume hyperbolic Riemann surface such that, in the notation of (9), $A_M=\exp(\ell_{M,0})$. Then, for all integers $k\geqslant 1$*

$$N_{\mathrm{w}}(T;(Z_M H_M)^{(k)})=N_{\mathrm{w}}(T;Z_{\widetilde{M}}^{(k)})+N_{\mathrm{w}}(T;H_M)+o(T)\quad \text{as } T\to\infty. \tag{13}$$

We find it very interesting that, in the case when M is such that $\exp(\ell_{M,0})<(\mathfrak{g}_2/\mathfrak{g}_1)^2$, the coefficients of the first two terms, namely $T\log T$ and T, in the asymptotic development of the counting function $N_{\mathrm{w}}(T;(Z_M H_M)^{(k)})$ for all $k\geqslant 1$ coincide with known results, namely Hejhal's theorem and (2).

1.6 Further comments

Weyl's law in its classical form evaluates the lead asymptotic behavior of the vertical counting function $N_{\mathrm{ver}}(T;Z_M)$ for compact M. As far as is known, the expansion in T involves $\mathrm{vol}(M)$ and no other information associated to the uniformizing group Γ. If M is noncompact, the generalization of Weyl's law addresses the asymptotic behavior of

$$\#\{\lambda_{j,M}<1/4+T^2\}-\frac{1}{4\pi}\int_{-T}^{T}\phi_M'/\phi_M(1/2+ir)\,dr \tag{14}$$

where $\lambda_{j,M}$ is the eigenvalue of an L^2 eigenfunction of the Laplacian on M. The asymptotic expansion of (14) is recalled below (see (68)) and, as in the compact case, all terms in the expansion involve elementary quantities associated to the uniformizing group Γ.

In § 9.1, we express the function in (14) in terms of $N_{\text{ver}}(T; Z_M H_M)$, obtaining an expression which involves the constant $\mathfrak{g}_1$. As a result, we accept the appearance of the term $\mathfrak{g}_2/\mathfrak{g}_1$ in our Main Theorem as being an appropriate generalization of a version of Weyl's law.

In a different direction, if one considers a degenerating family of finite volume hyperbolic Riemann surfaces, then it was shown in [15] that the asymptotic behavior of the associated sequence of vertical counting functions $N_{\text{ver}}(T; Z_M)$ has lead asymptotic behavior, for fixed T, which involves the lengths of the pinching geodesics; see [15, Theorem 5.5]. As a result, we do not view the appearance of the invariant $\ell_{M,0}$ in (1) and (2) as a new feature when using Weyl's laws to understand refined information associated to the uniformizing group Γ.

However, we find the appearance of the constants A_M and a_M, as defined in (9), (10) and (11) to be intriguing, specially since values of A_M and a_M are different for certain arithmetic groups of the same signature. In particular, for any given surface M, we do not know if there are conditions which determine the value taken by A_M. Consequently, we conclude that the study of the vertical counting function $N_{\text{ver}}(T; (Z_M H_M)')$ contains a term which provides new information associated to Γ which we do not see as being previously detected. In other words, if we are allowed to view the vertical counting function $N_{\text{ver}}(T; (Z_M H_M)')$ as another measure of the spectral analysis of the Laplacian on M, then our Main Theorem shows the existence of refined information, namely A_M with its conditional definition (9), about the uniformizing group Γ.

1.7 Computations for the modular group

After the completion of this article, W. Luo brought to our attention the unpublished article [23] from 2008 in which the author undertakes a related study in the case when $\Gamma = \text{PSL}(2, \mathbb{Z})$. There are a number of important differences between the results in the present paper and those in [23], which we now discuss.

In [23], as the title of the article states, the author studies the zeros of the derivative of the zeta function $Z_M(s)/\zeta_{\mathbb{Q}}(2s)$ where $M = \text{PSL}(2, \mathbb{Z})\backslash\mathbb{H}$. If we restrict our analysis to the case when $\Gamma = \text{PSL}(2, \mathbb{Z})$, then the function whose derivative we study is $Z_M(s)\zeta_{\mathbb{Q}}(2s-1)/\zeta_{\mathbb{Q}}(2s)$. Since the article [23] studies a different function than in the present article, one would expect that the statements of the main results are different, as, indeed, is the case. More importantly, however, the asymptotic expansions obtained in [23] has

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an error term of $O(T)$, whereas our error term is $o(T)$, which is significant since the coefficient of the T term contains the quantity A_M, which we view as a new spectral invariant. Finally, we note that the article [23] studies the single group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$.

The approach presented in [23] raise the question if a similar modification of the Selberg zeta function Z_M can occur in the general setting of the present paper. To do so, one can write the function $H_M(s)$ as a ratio $P_M(s)/Q_M(s)$ of two entire functions of order at most two and then consider $Z_M(s)/Q_M(s)$. It is true that the function $Z_M(s)/Q_M(s)$ has nontrivial zeros only at zeros of Z_M stemming from the discrete eigenvalues of the Laplacian. However, the Phillips–Sarnak conjecture/philosophy then asserts that for generic M, the quotient $Z_M(s)/Q_M(s)$ would have a finite number of nontrivial zeros, hence can be written as a polynomial times Gamma-type functions. Aside from this assertion, while focusing solely from the point of view of application of techniques developed in this paper, the function $Z_M(s)/Q_M(s)$ is suitable in the sense that it possesses no nonreal zeros in the half-plane $\mathrm{Re}(s) < 1/2$. However, there are two reasons why the investigation of $Z_M(s)H_M(s)$ is more natural. Firstly, the spectral information carried by $P_M(s)$, namely, the zeros of the scattering determinant, is lost when considering $Z_M(s)/Q_M(s)$. More importantly, unless the explicit expression for the scattering determinant is known, it is very difficult to determine $Q_M(s)$ explicitly and hence express $Z_M(s)/Q_M(s)$ in terms of the information related to the underlying group Γ, whereas $Z_M(s)H_M(s)$ has a general Dirichlet series representation in terms of the group information.

1.8 Outline of the paper

This article is organized as follows. In § 2, we establish notation and recall necessary results from the literature. The zero-free region for $(Z_M H_M)'$, as stated in part (a) of the Main Theorem, will be proved in § 3. Various lemmas leading up to the proof of parts (b) and (c) of the main Theorem will be given in § 4, the proof of parts (b) and (c) will be completed in § 5, and in § 6 we state and prove several corollaries of the Main Theorem. The examples of congruence groups and "moonshine" groups will be given in § 7. In § 8, we prove results analogous to our Main Theorem for higher derivatives of $Z_M H_M$. Finally, in § 9, we give various concluding remarks.

§2. Background material

2.1 Counting functions

Let F denote either a general Dirichlet series with a critical line; F itself may be the derivative of another general Dirichlet series. We assume that F is normalized to be convergent in the half-plane $\mathrm{Re}(s) > 1$ with critical line $\mathrm{Re}(s) = 1/2$. We define the vertical counting function of F as

$$N_{\mathrm{ver}}(T; F) = \sum_{\substack{F(\sigma+it)=0 \\ 0<t<T, 0<\sigma<1}} 1$$

and the weighted vertical counting function of F as

$$N_{\mathrm{w}}(T; F) = \sum_{\substack{F(\sigma+it)=0 \\ 0<t<T, 1/2<\sigma<1}} (\sigma - 1/2).$$

2.2 Additional identities

The logarithmic derivative $D_M(s) := \frac{Z_M'}{Z_M}(s)$ of the Selberg zeta function may be expressed, for $\mathrm{Re}(s) > 1$ as the absolutely convergent series

$$D_M(s) = \sum_{P \in \mathcal{H}(\Gamma)} \frac{\Lambda(P)}{N(P)^s}, \quad \text{where } \Lambda(P) := \frac{\log N(P_0)}{1 - N(P)^{-1}}. \tag{15}$$

Dirichlet series representation of the logarithmic derivative of the function $Z_M H_M$ is given by the following lemma.

Lemma 3. *There exists a constant $\sigma_0' \geqslant 1$ such that for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geqslant \sigma_0' + \epsilon > \sigma_0'$, we have that*

$$\frac{(Z_M H_M)'}{(Z_M H_M)}(s) = \sum_{P \in \mathcal{H}(\Gamma)} \frac{\Lambda(P)}{N(P)^s} + \sum_{i=1}^{\infty} \frac{b(q_i)}{q_i^s}. \tag{16}$$

In addition, the series converge absolutely and uniformly on every compact subset of the half-plane $\mathrm{Re}(s) > \sigma_0'$.

Proof. Equation (16) follows immediately from (8) and (15). We may take σ_0' to be equal to σ_0, which was defined in § 1.4. □

Lemma 4. *The derivative of the function $Z_M H_M$ satisfies the functional equation*

$$(Z_M H_M)'(s) = f_M(s)\eta_M(s)K_M^{-1}(s)\widetilde{Z}_M(1-s)Z_M(1-s), \tag{17}$$

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where

$$f_M(s) := \mathrm{vol}(M)(1/2-s)\left(\tan \pi(1/2-s)\right) \tag{18}$$

and

$$\widetilde{Z}_M(s) := \frac{1}{f_M(s)}\left(\frac{\eta_M'}{\eta_M}(s) - \frac{K_M'}{K_M}(1-s) - \frac{Z_M'}{Z_M}(s)\right). \tag{19}$$

Proof. Straightforward computation stemming from (7). ∎

2.3 An integral representation for $D_M(s)$

In this section, we recall results from [3] on the growth of the logarithmic derivative $D_M(s)$ and its derivatives $D_M^{(k)}(s)$ for $s = 1/2+\sigma+iT$, as $T \to \pm\infty$, for $\sigma \in (0, 1/2)$.

THEOREM 5. [3] *For $s = 1/2+\sigma+iT$, $0<\sigma<1/2$ and every nonnegative integer k, we have the asymptotic bound*

$$\begin{aligned} D_M^{(k)}(s) = O\Bigg(\min\Bigg\{&\frac{|T|}{\sigma^{k+1}\log|T|}, |T|^{1-2\sigma}\log^{k-2\sigma}|T| \\ &\cdot \max_{j=0,\dots,k}\left\{\frac{1}{\sigma^{j+1}\log^{j+1}|T|}, \log\left|\frac{T}{\sigma}\right|\right\}\Bigg\}\Bigg), \end{aligned} \tag{20}$$

as $|T| \to \infty$.

2.4 The completed function Ξ_M

In this section, we recall the notation and results from [7]. The notation of [7] is adjusted to our setting; we take $k = 0$, dimension $d = 1$ and $\tau^* = n_1$.

The completed function Ξ_M associated to the Selberg zeta function is defined by

$$\Xi_M(s) = \Xi_I(s)\Xi_{M,\mathrm{hyp}}(s)\Xi_{M,\mathrm{par}}(s)\Xi_{M,\mathrm{ell}}(s)$$

where $\Xi_{M,\mathrm{hyp}}(s) = Z_M(s)$ is the Selberg zeta function and the remaining functions are associated to the identity, parabolic, and elliptic elements in the underlying uniformizing group. The logarithmic derivative of the identity term Ξ_I is given by

$$-\frac{1}{2s-1}\frac{\Xi_I'(s)}{\Xi_I(s)} = \frac{\mathrm{vol}(M)}{2\pi}\frac{\Gamma'}{\Gamma}(s); \tag{21}$$

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see [7, Remark 3.1.3]. The function $\Xi_{M,\mathrm{ell}}(s)$ is computed in [7, Corollary 2.3.5]; using Stirling's formula, one can show that

$$\frac{1}{2s-1}\frac{\Xi'_{M,\mathrm{ell}}(s)}{\Xi_{M,\mathrm{ell}}(s)} = O\left(\frac{1}{|t|}\log|t|\right) \tag{22}$$

for any $s = \sigma + it$, $\sigma \leqslant 1/2$, as $|t| \to \infty$.

The function $\Xi_{M,\mathrm{par}}(s)$ is described in [7, Definition 3.1.4]. For our purposes, it suffices to relate $\Xi_{M,\mathrm{par}}(s)$ to the scattering determinant $\phi_M(s)$, so that we obtain an expression for $Z_M H_M(s)$. The following computations derive such an expression for $Z_M H_M(s)$.

Let $\{p_1, \dots, p_{N_0}\}$ denote the set of poles of ϕ_M lying in $(1/2, 1\,]$, counted with multiplicities; let $q_1, \dots, q_{N_1}$ denote the set of real zeros of ϕ_M larger than $1/2$ and let $\{q_n\}_{n>N_1}$ denote the set of zeros of ϕ_M with positive imaginary parts, counted with multiplicities. In the notation of [7, Definition 3.2.2], we set $\mathcal{P}_M \equiv 1$ if $n_1 = 0$, otherwise we define $\mathcal{P}_M(s) := f_1(s) f_2(s)$ where

$$f_1(s) := \prod_{n=1}^{N_1}\left(1 + \frac{s-1/2}{q_n - 1/2}\right)\exp\left[\frac{1}{2}\left(\frac{s-1/2}{q_n-1/2}\right)^2\right]$$

and

$$f_2(s) := \prod_{n \geqslant N_1+1}\left(1 + \frac{s-1/2}{q_n - 1/2}\right)\left(1 + \frac{s-1/2}{\overline{q_n} - 1/2}\right)$$
$$\times\ \exp\left[\frac{1}{2}\left(\frac{s-1/2}{q_n-1/2}\right)^2 + \frac{1}{2}\left(\frac{s-1/2}{\overline{q_n}-1/2}\right)^2\right].$$

The infinite product which defines f_2 converges uniformly on compact subsets of $\mathbb{C}$ and defines an entire function of finite order.

LEMMA 6. *For all $s \in \mathbb{C}$, the product $(\Xi_M \mathcal{P}_M)(1-s)$ can be expressed as*

$$(\Xi_M \mathcal{P}_M)(1-s) = (Z_M H_M)(s)\cdot \Xi_I(s)\cdot \Xi_{M,\mathrm{ell}}(s)\cdot \frac{\pi^{n_{1/2}} d(1)}{\phi_M(1/2)}\mathfrak{g}_1^{-s-1}$$
$$\cdot\left(s - \frac{1}{2}\right)^{(1/2)\mathrm{Tr}(I_{n_1} - \Phi_M(1/2)) - n_1}$$
$$\cdot\ \Gamma(s)^{-n_1}\prod_{m=1}^{N_0}\left(\frac{s-p_m}{1/2-p_m}\right). \tag{23}$$

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Proof. From the functional equation [7, (3.2.4), p. 123], we have, for all $s \in \mathbb{C}$

(24)

$$\left(\Xi_M \mathcal{P}_M\right)(1-s) = \left(\Xi_M \mathcal{P}_M\right)(s) \mathfrak{g}_1^{2s-1} \prod_{m=1}^{N_0} \left(\frac{s-p_m}{1-s-p_m}\right) \frac{1}{\phi_M(1/2)} \phi_M(s).$$

On the other hand, by [7, Corollary 2.4.22] it is easy to see that

$$\left(\Xi_{M,\mathrm{par}} \mathcal{P}_M\right)(s) = (s-1/2)^{(1/2)\mathrm{Tr} I_{n_1} - \Phi_M(1/2)} \mathfrak{g}_1^{-s} \left(\frac{1}{\Gamma(s+1/2)}\right)^{n_1} \cdot \prod_{m=1}^{N_0} \left(1 + \frac{s-1/2}{p_m - 1/2}\right).$$

We now write ϕ_M as

$$\phi_M(s) = \pi^{n_1/2} \mathfrak{g}_1^{-2s} d(1) (s-1/2)^{-n_1} \left(\frac{\Gamma(s+1/2)}{\Gamma(s)}\right)^{n_1} H_M(s).$$

The result follows through direct and straightforward computations involving the definition of Ξ_M together with (24). ∎

2.5 On generalized Backlund equivalent for the Lindelöf hypothesis

An important ingredient in the proof of the Main Theorem is a bound on the growth of the function $Z_M H_M$ on the critical line $\mathrm{Re}(s) = 1/2$. We obtain the bound using a slight modification of [10, Proposition 2], which we now state.

PROPOSITION 7. *Let $f(s)$ be a meromorphic function for all $s \in \mathbb{C}$ which is holomorphic in the region $|\mathrm{Im}(s)| \geqslant t_0 > 0$, for some fixed t_0. Let $P(t): \mathbb{R} \to \mathbb{R}$ be nondecreasing function such that $P(t) \geqslant 2$. Let $N(\sigma, f, T)$ denote the number of zeros ρ of f in the region $\mathrm{Re}(\rho) > \sigma$; $0 \leqslant \mathrm{Im}(\rho) \leqslant T$.*

Assume there exist constants $\sigma_0 > 1/2$ and $\omega > 0$ such that for $\sigma_0 - \omega \leqslant \mathrm{Re}(s) \leqslant \sigma_0 + \omega$ we have

$$|f(s)| \geqslant c > 0 \qquad \text{and,} \qquad (f'/f)(s) = o(P(t)) \quad \text{as} \quad t = \mathrm{Im}(s) \to \infty.$$

Furthermore, assume that $|f(s)| > 0$ for $\mathrm{Re}(s) \geqslant \sigma_0 + \omega$ and that for some fixed number $D > 0$ we have

$$f(s) = O((P(t))^D) \qquad \text{as } t = \mathrm{Im}(s) \to \infty, \text{ uniformly for } \mathrm{Re}(s) \geqslant 2 - 3\sigma_0.$$

Then if the estimate $N(\sigma, f, T+1) - N(\sigma, f, T) = o(P(T))$ holds true for all $\sigma \geqslant 1/2$ as $T \to \infty$, then $f(1/2 + it) = O_\epsilon((P(t))^\epsilon)$ as $t \to \infty$.

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There are slight differences between our statement above and [10, Proposition 2]. Firstly, we assume the function f depends on a single complex variable s, not necessarily a member of a family of meromorphic functions. Secondly, the author of [10] assumes that f has a finite number of poles which lie in a compact set. A review of the proof [10, Proposition 2] reveals that the argument is based on Landau's theorem (see [10, Lemma 8]) and Hadamard's three circles theorem (see [10, Lemma 9]). These classical results are applied to the function $f(s)$ in the neighborhood $|s - s_0| \leqslant 2(\sigma_0 - 1/2 - \delta)$ of the point $s_0 = \sigma_0 + iT$ for sufficiently large T. The proof given in [10] carries through without any changes whatsoever under the assumptions we state above.

We refer the reader to [10] for the proof and various interesting generalizations of Proposition 7.

§3. Zeros in a half-plane $\mathrm{Re}(s) < 1/2$

In this section, we prove part (a) of the Main Theorem. In fact, we prove more than stated, since our analysis will yield regions where each of the functions $\mathrm{Re}((Z_M H_M)')$ and $\mathrm{Im}((Z_M H_M)')$ are nonvanishing.

PROPOSITION 8.

(a) *For $\sigma < 1/2$, there exists $t_0 > 0$, which may depend on σ, such that*

$$\mathrm{Re}((Z_M H_M)'(\sigma + it)) \neq 0 \qquad \text{for all } t \text{ such that } |t| > t_0.$$

(b) *For every constant $C > 0$ and arbitrary $-C < \sigma_0' < 1/2$ there are at most finitely many zeros of $(Z_M H_M)'(s)$ inside the strip $-C \leqslant \mathrm{Re}(s) \leqslant \sigma_0'$.*

Proof. We first present the proof of part (a). By taking the logarithmic derivative of the functional equation (7) we get for $s = \sigma + it$ with $\sigma < 1/2$, the equation

$$\frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma + it) = \frac{\eta_M'}{\eta_M}(\sigma + it) - \frac{K_M'}{K_M}(\sigma + it) - \frac{Z_M'}{Z_M}(1 - \sigma - it). \tag{25}$$

From the definition (4) of η_M'/η_M and K_M, one can use Stirling's formula, together with the bound $0 < \theta < \pi$, to show that

$$\mathrm{Re}\left(\frac{\eta_M'}{\eta_M}(\sigma + it)\right) = -\mathrm{vol}(M)t + O(\log|t|) \qquad \text{and}$$
$$\frac{K_M'}{K_M}(\sigma + it) = O(\log|t|),$$

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for $\sigma < 1/2$ and as $|t| \to \infty$. Therefore,

$$\operatorname{Re}\left(-\frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma+it)\right) = \operatorname{vol}(M)t + O(\log|t|) + \operatorname{Re}\left(\frac{Z_M'}{Z_M}(1-\sigma-it)\right).$$

Replacing σ by $1/2-\sigma$ in (20) we get, for $\sigma < 1/2$

$$\operatorname{Re}\left(-\frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma+it)\right) = \operatorname{vol}(M)t + O\left(\frac{|t|}{(1/2-\sigma)\log|t|}\right),$$

as $t \to \pm\infty$. Therefore, there exists $t_0 > 0$ such that

$$\operatorname{Re}\left(\frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma+it)\right) \neq 0 \quad \text{for all } s=\sigma+it, \text{ with } |t|>t_0.$$

On the other hand, the nontrivial zeros of the function $Z_M H_M$ are either nontrivial zeros $\rho = \frac{1}{2} \pm ir_n$ of Z_M or zeros ρ of ϕ_M. All except finitely many zeros of ϕ_M have real part bigger than $1/2$; therefore, $Z_M H_M(\sigma+it) \neq 0$ for $\sigma < 1/2$ and $t > t_0$. Therefore, we conclude that $\operatorname{Re}((Z_M H_M)'(\sigma+it)) \neq 0$ for all $t > t_0$. With all this, the proof of part (a) is complete.

To prove part (b), we employ Lemma 4. Recall the function $\widetilde{Z}_M(s)$ which is defined in (19). Let us write $\widetilde{Z}_M(s) = 1 + Z_{M,1}(s)$. Then

$$\begin{aligned} -f_M(s)Z_{M,1}(s) &= \pi \sum_{\substack{\{R\}\\ 0<\theta(R)<\pi}} \frac{1}{M_R \sin\theta} \frac{\cos(2\theta-\pi)(s-1/2)}{\cos\pi(s-1/2)} - 2n_1 \log 2 \\ &\quad - n_1\left(\frac{\Gamma'}{\Gamma}(1/2+s) + \frac{\Gamma'}{\Gamma}(3/2-s)\right. \\ &\quad - \left.\frac{\Gamma'}{\Gamma}(1/2-s) + \frac{\Gamma'}{\Gamma}(1-s)\right) + \frac{Z_M'}{Z_M}(s), \end{aligned} \tag{26}$$

where f_M is defined in (18).

As in the proof of part (a), we can use Stirling's formula and (20) to arrive at the bound

$$Z_{M,1}(\sigma_1+it) = O\left(\frac{(|t|\log|t|)^{2-2\sigma_1}}{(\sigma_1-1/2)|t|}\right) \quad \text{as } |t| \to \infty$$

for $\sigma_1 > 1/2$ and $(\sigma_1+it) \in \mathbb{C}\setminus \cup_{n\in\mathbb{Z}} B_n$ where B_n are small circles of fixed radius centered at integers. In particular, for $\sigma_1 > 1/2$ and $(\sigma_1+it) \in \mathbb{C}\setminus \cup_{n\in\mathbb{Z}} B_n$, function $Z_{M,1}(\sigma_1+iT)$ is uniformly bounded in T. Therefore,

$\widetilde{Z}_M(1-s) = 1 + o(1)$, as $\mathrm{Im}(s) \to \pm\infty$ in the strip $-C \leqslant \mathrm{Re}(s) \leqslant \sigma_0' < 1/2$, hence $\widetilde{Z}_M(1-s)$ has finitely many zeros in this strip.

Since $Z_M(s)$ has only finitely many zeros for $\mathrm{Re}(s) > 1/2$, the function $\widetilde{Z}_M(1-s)Z_M(1-s)$ will have finitely many zeros in the strip $-C \leqslant \mathrm{Re}(s) \leqslant \sigma_0' < 1/2$. Equation (17) then implies that the set of zeros of $(Z_M H_M)'(s)$ in the strip $-C \leqslant \mathrm{Re}(s) \leqslant \sigma_0' < 1/2$ is finite since the factor $\Phi_M(s) := f_M(s)\eta_M(s)K_M^{-1}(s)$ of the functional equation (17) also has at most finitely many zeros in this strip. □

We note that the zeros of $(Z_M H_M)'(s)$ which arise from zeros of Φ_M are viewed as trivial zeros. They are located in the region $\mathrm{Re}(s) < 1/2$ and arise at all negative integers.

The above method of examining zeros of the function $(Z_M H_M)'$ has the critical line as its limitation, since the bounds for the logarithmic derivative (20) hold true only in the half-plane $\mathrm{Re}(s) > 1/2$. In order to derive results valid on the critical line we need a representation on the critical line. Such a representation exists for the complete zeta function $\Xi_M(s)$ (see § 2.4).

Proposition 9. *There exists a number $t_0 > 0$ such that the following statements hold:*

(a) $\frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma + it) \neq 0$ *for all* $\sigma < 1/2$ *and all* $|t| > t_0$*;*

(b) $\frac{(Z_M H_M)'}{(Z_M H_M)}(1/2 \pm it) \neq 0$ *for all* $|t| > t_0$, $t \neq r_n$ *for all* $n \geqslant 0$.

Proof. Let $\sigma < 0$. By Proposition 8, there exists a constant $t_0' > 0$ such that for all $\sigma < 0$ and all $|t| > t_0'$ we have $\frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma + it) \neq 0$. Therefore, it is enough to prove the statement when $0 \leqslant \sigma < 1/2$. Without loss of generality, we assume that $t > 0$.

Taking the logarithmic derivatives of the both sides of the equation (23), we get

$$\begin{aligned} \frac{1}{2s-1}\frac{(Z_M H_M)'}{(Z_M H_M)}(s) = &-\frac{1}{2s-1}\frac{\Xi_I'(s)}{\Xi_I(s)} - \frac{1}{2s-1}\frac{\Xi_{M,\mathrm{ell}}'(s)}{\Xi_{M,\mathrm{ell}}(s)} \\ &+ \frac{\log \mathfrak{g}_1}{2s-1} + \frac{n_1}{2s-1}\frac{\Gamma'}{\Gamma}(s) \\ &+ \frac{n_1 - \frac{1}{2}\mathrm{Tr}\left(I_{n_1} - \Phi(\frac{1}{2})\right)}{2(s-1/2)^2} - \frac{1}{2s-1}\sum_{m=1}^{N_0}\frac{1}{s-p_m} \\ &- \frac{1}{2s-1}\frac{(\Xi_M \mathcal{P}_M)'}{(\Xi_M \mathcal{P}_M)}(1-s), \end{aligned} \tag{27}$$

for all $s \in \mathbb{C}$ different from zeros and poles of Z_M and ϕ_M. Applying [7, formula (3.4.1)] yields

$$
\begin{aligned}
(\Xi_M \mathcal{P}_M)(s) = e^{Q(s)}(s-1/2)^{2d_{1/4}} \\
\cdot \prod_{n\geqslant 0,\, r_n\neq 0} \left(1+\frac{(s-1/2)^2}{r_n^2}\right) \exp\left(-\frac{(s-1/2)^2}{r_n^2}\right) \\
\cdot \prod_{n=1}^{N_1} \left(1-\frac{s-1/2}{\eta_n}\right) \exp\left(-\frac{s-1/2}{\eta_n}+\frac{(s-1/2)^2}{2\eta_n^2}\right) \\
\cdot \prod_{n\geqslant N_1+1} \left(1+\frac{s-1/2}{\eta_n+i\gamma_n}\right)\left(1+\frac{s-1/2}{\eta_n-i\gamma_n}\right) \\
\cdot \exp\left(-\frac{2\eta_n\left(s-\frac{1}{2}\right)}{\eta_n^2+\gamma_n^2}+\left(s-\frac{1}{2}\right)^2\frac{\eta_n^2-\gamma_n^2}{(\eta_n^2+\gamma_n^2)^2}\right),
\end{aligned}
$$

for all $s \in \mathbb{C}$ and where the notation is as follows: $\eta_n := \mathrm{Re}(q_n)$; $\gamma_n := \mathrm{Im}(q_n)$, $d_{1/4}$ is the multiplicity of $\lambda = 1/4$ as an eigenvalue; and $Q(s) = a_2(s-1/2)^2 + a_1(s-1/2) + a_0$ for some constants a_i, $i = 0, 1, 2$ computed in [7]. The constants a_1 and a_2 are defined by [7, formulas (3.4.8) and (3.4.9)]. For our purposes it is important to know that a_1 and a_2 are real.

We now compute the logarithmic derivative of $(\Xi_M\mathcal{P}_M)(s)$ and substitute the expression into (27). After some elementary calculations, employing the Stirling formula and having in mind (21) and (22) we end up with

$$
\begin{aligned}
\frac{1}{2s-1}\frac{(Z_M H_M)'}{(Z_M H_M)}(s) = \frac{\mathrm{vol}(M)}{2\pi}\frac{\Gamma'}{\Gamma}(s) + \sum_{n\geqslant 0,\, r_n\neq 0}\left(\frac{1}{(s-\frac{1}{2})^2+r_n^2}-\frac{1}{r_n^2}\right) \\
+ a_2 + \sum_{n\geqslant N_1+1}\left[\frac{\eta_n^2-\gamma_n^2}{(\eta_n^2+\gamma_n^2)^2}\right. \\
\left. + \frac{\gamma_n^2-\eta_n^2+\eta_n(s-1/2)}{\left((\eta_n-s+1/2)^2+\gamma_n^2\right)(\eta_n^2+\gamma_n^2)}\right] \\
+ \frac{1}{2}\sum_{n=1}^{N_1}\left(\frac{1}{\eta_n^2}-\frac{1}{\eta_n(\eta_n-s+1/2)}\right) + O\left(\frac{\log t}{t}\right),
\end{aligned}
\tag{28}
$$

as $t = \mathrm{Im}(s) \to \infty$. We now set $s = \sigma + it$ with $t > 0$ and $0 \leqslant \sigma < 1/2$. By computing the imaginary parts of both sides (28) we get

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$$\begin{aligned}
\operatorname{Im}&\left(\frac{1}{2\sigma-1+2it}\frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma+it)\right)\\
&=\frac{\operatorname{vol}(M)}{2\pi}\cdot\left[\frac{t}{\sigma^2+t^2}+\sum_{n=1}^{\infty}\frac{t}{(n+\sigma)^2+t^2}\right]\\
&\quad+\sum_{n\geqslant 0,\ r_n\neq 0}\frac{t(1/2-\sigma)}{\left((\sigma-1/2)^2-t^2+r_n^2\right)^2+4t^2(\sigma-1/2)^2}\\
&\quad+O\left(\frac{1}{t}\right)+O\left(\frac{\log t}{t}\right)\\
&\quad+\sum_{n\geqslant N_1+1}\frac{t\eta_n(3\gamma_n^2-t^2)-t\eta_n^3+t(1/2-\sigma)(2\gamma_n^2-2\eta_n^2-\eta_n(1/2-\sigma))}{[((\eta_n-\sigma+1/2)^2+\gamma_n^2-t^2)^2+4t^2(\eta_n+1/2-\sigma)^2]\,(\eta_n^2+\gamma_n^2)}.
\end{aligned}\tag{29}$$

Since $0\leqslant\sigma<1/2$ we have that $(n+\sigma)^2<(n+1/2)^2$, for all $n\geqslant 0$. Therefore

$$\frac{t}{\sigma^2+t^2}+\sum_{n=1}^{\infty}\frac{t}{(n+\sigma)^2+t^2}>\frac{t}{1/4+t^2}+\sum_{n=1}^{\infty}\frac{t}{(n+1/2)^2+t^2}=\frac{\pi}{2}\tanh(\pi t).$$

Furthermore, since $0<\eta_n<c$, for some positive constant c and all $n\geqslant 1$ and $\gamma_n\to\infty$, as $n\to\infty$, by the choice of σ we have that

$$(1/2-\sigma)(2\gamma_n^2-2\eta_n^2-\eta_n(1/2-\sigma))\geqslant 2\gamma_n^2-2\eta_n^2-\eta_n\geqslant 0$$

for all but finitely many $n\geqslant(N_1+1)$. Let $n_1\geqslant(N_1+1)$ be an integer such that $2\gamma_n^2-2\eta_n^2-\eta_n\geqslant 0$ for all $n\geqslant n_1$. For simplicity, we introduce the notation

$$D(n,\sigma,t)=\left((\eta_n-\sigma+1/2)^2+\gamma_n^2-t^2\right)^2+4t^2(\eta_n+1/2-\sigma)^2.$$

From (29), we conclude the existence a constant $C_1>0$ and a positive number $t_0>t_0'$ such that for all $t>t_0$,

$$\begin{aligned}
\operatorname{Im}&\left(\frac{1}{2\sigma-1+2it}\frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma+it)\right)\\
&\geqslant\frac{\operatorname{vol}(M)}{4}\tanh(\pi t)-C_1\frac{\log t}{t}+\sum_{|\gamma_n|<t/\sqrt{3}}\frac{t\eta_n(3\gamma_n^2-t^2)}{D(n,\sigma,t)(\eta_n^2+\gamma_n^2)}\\
&\quad+\sum_{N_1+1\leqslant n\leqslant n_1}\frac{(1/2-\sigma)t(2\gamma_n^2-2\eta_n^2-\eta_n)}{D(n,\sigma,t)(\eta_n^2+\gamma_n^2)}+\sum_{n\geqslant N_1+1}\frac{-t\eta_n^3}{D(n,\sigma,t)\,(\eta_n^2+\gamma_n^2)}.
\end{aligned}\tag{30}$$

Observe that each term in each summand in (30) is negative. We investigate separately each of the three sums on the right-hand side of (30).

Since $(\eta_n - \sigma + 1/2)^2$ is bounded by some constant, by enlarging t_0 if necessary, we get that $D(n,\sigma,t) \geqslant t^4/4$ for all n such that $|\gamma_n| < t/\sqrt{3}$ and all $t > t_0$. Therefore

$$0 \leqslant \sum_{|\gamma_n|<t/\sqrt{3}} \frac{t\eta_n(t^2-3\gamma_n^2)}{D(n,\sigma,t)\,(\eta_n^2+\gamma_n^2)} \leqslant \frac{2}{t} \sum_{|\gamma_n|<t/\sqrt{3}} \frac{2\eta_n}{(\eta_n^2+\gamma_n^2)} = O(1/t),$$

as $t\to\infty$ because the series $\sum_{n\geqslant N_1+1} 2\eta_n(\eta_n^2+\gamma_n^2)^{-1}$ converges; see [7, Corollary 2.4.17].

For the second and the third sum in (30) we use the elementary inequality $D(n,\sigma,t) \geqslant 4t^2\eta_n^2$ to deduce that both sums are $O(1/t)$ as $t\to\infty$, hence all sums on the right-hand side of (30) are $O(1/t)$, as $t\to\infty$. Therefore, there exists a constant $C_2 > 0$, such that for all $t > t_0$, $t \neq r_n$ and $0 \leqslant \sigma \leqslant 1/2$ one has

$$\operatorname{Im}\left(\frac{1}{2\sigma-1+2it}\frac{(Z_M H_M)'}{(Z_M H_M)}(\sigma+it)\right) \geqslant \frac{\operatorname{vol}(M)}{4}\cdot \tanh \pi t - \frac{C_1 \log t + C_2}{t}. \tag{31}$$

Since $\tanh \pi t = 1 + O(e^{-\pi t})$, as $t\to\infty$ we conclude that statement (a) holds true.

We now prove part (b). We put $s = 1/2 + it$ for $t > 0$ with $t \neq r_n$ in (29) to get

$$\operatorname{Im}\left(\frac{1}{2it}\frac{(Z_M H_M)'}{(Z_M H_M)}\left(\frac{1}{2}+it\right)\right) = \frac{\operatorname{vol}(M)}{4}\tanh(\pi t) + O\left(\frac{\log t}{t}\right) + \sum_{n\geqslant N_1+1} \frac{t\eta_n(3\gamma_n^2-t^2) - t\eta_n^3}{D(n,1/2,t)\,(\eta_n^2+\gamma_n^2)}.$$

Analogously as in the proof of part (a) we deduce that there exist a constant $t_1 > 0$ such that (31) holds true with $\sigma = 1/2$ and some constants C_1 and C_2, for all $t > t_1$, $t \neq r_n$. The proof of part (b) is complete. □

We can now give a proof of part (a) of the Main Theorem.

The function $(Z_M H_M)(s)$ has finitely many nontrivial zeros in the region $\operatorname{Re}(s) < 1/2$. Combining this statement with Proposition 9(a) immediately implies the existence a of constant t_0 such that $(Z_M H_M)'(\sigma+it) \neq 0$ for $\sigma < 1/2$ and $|t| > t_0$.

Proposition 9(b) yields that $\frac{(Z_M H_M)'}{(Z_M H_M)}(1/2 \pm it) \neq 0$ for $|t| > t_0$, $t \neq r_n$ for all $n \geqslant 1$. Therefore, the only zeros of $(Z_M H_M)'$ on the line $\mathrm{Re}(s) = 1/2$, with at most a finite number of exceptions, are multiple zeros of $(Z_M H_M)$, or, equivalently, multiple zeros of Z_M.

§4. Preliminary lemmas

In this section, we prove some preliminary results needed in the proof of the Main Theorem. Let quantity A_M, respectively a_M, be defined by (9) and (10), respectively (11) and let $X_M(s)$ be defined by (12). Let $P_{00} \in \mathcal{H}(\Gamma)$ denote the primitive hyperbolic element of Γ with the property that $N(P_{00}) = e^{\ell_{M,0}}$; or equivalently, with the property that $N(P_{00}) = \min\{N(P) : P \in \mathcal{H}(\Gamma)\}$. In the case when $A_M = e^{\ell_{M,0}}$ we may write a_M in terms of the norm of P_{00}, namely $a_M = m_{M,0}\Lambda(P_{00})$.

LEMMA 10. *There exist $\sigma_1 > 1$ and a constant $0 < c_\Gamma < 1$ such that for $\sigma = \mathrm{Re}(s) \geqslant \sigma_1$, we have the asymptotic formula*

$$X_M(s) = 1 + O(c_\Gamma^\sigma) \neq 0, \quad \textit{as } \sigma \to +\infty.$$

Proof. From the Euler product definition (3) of Z_M and from (6), we have that

$$(32)\quad Z_M(s) = 1 + O(N(P_{00})^{-\mathrm{Re}(s)}) \quad \text{and} \quad H_M(s) = 1 + O(r_2^{-2\,\mathrm{Re}(s)})$$

as $\mathrm{Re}(s) \to \infty$. Furthermore, by the definition of A_M and a_M

$$\sum_{\{P\}\in\mathcal{H}(\Gamma)} \frac{\Lambda(P)}{N(P)^s} + \sum_{i=1}^{\infty} \frac{b\,(q_i)}{q_i^s} = \frac{a_M}{A_M^s}(1 + O(A_{\Gamma,1}^{-\mathrm{Re}(s)})),$$

as $\mathrm{Re}(s) \to +\infty$, for some constant $A_{\Gamma,1} > 1$. Multiplying the formula (16) by $(Z_M H_M)(s)$ and employing the equation (32) we complete the proof. □

The following lemma provides the bound for the growth of the function $Z_{M,1}(s)$; recall that $Z_{M,1}(s)$ is defined by (26).

LEMMA 11. *Let $0 < a < 1/2$ be an arbitrary real number and let $\sigma \geqslant 1 - a$. Then*

$$\log|1 + Z_{M,1}(\sigma \pm iT)| = O(|Z_{M,1}(\sigma \pm iT)|) = O(T^{2a-1}\log^{2a} T),$$
$$(33)\qquad \textit{as } T \to \infty.$$

Proof. From the bound (20) with $k=0$ and $\sigma \geqslant 1-a$, we get

$$\frac{Z'_M}{Z_M}(\sigma \pm iT) = O((T\log T)^{1-2(\sigma-1/2)}) = O((T\log T)^{2a}), \quad \text{as } T\to\infty,$$

where the implied constant depends only upon M and a. We can then argue in the same manner as in the proof of Proposition 8(b). Namely, applying Stirling's formula and the above estimate, we get, for $s=\sigma\pm iT$ and $T\geqslant 1$, the estimate

$$\left|\frac{1}{f_M(s)}\left(\frac{\eta'_M}{\eta_M}(s) - \frac{K'_M}{K_M}(1-s) - \frac{Z'_M}{Z_M}(s)\right) - 1\right| = O\left(\frac{\log T}{T} + \frac{(T\log T)^{2a}}{T}\right),$$

as $T\to\infty$. This implies the bound (33) as claimed. ∎

The following lemma is a Phragmén–Lindelöf type bound for $(Z_M H_M)$. The bound will be used to derive a similar bound for $(Z_M H_M)'$ using the Cauchy formula.

LEMMA 12. *Let $\sigma_2 \geqslant 1$ be a fixed real number, such that $-\sigma_2$ is not a pole of $(Z_M H_M)$. Then, for an arbitrary $\delta > 0$*

(a)
$$(Z_M H_M)(\sigma+it) = O_\Gamma\left(\exp\left(1/2+\sigma_2+\delta\right)\mathrm{vol}(M)t\right),$$

(b)
$$Z_M(\sigma+it) = O_\Gamma\left(\exp(1/2+\sigma_2)\mathrm{vol}(M)t\right)$$

for $t\geqslant 1$, uniformly in $\sigma \leqslant -\sigma_2$.

Proof. To prove part (a), we apply the Phragmen–Lindelöf theorem to the function

$$F(s) = (Z_M H_M)(s)\exp\left[\mathrm{vol}(M)(1/2+\sigma_2+\delta)is\right]$$

which is an entire function of finite order at most two in the sector $D := \{\pi/4 \leqslant \arg(s+\sigma_2) \leqslant \pi/2\}$. Obviously, $(Z_M H_M)(s) = O(1)$ along the line $\arg(s+\sigma_2)=\pi/4$, since $(Z_M H_M)(\sigma+it) = O(1)$, for $\sigma>\sigma_1$ and $t\geqslant 1$; see the proof of Lemma 10. Therefore,

$$|F(s)| = O\left(1\right) \qquad \text{along the line } \arg(s+\sigma_2)=\pi/4.$$

To determine the behavior of the function $F(s)$ along the vertical line $\arg(s+\sigma_2)=\pi/2$, that is, for $s=-\sigma_2+it$, $t\geqslant 0$, we use the functional

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equation (7) to get

$$|F(-\sigma_2+it)| = \exp\left(-\,(1/2+\sigma_2+\delta)\,\mathrm{vol}(M)t\right)|\eta(-\sigma_2+it)| \times \left|K_M^{-1}(-\sigma_2+it)\right|\cdot O(1), \tag{34}$$

since $1+\sigma_2 \geqslant 2$. It remains to estimate $|\eta_M(-\sigma_2+it)|$. Applying (4), [13, formula (4.4), p. 76], Stirling's formula and the bound $0<\theta<\pi$, elementary computations show that

$$|\eta_M(-\sigma_2+it)| = O\left(\exp\left(\mathrm{vol}(M)(1/2+\sigma_2)t+O(1)\right)\right) \quad \text{as } t\to+\infty. \tag{35}$$

Formula (6.1.45) from [1], which itself is an application of Stirling's formula, yields

$$\left|K_M^{-1}(-\sigma_2+it)\right| = O\left(\exp\left(\frac{n_1}{2}\log t\right)\right) \quad \text{as } t\to+\infty. \tag{36}$$

Substituting the bound (36) together with (35) into (34) we get

$$|F(-\sigma_2+it)| = O\left(\exp\left(-\mathrm{vol}(M)\delta t+\frac{n_1}{2}\log t+O(\log(t))\right)\right) = o(1),$$

as $t\to+\infty$. One now can apply the Phragmen–Lindelöf theorem, which implies that $F(s)=O(1)$ in the sector $D:=\{\pi/4\leqslant\arg(s+\sigma_2)\leqslant\pi/2\}$ and the proof of (a) is complete.

To prove (b), we repeat the proof given above for the function $G(s)=Z_M(s)\exp[(1/2+\sigma_2)\mathrm{vol}(M)is]$, which is an entire function of finite order in the sector $D:=\{\pi/4\leqslant\arg(s+\sigma_2)\leqslant\pi/2\}$. We omit the details. □

The following lemma is a Lindelöf type bound for the function $Z_M H_M$ which will be used to deduce a sharper bound for the function $\arg X_M(\sigma+iT)$, when σ is close to $1/2$.

LEMMA 13. *For $\epsilon>0$ and $t\geqslant 1$*

$$(Z_M H_M)(1/2+it) = O(\exp(\epsilon t)) \quad \textit{as } t\to+\infty.$$

Proof. Since

$$|H_M(1/2+it)| = |\phi_M(1/2+it)||K_M^{-1}(1/2+it)| = O(\exp(n_1\log t/2)),$$

as $t\to+\infty$, it is enough to prove that $Z_M(1/2+it)=O(\exp(\epsilon t))$ as $t\to\infty$.

We apply Proposition 7. In the notation of Proposition 7 we take

$$f(s) = Z_M(s) \quad \text{with } \sigma_0 = \sigma_1 + \omega > \sigma_1 \qquad \text{and} \qquad P(t) = 2\exp(t),$$

where σ_1 is defined in Lemma 10. Let us verify that all assumptions of Proposition 7 are fulfilled.

The function Z_M is meromorphic function of finite order, with poles at points on the real line; see [14, p. 498]. Hence $Z_M(s)$ is holomorphic function for $|\operatorname{Im}(s)| \geqslant t_0 > 0$, for any $t_0 > 0$.

From the proof of Lemma 10 it is obvious that $|Z_M(s)| \geqslant c > 0$ and $Z'_M/Z_M(s) = O(1)$ as $t \to \infty$, for $s = \sigma + it$ and with $\sigma_0 - \omega \leqslant \sigma \leqslant \sigma_0 + \omega$. Furthermore, $|Z_M(s)| > 0$ for $\operatorname{Re}(s) > \sigma_0 + \omega$.

From Lemma 12(b), we have that

$$Z_M(\sigma + it) = O_\Gamma(\exp(1/2 + 3\sigma_0 - 2)\operatorname{vol}(M)t) = O_\Gamma(P(t)^D),$$

for a fixed $D = (3\sigma_0 - 3/2)\operatorname{vol}(M)$, uniformly in $\sigma \geqslant 2 - 3\sigma_0$.

Since Z_M has no zeros in the half-plane $\operatorname{Re}(s) > 1/2$, the Lindelöf condition on the vertical distribution of zeros of $Z_M(s)$ in the half-plane $\operatorname{Re}(s) > 1/2$, as required in Proposition 7, is trivially fulfilled.

Therefore, all the assumptions of Proposition 7 are satisfied, hence $Z_M(1/2 + it) = O(\exp(\epsilon t))$ as $t \to \infty$. ∎

LEMMA 14. *For an arbitrary $\epsilon > 0$, $t \geqslant 1$ and σ_2 defined in Lemma 12 we have*

$$(Z_M H_M)'(\sigma + it) = \begin{cases} O(\exp \epsilon t) & \text{for } \frac{1}{2} \leqslant \sigma \leqslant \sigma_0 \\ O(\exp(1/2 - \sigma + \epsilon)t) & \text{for } -\sigma_2 \leqslant \sigma < 1/2, \end{cases}$$

as $t \to \infty$.

Proof. The proof involves an application of the Phragmen–Lindelöf theorem to the open sector bounded by the lines $\operatorname{Re}(s) = -\sigma_2$, $\operatorname{Re}(s) = \frac{1}{2}$ and $\operatorname{Im}(s) = 1$. The bounds to be used come from Lemma 12, with $\delta = \epsilon$, and from Lemma 13. A direct application of the Phragmen–Lindelöf theorem yields the bound

$$(Z_M H_M)(\sigma + it) = O(\exp(1/2 - \sigma + \epsilon)t), \tag{37}$$

for $t \geqslant 1$ and $-\sigma_2 \leqslant \sigma \leqslant 1/2$. Similarly, for σ_0 defined as in Lemma 12, one can apply the Phragmen–Lindelöf theorem in the open sector bounded by the lines $\mathrm{Re}(s)=\sigma_0$, $\mathrm{Re}(s)=\frac{1}{2}$ and $\mathrm{Im}(s)=1$, from which one gets

$$(Z_M H_M)(\sigma+it) = O(\exp \epsilon t), \tag{38}$$

for $1/2 \leqslant \sigma \leqslant \sigma_0$. The Cauchy integral formula can be applied, from which we have the equation

$$(Z_M H_M)'(s) = \frac{1}{2\pi i}\int_C \frac{(Z_M H_M)(z)}{(z-s)^2}\,dz$$

where C is a circle of a small, fixed radius $r<\epsilon$, centered at $s=\sigma+it$. Applying (38) to $(Z_M H_M)(z)$, when $1/2 \leqslant \mathrm{Re}(z) \leqslant \sigma_0$ and (37) when $\mathrm{Re}(z) < 1/2$, we deduce that

$$(Z_M H_M)'(\sigma+it) = O\left(\exp((r+\epsilon)t)/r\right) = O(\exp(2\epsilon t))$$

for $1/2 \leqslant \sigma \leqslant \sigma_0$ and $t \geqslant 1$. This proves the first part of the Lemma when replacing ε by $\varepsilon/2$.

In the case when $\sigma < 1/2$, we can use the functional equation for $(Z_M H_M)'$ to arrive at the expression

$$\begin{aligned}|(Z_M H_M)'(-\sigma_2+it)| = {} & |\eta_M(-\sigma_2+it)||K_M^{-1}(-\sigma_2+it)||Z_M(1+\sigma_2-it)| \\ & \cdot \left|\frac{\eta_M'}{\eta_M}(-\sigma_2+it) - \frac{K_M'}{K_M}(-\sigma_2+it)\right. \\ & \left. - \frac{Z_M'}{Z_M}(1+\sigma_2-it)\right|.\end{aligned}$$

Since $\sigma_2 \geqslant 1$, we have $(Z_M'/Z_M)(1+\sigma_2-it) = O(1)$, as $t \to +\infty$. Elementary computations involving the definition of the function η_M'/η_M and the Stirling formula imply that

$$\frac{\eta_M'}{\eta_M}(-\sigma_2+it) - \frac{K_M'}{K_M}(-\sigma_2+it) - \frac{Z'}{Z}(1+\sigma_2-it) = O(t) \quad \text{as } t \to \infty;$$

in brief, one sees the asymptotic bound by observing that the leading term in the above expression is $\mathrm{vol}(M)(1/2+\sigma_2-it)\tan(\pi(1/2+\sigma_2-it))$. From the bounds (35) and (36) obtained in the proof of Lemma 12, we arrive at the bound

$$\left|(Z_M H_M)'(-\sigma_2+it)\right| = O\left(\exp\left((1/2+\sigma_2+\epsilon)\,\mathrm{vol}(M)t\right)\right) \quad \text{as } t \to \infty.$$

The bound claimed in the statement of the Lemma follows by applying the Phragmen–Lindelöf theorem to the function $(Z_M H_M)'$ in the open sector bounded by the lines $\mathrm{Im}(s)=1$, $\mathrm{Re}(s)=-\sigma_2$ and $\mathrm{Re}(s)=1/2$, keeping in mind that $-\sigma_2 \leqslant \sigma < 1/2$. ∎

§5. Vertical and weighted vertical distribution of zeros

In this section, we prove parts (b) and (c) of the Main Theorem.

We fix a large positive number T and choose number T' such that $|T'-T|=O(1)$ independently of T where no zero of $Z_M H_M$ has imaginary part equal to T'. Let $t_0>0$ be a number such that $(Z_M H_M)'/(Z_M H_M)(\sigma+it)\neq 0$ for all $\sigma<1/2$ and $|t|>t_0$; the existence of such t_0 is established by Proposition 9. Let $\sigma_0\geqslant 1$ be a constant chosen so that $\sigma_0\geqslant \max\{\sigma_0',\sigma_1\}$, where σ_0' is defined in Lemma 3 and σ_1 is defined in Lemma 10. Let $0<a<1/2$ be arbitrary.

The function $X_M(s)$, which was defined in (12), is holomorphic in the rectangle $R(a,T')$ with vertices $a+it_0$, σ_0+it_0, σ_0+iT' and $a+iT'$. As in [22], we use Littlewood's theorem from which we get the formula

$$\begin{aligned}2\pi \sum_{\substack{\rho'=\beta'+i\gamma\\ t_0<\gamma<T',\beta'>a}} (\beta'-a) &= \int_{t_0}^{T'} \log|X_M(a+it)|dt - \int_{t_0}^{T'} \log|X_M(\sigma_0+it)|dt \\ &\quad - \int_a^{\sigma_0} \arg X_M(\sigma+it_0)\,d\sigma + \int_a^{\sigma_0} \arg X_M(\sigma+iT')\,d\sigma \\ &= I_1 - I_2 - I_3 + I_4. \end{aligned} \tag{39}$$

The variable ρ' denotes a zero of $(Z_M H_M)'$, and the integrals I_1, I_2, I_3 and I_4 are defined to be the four integrals in (39), in obvious notation. By Proposition 9, the condition that $\mathrm{Im}(\rho')>t_0$ implies that $\mathrm{Re}(\rho')\geqslant 1/2$, hence the sum on the left-hand side of (39) is actually taken over all zeros of $(Z_M H_M)'$ with imaginary part in the interval (t_0,T').

We investigate integrals I_1, I_2, I_3 and I_4 separately.

Obviously, $I_3=O(1)$ as $T\to\infty$ since, in fact, I_3 is independent of T. As for I_2, the function $\log X_M$ is holomorphic and bounded in the infinite strip $\{s\in\mathbb{C}: t_0\leqslant \mathrm{Im}(s)\leqslant T', \mathrm{Re}(s)\geqslant\sigma_0\}$, hence following the argument from [22] we get that $I_2=O(1)$ as $T\to\infty$.

The evaluation of I_4 closely follows the lines of the proof treating the analogous integral in the compact case considered by Garunkštis in [11], the new input being our Lemma 14. In order to show that $I_4=o(T)$, it is sufficient to prove that

$$\arg X(\sigma+iT')=o(T) \quad \text{for } a\leqslant\sigma\leqslant\sigma_0 \quad \text{and} \quad \text{as } T\to\infty. \tag{40}$$

The proof of (40) is very similar to the proof of [11, formula (3.4)], hence we omit the details. It remains to evaluate I_1.

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5.1 Evaluation of I_1

We shall break apart further I_1 by using the functional equation (17) for $(Z_M H_M)'$, the definition (12) of X_M, and representation of $\widetilde{Z}_M(s) = 1 + Z_{M,1}(s)$ which was used in the proof of Proposition 8(b). By doing so, we arrive at the expression

$$\begin{aligned} I_1 = &- \int_{t_0}^{T'} \log \left| a_M A_M^{-(a+it)} \right| dt \\ &+ \int_{t_0}^{T'} \log |f_M(a+it)\eta_M(a+it)K_M^{-1}(a+it)| dt \\ &+ \int_{t_0}^{T'} \log |Z_M(1-(a+it))| \, dt + \int_{t_0}^{T'} \log |1 + Z_{M,1}(1-(a+it))| \, dt \\ &= I_{11} + I_{12} + I_{13} + I_{14}, \end{aligned}$$

with the obvious notation for the integrals I_{11}, I_{12}, I_{13} and I_{14}. Clearly, we have that

$$I_{11} = -T(\log|a_M| - a \log A_M) + O(1) \quad \text{as } T \to \infty. \tag{41}$$

From the computations on the bottom of [22, p. 1146], we have that

$$\int_{t_0}^{T'} \log |f_M(a+it)| dt = T \log T + T(\log \operatorname{vol}(M) - 1) + O(\log T) \quad \text{as } T \to \infty, \tag{42}$$

hence

$$\begin{aligned} I_{12} &= T \log T + T(\log \operatorname{vol}(M) - 1) + \int_{t_0}^{T'} \log |\eta_M(a+it)| \, dt \\ &\quad + \int_{t_0}^{T'} \log \left| K_M^{-1}(a+it) \right| dt + O(\log T) \\ &= T \log T + T(\log \operatorname{vol}(M) - 1) + I_{121} + I_{122} + O(\log T) \quad \text{as } T \to \infty, \end{aligned} \tag{43}$$

with obvious notation for I_{121} and I_{122}. Stirling's formula implies that

$$\begin{aligned} |K_M^{-1}(a+it)| &= \pi^{-n_1/2} \exp(-c_1 a - \operatorname{Re}(c_2)) \\ &\quad \times \exp\left[n_1 \left(\frac{1}{2} \log |a - 1/2 + it| + O\left(\frac{1}{t}\right) \right) \right] \left(1 + O\left(\frac{1}{t^2}\right) \right), \end{aligned}$$

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as $t \to \infty$, where c_1 and c_2 are constants defined in (5). Therefore,

$$I_{122} = \left(-c_1 a - \mathrm{Re}(c_2) - \frac{n_1}{2}\log\pi\right) T + \frac{n_1}{2}\int_{t_0}^{T'} \log|a - 1/2 + it| + O(\log T)$$
$$= \frac{n_1}{2} T \log T - T\left(c_1 a + \mathrm{Re}(c_2) + \frac{n_1}{2}\log\pi + \frac{n_1}{2}\right) + O(\log T)$$
$$\text{as } T \to \infty. \tag{44}$$

As in the proof of Lemma 12, one can use (4) and Stirling's formula to get

$$I_{121} = \left(\frac{1}{2} - a\right)\frac{\mathrm{vol}(M)}{2} T^2 + 2n_1(\log 2)(a - 1/2)T + O(\log T) \quad \text{as } T \to \infty. \tag{45}$$

By substituting (45) and (44) into (43), we arrive at

$$I_{12} = \left(\frac{1}{2} - a\right)\frac{\mathrm{vol}(M)}{2} T^2 + \left(\frac{n_1}{2} + 1\right) T \log T + O(\log T)$$
$$+ T\left[2n_1 \log 2(a - 1/2) - c_1 a + \log \mathrm{vol}(M) - 1 - \mathrm{Re}(c_2)\right.$$
$$\left. - \frac{n_1}{2}\log\pi - \frac{n_1}{2}\right] \quad \text{as } T \to \infty. \tag{46}$$

The integral I_{13} is estimated by applying the Cauchy's theorem to the function $\log Z_M(s)$ within in the rectangle with vertices $1 - a - iT'$, $2 - iT'$, $2 - it_0$ and $1 - a - it_0$. As in [22], it is easily shown that

$$I_{13} = -\int_{1-a}^{2} \arg Z_M(\sigma - iT')\, d\sigma + O(1) = O\left(\max_{1-a \leqslant \sigma \leqslant 2} \left|\log Z_M(\sigma - iT')\right|\right).$$

From

$$\log Z_M(\sigma - iT') = \log Z_M(2 - iT') - \int_{\sigma - iT'}^{2 - iT'} \frac{Z_M'}{Z_M}(\xi)\, d\xi,$$

and the bound in (20) we obtain the expression

$$I_{13} = \int_{t_0}^{T'} \log|Z_M(1 - a - it)| dt = O((T\log T)^{2-2(1-a)}) = O((T\log T)^{2a})$$
$$\text{as } T \to \infty. \tag{47}$$

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Directly from Lemma 11, we have the estimate

$$I_{14} = \int_{t_0}^{T'} \log|1 + Z_{M,1}(1-(a+it))|\, dt = O\left((T\log T)^{2a}\right) \qquad \text{as } T \to \infty. \tag{48}$$

Combining (41), (46), (47) and (48) yields

$$I_1 = \left(\frac{1}{2} - a\right)\frac{\operatorname{vol}(M)}{2}T^2 + \left(\frac{n_1}{2}+1\right)T\log T + O((T\log T)^{2a}) + TC_{M,a} \qquad \text{as } T \to \infty, \tag{49}$$

where

$$C_{M,a} = (a-1/2)\cdot 2n_1\log 2 + a(\log A_M - c_1) - \log|a_M| + \log \operatorname{vol}(M) - 1 - \operatorname{Re}(c_2) - \frac{n_1}{2}\log\pi - \frac{n_1}{2}.$$

Finally, we have arrived at our estimate for I_1.

5.2 Proof of the Main Theorem

Since $0 < a < 1/2$, we have that $(T\log T)^{2a} = o(T)$. We have shown that I_2 and I_3 are $O(1)$ as $T \to \infty$ and that $I_4 = o(T)$ as $T \to \infty$. Hence, by substituting equation (49) into (39) we get

$$2\pi \sum_{\substack{\rho'=\beta'+i\gamma \\ t_0<\gamma<T'}} (\beta' - a) = \left(\frac{1}{2}-a\right)\frac{\operatorname{vol}(M)}{2}T^2 + \left(\frac{n_1}{2}+1\right)T\log T + TC_{M,a} + o(T) \qquad \text{as } T \to \infty, \tag{50}$$

where $C_{M,a}$ is defined above.

Substituting $a/2$ instead of a into (50), subtracting the obtained formulas, and then dividing by $a/2$ yields the statement (b) of the Main Theorem.

As for part (c) of the Main Theorem, we begin with the formula

$$\sum_{\substack{\rho'=\beta'+i\gamma \\ 0<\gamma\leqslant T}} (\beta' - 1/2) = \sum_{\substack{\rho'=\beta'+i\gamma \\ 0<\gamma<T'}} (\beta' - a) + (a-1/2)\sum_{\substack{\rho'=\beta'+i\gamma \\ 0<\gamma<T'}} 1. \tag{51}$$

The first sum on the right-hand side of (51) is estimated by (50). The second sum on the right-hand side of (51) is estimated by part (b) of the

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Main Theorem, keeping in mind that the difference between the second sum in (51) and the sum in part (b) is the finite number of zeros in the half-plane $\mathrm{Re}(s) < 1/2$.

With all this, the proof of the Main Theorem is complete.

In the case when the surface is co-compact the statement of the Main Theorem is easily deduced, since, in that case $n_1 = c_1 = c_2 = 0$, $H_M = 1$, $A_M = \exp(\ell_{M,0})$ and

$$\frac{\eta_M'}{\eta_M}(s) = \mathrm{vol}(M)(s-1/2)\tan(\pi(s-1/2)) - \pi \sum_{\substack{\{R\} \\ 0<\theta(R)<\pi}} \frac{1}{M_R \sin\theta} \frac{\cos(2\theta-\pi)(s-1/2)}{\cos\pi(s-1/2)}.$$

§6. Corollaries of the Main Theorem

In this section, we deduce three corollaries of our Main Theorem. The results we prove are analogous to [21, Theorem 2 and Theorem 3], with, in their notation, $k = 1$. Similar results may be deduced for the weighted vertical distribution of zeros of the kth derivative, based on the results of § 8, with suitably replaced constants.

COROLLARY 15. *For $\delta > 1/2$, let $N_{\mathrm{ver}}(\delta, T; (Z_M H_M)')$ denote the number of zeros ρ' of $(Z_M H_M)'$ such that $\mathrm{Re}(\rho') > \delta$ and $0 < \mathrm{Im}(\rho') < T$. Then, for an arbitrary $\epsilon > 0$*

$$N_{\mathrm{ver}}\left(\frac{1}{2}+\epsilon, T; (Z_M H_M)'\right) < \frac{1}{\epsilon} N_{\mathrm{w}}(T; (Z_M H_M)').$$

Proof. Trivially, we have the bounds

$$\begin{aligned} N_{\mathrm{ver}}\left(\frac{1}{2}+\epsilon, T; (Z_M H_M)'\right) &< \frac{1}{1/2+\epsilon} \sum_{\substack{(Z_M H_M)'(\sigma+it)=0 \\ \sigma>1/2+\epsilon,\ 0<t<T}} \sigma \\ &= \frac{1}{1/2+\epsilon} \sum_{\substack{(Z_M H_M)'(\sigma+it)=0 \\ \sigma>1/2+\epsilon,\ 0<t<T}} \left(\sigma - \frac{1}{2}\right) \\ &\quad + \frac{1/2}{1/2+\epsilon} N_{\mathrm{ver}}\left(\frac{1}{2}+\epsilon, T; (Z_M H_M)'\right) \end{aligned} \tag{52}$$

Therefore,

$$\frac{2\epsilon}{1+2\epsilon}N_{\mathrm{ver}}\left(\frac{1}{2}+\epsilon, T; (Z_M H_M)'\right) < \frac{2}{1+2\epsilon}N_{\mathrm{w}}(T; (Z_M H_M)'),$$

from which the result immediately follows. ∎

Observe that the lead term in the asymptotic expansion in part (b) of the Main Theorem is $O(T^2)$, whereas the lead term in the asymptotic expansion in part (c) of the Main Theorem is $O(T\log(T))$. Consequently, Corollary 15 shows that zeros of $(Z_M H_M)'$ are concentrated very close the critical line $\mathrm{Re}(s) = 1/2$. The following corollary further quantifies this observation.

Corollary 16. *For any $\delta > 1/2$, let $N^{-}_{\mathrm{ver}}(\delta, T; (Z_M H_M)')$ denote the number of nontrivial zeros $\rho = \sigma + it$ of $(Z_M H_M)'$ with $\sigma < \delta$ and $0 < t < T$. Then, for any constant $\epsilon > 0$,*

$$\lim_{T\to\infty}\frac{N^{-}_{\mathrm{ver}}(1/2+\epsilon, T; (Z_M H_M)')}{N_{\mathrm{vert}}(T; (Z_M H_M)')} = 1.$$

Proof. Corollary 15 implies that

$$1 \geqslant \frac{N^{-}_{\mathrm{ver}}(1/2+\epsilon, T; (Z_M H_M)')}{N_{\mathrm{vert}}(T; (Z_M H_M)')} > 1 - \frac{1}{\epsilon}\frac{N_{\mathrm{w}}(T; (Z_M H_M)')}{N_{\mathrm{vert}}(T; (Z_M H_M)')}. \tag{53}$$

From the Main Theorem (b) and (c) we deduce that

$$\frac{N_{\mathrm{w}}(T; (Z_M H_M)')}{N_{\mathrm{vert}}(T; (Z_M H_M)')} \to 0 \qquad \text{as } T\to\infty.$$

Therefore, by passing to the limit as $T\to\infty$ in (53), the claimed result follows. ∎

The following corollary gives estimates of short sums of distances $(\sigma - 1/2)$.

Corollary 17. *Let $0 < U < T$. Then,*

$$\begin{aligned}2\pi \sum_{\substack{(Z_M H_M)'(\sigma+it)=0\\ \sigma>1/2, T<t\leqslant T+U}}\left(\sigma-\frac{1}{2}\right) &= \left(\frac{n_1}{2}+1\right)U\log(T+U)\\ &\quad + \left(\log\frac{\mathfrak{g}_1 \mathrm{vol}(M) A_M^{1/2}}{\pi^{n_1/2}|d(1)a_M|}\right)U\\ &\quad + o(T) + O(U^2/T) \qquad \textit{as } T\to\infty.\end{aligned} \tag{54}$$

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Proof. The left-hand side of the (54) is equal to $2\pi\,(N_{\rm w}(T+U;(Z_M H_M)') - N_{\rm w}(T;(Z_M H_M)')$, hence part (c) of the Main Theorem yields

$$2\pi \sum_{\substack{(Z_M H_M)'(\sigma+it)=0\\ \sigma>1/2, T<t\leqslant T+U}} \left(\sigma-\frac{1}{2}\right) = \left(\frac{n_1}{2}+1\right)\left(T\log\left(1+\frac{U}{T}\right) - U + U\log(T+U)\right)$$

$$+\left(\log\frac{\mathfrak{g}_1 \mathrm{vol}(M) A_M^{1/2}}{\pi^{n_1/2}|d(1)a_M|}\right)U + o(T) \quad \text{as } T\to\infty. \tag{55}$$

The elementary observation that $T\log(1+U/T) - U = O(U^2/T)$ completes the proof. ∎

§7. Examples

The Main Theorem naturally leads to the following question: Are there examples of groups Γ where $e^{\ell_{M,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$ as well as groups where $e^{\ell_{M,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2$? The purpose of this section is to prove Proposition 1 and present examples of groups in each category. In fact, there are examples of both arithmetic and nonarithmetic groups in each category.

7.1 Congruence subgroups

In this subsection we prove part (i) of Proposition 1.

Let $\Gamma = \overline{\Gamma_0(N)}$ be the congruence subgroup defined by the arithmetic condition

$$\overline{\Gamma_0(N)} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : c \equiv 0 \ (\mathrm{mod}\ N) \right\} \Big/ \pm I,$$

where I denotes the identity matrix and N is a positive integer. If $N = p_1 \cdots p_r$, for distinct primes $p_1, \ldots, p_r$; then, it is proved in [16], that the corresponding surface has $n_1 = 2^r$ cusps and the scattering determinant is given by the formula

$$\varphi_N(s) = \left[\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\right]^{n_1} \left[\frac{\zeta_{\mathbb{Q}}(2s-1)}{\zeta_{\mathbb{Q}}(2s)}\right]^{n_1} \prod_{p|N}\left(\frac{1-p^{2-2s}}{1-p^{2s}}\right)^{n_1/2},$$

where $\zeta_{\mathbb{Q}}$ is the Riemann zeta function. Now, it is easy to show that $(\mathfrak{g}_2/\mathfrak{g}_1)^2 = 4$. In the case N is not square-free, an application of [14], formula (4.2), page 536 yields the same conclusion.

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All elements of $\overline{\Gamma_0(N)}$ have integer entries, so any hyperbolic element has trace whose absolute value is at least equal to 3. Therefore, $e^{\ell_{M,0}} \geqslant u$ where u is a solution to $u^{1/2} + u^{-1/2} = 3$. Solving, we get that $u = ((3+\sqrt{5})/2)^2 > 4$. Therefore, for any such group $\overline{\Gamma_0(N)}$, one has that $e^{\ell_{M,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2$.

In the case of the principal congruence subgroups $\overline{\Gamma(N)}$ the scattering determinant can be computed using the analysis presented in [14] and [16]. As above, one shows that $(\mathfrak{g}_2/\mathfrak{g}_1)^2 = 4$ because the Dirichlet series portion of the scattering determinant is shown to be given by ratios of classical Dirichlet series. Furthermore, the matrices in $\overline{\Gamma(N)}$ also have integral entries, so $e^{\ell_{M,0}} \geqslant ((3+\sqrt{(5)})/2)^2 > 4$.

7.2 Moonshine subgroups

We now prove part (ii) of Proposition 1.

Following [8], we use the term "moonshine group" for any subgroup Γ of $\mathrm{PSL}(2,\mathbb{R})$ which satisfies the following two conditions. First, there exists an integer $N \geqslant 1$ such that Γ contains $\overline{\Gamma_0(N)}$. Second, Γ contains the element $\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ if and only if $k \in \mathbb{Z}$.

Following [6, p. 363], let f be a square-free, nonnegative integer, and consider the group

$$\Gamma_0(f)^+ := \left\{ e^{-1/2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) : a, b, c, d, \right.$$
$$\left. e \in \mathbb{Z}, e \mid f, e \mid a, e \mid d, f \mid c, ad - bc = e \right\}.$$

In [6, Lemma 2.20] it is proved that the parabolic elements of $\Gamma_0(f)^+$ have integral entries. Therefore, $\Gamma_0(f)^+$ is a moonshine group. Let $\overline{\Gamma_0(f)^+} = \Gamma_0(f)^+/\pm I$. In [20] it is proved that the Riemann surface $\overline{\Gamma_0(f)^+}\backslash\mathbb{H}$ for all square-free f has finite volume and one cusp at infinity.

Consider the case when $f = 5$. The scattering matrix in this case has a single entry which, as proved in [20] is given by

$$\Phi_5(s) = \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)} \left(\frac{5^s + 5}{5^s(5^s+1)} \right) \cdot \frac{\zeta_{\mathbb{Q}}(2s-1)}{\zeta_{\mathbb{Q}}(2s)},$$

hence, one immediately can show that $(\mathfrak{g}_2/\mathfrak{g}_1)^2 = 4$.

It is easy to confirm that $\gamma = \left(\begin{smallmatrix} 0 & -1/\sqrt{5} \\ \sqrt{5} & \sqrt{5} \end{smallmatrix}\right) \in \Gamma_0(5)^+$. The trace of γ is $\sqrt{5} > 2$, hence γ is hyperbolic. Therefore, $e^{\ell_{M_{5},0}} \leqslant u$ where u is a positive solution of $u^{1/2} + u^{-1/2} = \sqrt{5}$. Solving, we have that $u = ((1+\sqrt{5})/2)^2 < 4$.

With all this, we have proved that $e^{\ell_{M_{5},0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$.

The surface $\overline{\Gamma_0(5)^+}\backslash\mathbb{H}$ has a signature (0;2,2,2;1) meaning that its genus is zero, it has three inequivalent elliptic points of order two and one cusp. The surface $\overline{\Gamma_0(6)^+}\backslash\mathbb{H}$ has the same signature, as shown in [6, Table C]. The scattering matrix in this case has a single entry which is given by

$$\Phi_6(s) = \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\left(\frac{(2^s+2)(3^s+3)}{6^s(2^s+1)(3^s+1)}\right)\cdot\frac{\zeta_{\mathbb{Q}}(2s-1)}{\zeta_{\mathbb{Q}}(2s)}.$$

Obviously, $\mathfrak{g}_1 = \sqrt{6}$, $\mathfrak{g}_2 = 2\sqrt{3}$, hence $(\mathfrak{g}_2/\mathfrak{g}_1)^2 = 2$.

On the other hand, $\min\{|\mathrm{Tr}A| : A \in \mathcal{H}(\Gamma_0(6)^+)\} = \sqrt{6}$, hence $e^{\ell_{M_6,0}} \geqslant u$ where $u > 1$ is a solution of the equation $u^{1/2} + u^{-1/2} = \sqrt{6}$. Since $u = ((\sqrt{6}+\sqrt{2})/2)^2 > 2$, we see that $e^{\ell_{M_6,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2$. This completes the proof of Proposition 1(ii).

7.3 On existence of surfaces where $e^{\ell_{M,0}} = (\mathfrak{g}_2/\mathfrak{g}_1)^2$

We now argue the existence of an abundance of surfaces for which $e^{\ell_{M,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$ and prove part (iii) of Proposition 1.

Let M_τ denote a degenerating family of Riemann surfaces, parameterized by the holomorphic parameter τ, which approach the Deligne–Mumford boundary of moduli space when τ approaches zero. One can select distinguished points of M_τ which are either removed or whose local coordinates z are replaced by fractional powers $z^{1/n}$. By doing so, one obtains a degenerating sequence of hyperbolic Riemann surfaces of any signature; we refer the reader to [15] and references therein for further details regarding the construction of the sequence of degenerating hyperbolic Riemann surfaces.

By construction, the length of the smallest geodesic on M_ℓ approaches zero, so then $\exp(\ell_{M_\tau,0})$ approaches one as τ approaches zero. In [9], the authors prove that through degeneration, parabolic Eisenstein series on M_τ converge to parabolic Eisenstein series on the limit surface; see part (ii) of the Main Theorem of [9]. To be precise, one needs that the holomorphic parameter s of the parabolic Eisenstein series lies in the half-plane $\mathrm{Re}(s) > 1$ and the spatial parameter z to lie in a bounded region of M_τ. However, in these ranges, one can compute the scattering matrix by computing the zeroth Fourier coefficient of the parabolic Eisenstein series, and, subsequently, compute the ratio $\mathfrak{g}_2/\mathfrak{g}_1$ on M_τ. Since the parabolic Eisenstein series converge through degeneration to the parabolic Eisenstein series on the limit surface, the associated scattering matrix converges to a submatrix Φ of the full scattering matrix on the limit surface. Clearly, the determinant of Φ can be decomposed into a product of Gamma functions and a Dirichlet series, where the Dirichlet series is such that $\mathfrak{g}_2/\mathfrak{g}_1 > 1$.

Therefore, we conclude that for all τ sufficiently close to zero, we have that $e^{\ell_{M_\tau,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$. In fact, all surfaces near the Deligne–Mumford boundary of any given moduli space satisfy the inequality $e^{\ell_{M_\tau,0}} < (\mathfrak{g}_2/\mathfrak{g}_1)^2$.

In addition, let us assume that one is considering a moduli space which contains a congruence subgroup so then there exists a surface where $e^{\ell_{M_\tau,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2$. Then by combining the above argument with the computations from § 7.1, there exists surfaces for which $e^{\ell_{M_\tau,0}} = (\mathfrak{g}_2/\mathfrak{g}_1)^2$. However, we have not been successful in our attempts to explicitly construct such a surface. In a sense, our Main Theorem shows that surfaces for which $e^{\ell_{M_\tau,0}} = (\mathfrak{g}_2/\mathfrak{g}_1)^2$ have a larger number of zeros of $(Z_M H_M)'$ than nearby surfaces for which the inequality holds.

§8. Higher derivatives

In this section, we outline the proof of the Main Theorem for higher order derivatives of $Z_M H_M$. The results are analogous to theorems proved for the zeros of the higher order derivatives of the Riemann zeta function; see [4] and [21].

8.1 Preliminary lemmas on higher derivatives

In order to deduce the vertical and weighted vertical distribution of zeros of the higher order derivatives of $(Z_M H_M)$ we prove some preliminary lemmas, analogous to lemmas in § 4.

LEMMA 18. *Let $f_M(s)$ be defined by (18) and $\widetilde{Z}_M(s)$ defined by (19). Let us define, inductively, the functions $\widetilde{Z}_{M,j}(s)$ as $\widetilde{Z}_{M,0}(s) := Z_M(s)$, $\widetilde{Z}_{M,1}(s) := \widetilde{Z}_M(s)$ and, for $j \geqslant 2$,*

$$\widetilde{Z}_{M,j}(1-s) = \frac{1}{f_M(s)}\left((j-1)\frac{f_M'}{f_M}(s) + \frac{\eta_M'}{\eta_M}(s) - \frac{K_M'}{K_M}(s) - \sum_{i=0}^{j-1}\frac{\widetilde{Z}_{M,i}'}{\widetilde{Z}_{M,i}}(1-s)\right). \tag{56}$$

Then for every positive integer k the kth derivative of the function $Z_M H_M$ can be represented as

$$(Z_M H_M)^{(k)}(s) = (f_M(s))^k\, \eta_M(s) K_M^{-1}(s) Z_M(1-s) \prod_{i=1}^{k} \widetilde{Z}_{M,i}(1-s). \tag{57}$$

Proof. The proof is based on a rather obvious induction argument. □

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LEMMA 19. *For* $j \geqslant 1$, *let* $Z_{M,j}(s) := \widetilde{Z}_{M,j}(s) - 1$. *For small* $\delta > 0$ *and* $\delta_1 > 0$, *let* σ_1 *be a real number such that* $\sigma_1 \geqslant 1/2 + \delta_1 > 1/2$ *and* $(\sigma_1 \pm iT)$ *is away from circles of a fixed, small radius* $\delta > 0$, *centered at integers. Then for* $k = 0, 1$

$$Z^{(k)}_{M,j}(\sigma_1 \pm iT) = O\left(\frac{(T\log T)^{2-2\sigma_1}\log^k T}{(\sigma_1 - 1/2)T}\right) \quad \textit{as } T \to \infty, \tag{58}$$

and

$$\frac{\widetilde{Z}'_{M,j}}{\widetilde{Z}_{M,j}}(\sigma_1 \pm iT) = O\left(\frac{(T\log T)^{2-2\sigma_1}\log T}{(\sigma_1 - 1/2)T}\right) \quad \textit{as } T \to \infty. \tag{59}$$

Proof. We prove the statement by induction in $j \geqslant 1$. When $j = 1$, we use formula (26), which we differentiate, use the bound on the growth of the derivative of the digamma function (see [1, formula 6.4.12.]) and the bound (20) with $k = 0$ or $k = 1$. These computations, which are elementary, allow one to prove (58) for $\sigma_1 \geqslant 1/2 + \delta_1 > 1/2$ in the case when $j = 1$. In addition,

$$\begin{aligned}\frac{\widetilde{Z}'_{M,1}}{\widetilde{Z}_{M,1}}(\sigma_1 \pm iT) &= \frac{Z'_{M,1}(\sigma_1 \pm iT)}{1 + Z_{M,1}(\sigma_1 \pm iT)} \\ &= O\left(\frac{(T\log T)^{2-2\sigma_1}\log T}{(\sigma_1 - 1/2)T}\right) \quad \text{as } T \to \infty.\end{aligned}$$

With all this, we have proved (59) for $j = 1$.

Assume now that (58) and (59) hold true for all $1 \leqslant m \leqslant j$. Then, by (56) we get

$$1 + Z_{M,j+1}(s) = \widetilde{Z}_{M,j+1}(s) = 1 + Z_{M,j}(s) + \frac{1}{f_M(s)}\left(\frac{f'_M}{f_M}(s) - \frac{\widetilde{Z}'_{M,k,j}}{\widetilde{Z}_{M,j}}(s)\right).$$

Therefore, by the inductive assumption on $\widetilde{Z}'_{M,j}/\widetilde{Z}_{M,j}$ and $Z_{M,j}$, we have for $k = 0, 1$,

$$Z^{(k)}_{M,j+1}(\sigma_1 \pm iT) = O\left(\frac{(T\log T)^{2-2\sigma_1}\log^k T}{(\sigma_1 - 1/2)T}\right) \quad \text{as } T \to \infty.$$

In other words, (58) holds true with $m = j + 1$. In addition,

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$$\frac{\widetilde{Z}'_{M,j+1}}{\widetilde{Z}_{M,j+1}}(\sigma_1 \pm iT) = \frac{Z'_{M,j+1}(\sigma_1 \pm iT)}{1 + Z_{M,j+1}(\sigma_1 \pm iT)}$$
$$= O\left(\frac{(T \log T)^{2-2\sigma_1} \log T}{(\sigma_1 - 1/2)T}\right) \qquad \text{as } T \to \infty.$$

The proof is complete. ∎

For any integer $k \geqslant 2$, let us define $a_{M,k} := (-1)^{k-1} a_M \log^{k-1} A_M$, where we set $a_{M,1} := a_M$. The analogue of the function $X_M(s)$, defined by (12), is

$$X_{M,k}(s) := \frac{A_M^s}{a_{M,k}} (Z_M H_M)^{(k)}(s), \tag{60}$$

where, of course, $X_{M,1}(s) = X_M(s)$.

LEMMA 20. *For any integer $k \geqslant 1$, there exists constants $\sigma_k > 1$ and $0 < c_{\Gamma,k} < 1$ such that for all $\sigma = \mathrm{Re}(s) \geqslant \sigma_k$,*

$$X_{M,k}(s) = 1 + O(c_{\Gamma,k}^{\sigma}) \neq 0 \qquad \textit{as } \sigma \to +\infty.$$

Proof. For $k = 1$, the statement is Lemma 10. Furthermore, from the proof of Lemma 10 and the definition of constants A_M and $a_{M,1}$, we see that

$$(Z_M H_M)'(s) = Z_M(s) H_M(s) \mathcal{D}_1(s), \tag{61}$$

where $\mathcal{D}_1(s)$ is a Dirichlet series, converging absolutely for $\mathrm{Re}(s) > \sigma_1$, for sufficiently large σ_1, with the leading term equal to $a_{M,1} \cdot A_M^{-s}$ as $\mathrm{Re}(s) \to +\infty$.

Let us define, for $k \geqslant 1$ and $\mathrm{Re}(s) \gg 0$

$$(Z_M H_M)^{(k)}(s) = Z_M(s) H_M(s) \mathcal{D}_k(s).$$

We claim that $\mathcal{D}_k(s)$ is a Dirichlet series with the leading term equal to $a_{M,k} \cdot A_M^{-s}$ as $\mathrm{Re}(s) \to +\infty$. The statement is obviously true for $k = 1$. A simple inductive argument shows that the statement is true for all $k \geqslant 1$. Therefore, for $\mathrm{Re}(s) = \sigma \gg 0$, we may write

$$(Z_M H_M)^{(k)}(s) = Z_M(s) H_M(s) \frac{a_{M,k}}{A_M^s} \left(1 + O(A_{\Gamma,k}^{-\sigma})\right) \qquad \text{as } \mathrm{Re}(s) \to \infty.$$

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Equation (32) implies that there exists $\sigma_k \geqslant 1$ and a constant $C_{\Gamma,k} > 1$ such that for $\operatorname{Re} s > \sigma_k$, we have

$$(Z_M H_M)^{(k)}(s) = \frac{a_{M,k}}{A_M^s}\left[1 + O\left(\frac{1}{C_{\Gamma,k}^{\operatorname{Re}(s)}}\right)\right] \quad \text{as } \operatorname{Re}(s) \to \infty.$$

Setting $c_{\Gamma,k} = 1/C_{\Gamma,k}$ completes the proof. ∎

LEMMA 21. *For arbitrary $\epsilon > 0$, $t \geqslant 1$ and $\sigma_2 \geqslant 1$ such that $-\sigma_2$ is not a pole of $(Z_M H_M)$ we have, for any positive integer k*

$$(Z_M H_M)^{(k)}(\sigma + it) = \begin{cases} O(\exp \epsilon t) & \text{for } \frac{1}{2} \leqslant \sigma \leqslant \sigma_0, \\ O\left(\exp(1/2 - \sigma + \epsilon)t\right) & \text{for } -\sigma_2 \leqslant \sigma < 1/2, \end{cases}$$

as $t \to \infty$.

Proof. When $k = 1$, the statement is proved in Lemma 14. Assume that the statement of Lemma holds for an integer $k \geqslant 1$. Then for $1/2 \leqslant \operatorname{Re}(s) = \sigma \leqslant \sigma_0$ the Cauchy integral formula yields

$$(Z_M H_M)^{(k+1)}(s) = \frac{1}{2\pi i}\int_C \frac{(Z_M H_M)^{(k)}(z)}{(z - s)^2} dz$$

where C is a circle of a small, fixed radius $r < \epsilon$, centered at s. Using the inductive assumption on $(Z_M H_M)^{(k)}(z)$, we then get the bounds

$$(Z_M H_M)^{(k+1)}(\sigma + it) = O\left(\exp((r + \epsilon)t)/r\right) = O\left(\exp(2\epsilon t)\right),$$

for $1/2 \leqslant \sigma \leqslant \sigma_0$ and $t \geqslant 1$. This proves the first part of Lemma for $(Z_M H_M)^{(k+1)}(z)$, hence, the first part of the Lemma holds true for all $k \geqslant 1$.

In the case when $\sigma < 1/2$, we employ the functional equation (57) for $(Z_M H_M)^{(k)}$ to deduce that

$$\begin{aligned} |(Z_M H_M)^{(k+1)}(-\sigma_2 + it)| = |(Z_M H_M)^{(k)}(-\sigma_2 + it)| \\ \cdot \left|\left[k\frac{f'}{f}(-\sigma_2 + it) + \frac{\eta_M'}{\eta_M}(-\sigma_2 + it) \right.\right. \\ \left.\left. - \frac{K_M'}{K_M}(-\sigma_2 + it) - \sum_{i=0}^{k} \frac{\widetilde{Z}_{M,i}'}{\widetilde{Z}_{M,i}}(1 + \sigma_2 - it)\right]\right|. \end{aligned}$$

Since $\sigma_2 \geqslant 1$, we have $Z_M'/Z_M(1 + \sigma_2 - it) = O(1)$ as $t \to +\infty$. Furthermore, formula (59) and the same computations as in the proof of Lemma 14

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imply that

$$k\frac{f'}{f}(-\sigma_2+it)+\frac{\eta'_M}{\eta_M}(-\sigma_2+it)-\frac{K'_M}{K_M}(-\sigma_2+it)$$
$$-\sum_{i=0}^{k}\frac{\widetilde{Z}'_{M,i}}{\widetilde{Z}_{M,i}}(1+\sigma_2-it)=O(t)\qquad \text{as } t\to\infty,$$

since the leading term in the above expression is $\mathrm{vol}(M)(1/2+\sigma_2-it)$ $\tan(\pi(1/2+\sigma_2-it))$. By the inductive assumption on $(Z_MH_M)^{(k)}(-\sigma_2+it)$, we get

$$|(Z_MH_M)^{(k+1)}(-\sigma_2+it)|=O\left(\exp\left(\left(\frac{1}{2}+\sigma_2+\epsilon\right)\mathrm{vol}(M)t\right)\right),$$
$$\text{as } t\to\infty.$$

As in the proof of Lemma 14, one applies the Phragmen–Lindelöf theorem to the function $(Z_MH_M)^{(k+1)}$ in the open sector bounded by the lines $\mathrm{Im}(s)=1$, $\mathrm{Re}(s)=-\sigma_2$ and $\mathrm{Re}(s)=1/2$. As a result, the proof of the second part of the Lemma is complete for $(Z_MH_M)^{(k+1)}$. ∎

8.2 Distribution of zeros of $(Z_MH_M)^{(k)}$

The following theorem is the analogue of the Main Theorem for zeros of higher derivatives of (Z_MH_M).

THEOREM 22. *With the notation as above, the following statements are true for any integer $k\geqslant 2$.*

(a) *For $\sigma<1/2$, there exist $t_0>0$ such that $(Z_MH_M)^{(k)}(\sigma+it)\neq 0$ for all $|t|>t_0$.*

(b)

$$N_{\mathrm{ver}}(T;(Z_MH_M)^{(k)})=N_{\mathrm{ver}}(T;(Z_MH_M)')+o(T)\qquad \text{as } T\to\infty. \tag{62}$$

(c)

$$N_{\mathrm{w}}(T;(Z_MH_M)^{(k)})=N_{\mathrm{w}}(T;(Z_MH_M)')+\frac{(k-1)T}{2\pi}[\log(T\cdot\mathrm{vol}(M))-1]$$
$$-\frac{T}{2\pi}\log((k-1)\log A_M)+o(T)\qquad \text{as } T\to\infty. \tag{63}$$

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Proof. We first outline the proof of part (a). For $k \geqslant 2$, $\sigma < 1/2$ and $s = \sigma \pm iT$ equation (57) yields

$$\begin{aligned} \frac{(Z_M H_M)^{(k)}}{(Z_M H_M)^{(k-1)}}(s) &= \log\left((Z_M H_M)^{(k-1)}(s)\right)' \\ &= (k-1)\frac{f'}{f}(s) + \frac{\eta_M'}{\eta_M}(s) - \frac{K_M'}{K_M}(s) - \frac{Z_M'}{Z_M}(1-s) \\ &\quad - \sum_{i=1}^{k-1} \frac{\widetilde{Z}_{M,i}'}{\widetilde{Z}_{M,i}}(1-s). \end{aligned} \tag{64}$$

We now apply (59) with $\sigma_1 = 1 - \sigma > 1/2$ and (20) to deduce that

$$\frac{Z_M'}{Z_M}(1-s) + \sum_{i=1}^{k-1} \frac{\widetilde{Z}_{M,i}'}{\widetilde{Z}_{M,i}}(1-s) = O\left(\frac{(T\log T)^{2\sigma}\log T}{(1/2-\sigma)}\right) \quad \text{as } T \to \infty.$$

Since $\mathrm{Re}(\eta_M'/\eta_M(\sigma \pm iT)) = -\mathrm{vol}(M)t + O(\log t)$ and $K_M'/K_M(\sigma \pm it) = O(\log t)$ as $t \to +\infty$, we immediately deduce from (64) that

$$\begin{aligned} \mathrm{Re}\left(-\frac{(Z_M H_M)^{(k)}}{(Z_M H_M)^{(k-1)}}(\sigma \pm it)\right) &= \mathrm{vol}(M)t \\ &\quad + O\left(\max\left\{\log t, \frac{(t\log t)^{2\sigma}\log t}{(1/2-\sigma)}\right\}\right) \\ &\quad \text{as } t \to +\infty, \end{aligned}$$

for any $\sigma < 1/2$. This proves part (a).

The proof of parts (b) and (c) closely follows lines of the proof of parts (b) and (c) of the Main Theorem. We fix a large positive number T and choose number T' to be a bounded distance from T such that T' is distinct from the imaginary part of any zero of $Z_M H_M$. We fix a number $a \in (0, 1/2)$ and use part (a) of the Theorem to choose $t_0 > 0$ to be the number such that $(Z_M H_M)^{(k)}(\sigma + it) \neq 0$ for all $\sigma \leqslant a$ and $|t| > t_0$. Let σ_0 be a constant such that $\sigma_0 \geqslant \max\{\sigma_0', \sigma_k\}$, where σ_0' is defined in Lemma 3 and σ_k is defined in Lemma 20.

We apply Littlewood's theorem to the function $X_{M,k}(s)$, defined by (60) which is holomorphic in the rectangle $R(a, T')$ with vertices

$a+it_0$, σ_0+it_0, σ_0+iT', $a+iT'$. The resulting formula is

$$\begin{aligned}2\pi \sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)}\\ t_0<\gamma^{(k)}<T',\beta^{(k)}>a}} \left(\beta^{(k)}-a\right) &= \int_{t_0}^{T'} \log\left|X_{M,k}(a+it)\right| dt \\ &\quad - \int_{t_0}^{T'} \log\left|X_{M,k}(\sigma_0+it)\right| dt \\ &\quad - \int_{a}^{\sigma_0} \arg X_{M,k}(\sigma+it_0)\, d\sigma \\ &\quad + \int_{a}^{\sigma_0} \arg X_{M,k}(\sigma+iT')\, d\sigma \\ &= I_{1,k}+I_{2,k}+I_{3,k}+I_{4,k}, \end{aligned} \tag{65}$$

where $\rho^{(k)}$ denotes the zero of $(Z_M H_M)^{(k)}$. By the choice of t_0, the sum on the left-hand side of (65) is actually taken over all zeros $\rho^{(k)}$ of $(Z_M H_M)^{(k)}$ with imaginary part in the interval (t_0, T').

Trivially, $I_{3,k}=O(1)$ as $T\to+\infty$. The application of Lemma 20 immediately yields that $I_{2,k}=O(1)$ as $T\to+\infty$, once we apply the same method as in evaluation of I_2.

One can follow the steps of the proof that $|\arg X_M(\sigma+iT')|=o(T)$ as $T\to+\infty$ in the present setting. One uses function $X_{M,k}$ instead of X_M and Lemma 21 instead of Lemma 14. From this, we deduce that $|\arg X_{M,k}(\sigma+iT')|=o(T)$ as $T\to+\infty$. Therefore, it is left to evaluate $I_{1,k}$.

From definition of $X_{M,k}$, using the functional equation (57) for $(Z_M H_M)^{(k)}$, we get for $k\geqslant 2$, the expression

$$\begin{aligned} I_{1,k} = &\int_{t_0}^{T'} \log\left|A_M^{(a+it)} a_{M,k}^{-1}\right| dt + k\int_{t_0}^{T'} \log\left|f_M(a+it)\right| dt \\ &+ \int_{t_0}^{T'} \log\left|\eta_M(a+it)\right| dt \\ &+ \int_{t_0}^{T'} \log\left|K_M^{-1}(a+it)\right| dt + \int_{t_0}^{T'} \log\left|Z_M(1-a-it)\right| dt \\ &+ \sum_{i=1}^{k-1} \int_{t_0}^{T'} \log\left|1+Z_{M,i}(1-a-it)\right| dt. \end{aligned} \tag{66}$$

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By employing equation (58) with $k=0$, we get

$$\int_{t_0}^{T'} \log|1+Z_{M,i}(1-a-it)|dt = O((T\log T)^{2a}) \quad \text{as } T\to\infty$$

for all $i=1,\ldots,k-1$. Substituting this equation, together with (42), (44), (45) and (47) into (66), we immediately deduce that

$$I_{1,k} = \left(\frac{1}{2}-a\right)\frac{\text{vol}(M)}{2}T^2 + \left(\frac{n_1}{2}+k\right)T\log T \\ + C_{M,a,k}T + O((T\log T)^{2a}) \quad \text{as } T\to\infty,$$

where

$$C_{M,a,k} = 2\left(a-\frac{1}{2}\right)n_1\log 2 + a\log A_M - \log|a_{M,k}| \\ + k(\log(\text{vol}(M))-1) + 2a\log\mathfrak{g}_1 - \log|d(1)| - \frac{n_1}{2}(\log\pi+1).$$

Combining this equation with the bounds on $I_{2,k}$, $I_{3,k}$ and $I_{4,k}$ and (65), we get

$$2\pi\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)}\\ t_0<\gamma^{(k)}<T'}}\left(\beta^{(k)}-a\right) = \left(\frac{1}{2}-a\right)\frac{\text{vol}(M)}{2}T^2 + \left(\frac{n_1}{2}+k\right)T\log T \\ + C_{M,a,k}T + o(T) \quad \text{as } T\to\infty. \tag{67}$$

Replacing a by $a/2$ in (67) and subtracting proves part (b). Part (c) is proved by employing an analogue of equation (51), with β' and ρ' replaced by $\beta^{(k)}$ and $\rho^{(k)}$. □

REMARK 23. The statement of Theorem 22 is true in the case of co-compact Riemann surfaces $\Gamma\setminus\mathbb{H}$ when taking $H_M=1$ and $A_M=\exp(\ell_{M,0})$ in (62) and (63).

In the case when $\Gamma\setminus\mathbb{H}$ is compact the statement (b) of Theorem 22 was announced by Luo in [22], with the weaker error term $O(T)$. As one can see, we put considerable effort into the analysis yielding the error term $o(T)$, and the structure of the constant $C_{M,a,k}$ is, in our opinion, fascinating.

REMARK 24. From the formula (63) for the weighted vertical distribution of zeros of $(Z_MH_M)^{(k)}$, we see that the differentiation of $(Z_MH_M)^{(k)}$

increases the sum $N_{\mathrm{w}}(T;(Z_M H_M)^{(k)})$ by the quantity $[(1/(2\pi))\cdot T\log T + O(T)]$ as $T\to\infty$. Hence, after each differentiation, zeros of $(Z_M H_M)'$ move further to the right of $1/2$. Since every zero of $(Z_M H_M)'$ on the line $\mathrm{Re}(s)=1/2$ (up to finitely many of them) is a multiple zero of Z_M, this result fully supports the "bounded multiplicities conjecture". To recall, the "bounded multiplicities conjecture" asserts that the order of every multiple zero of Z_M is uniformly bounded, or, equivalently, that the dimension of every eigenspace associated to the discrete eigenvalue of the Laplacian on M is uniformly bounded, with a bound depending solely upon M.

§9. Concluding remarks

9.1 Revisiting Weyl's law

Weyl's law for an arbitrary finite volume hyperbolic Riemann surface M is the following asymptotic formula, which we quote from [14, p. 466]:

$$N_{M,\mathrm{dis}}(T)+N_{M,\mathrm{con}}(T)=\frac{\mathrm{vol}(M)}{4\pi}T^2-\frac{n_1}{\pi}T\log T+\frac{n_1T}{\pi}(1-\log 2) + O\left(T/\log T\right) \quad \text{as } T\to\infty, \tag{68}$$

where

$$N_{M,\mathrm{dis}}(T)=\#\{s=1/2+it \mid Z_M(s)=0 \text{ and } 0\leqslant t\leqslant T\}$$

and

$$N_{M,\mathrm{con}}(T)=\frac{1}{4\pi}\int_{-T}^{T}\frac{-\phi_M'}{\phi_M}(1/2+it)\,dt.$$

The term $N_{M,\mathrm{dis}}(T)$ counts the number of zeros of the Selberg zeta function $Z_M(s)$ on the critical line $\mathrm{Re}(s)=1/2$, whereas the term $N_{M,\mathrm{con}}(T)$ is related to the number of zeros of $Z_M(s)$ off the critical line but within the critical strip. In the following proposition, we relate the counting function $N_{\mathrm{ver}}(T;\phi_M)$ with the function $N_{M,\mathrm{con}}(T)$, showing that the constant $\mathfrak{g}_1$ appears in the resulting asymptotic formula.

PROPOSITION 25. *There exists a sequence $\{T_n\}$ of positive numbers tending toward infinity such that, with the notation as above, we have the asymptotic formula*

$$N_{\mathrm{ver}}(T_n;\phi_M)=N_{M,\mathrm{con}}(T_n)-\frac{\log\mathfrak{g}_1}{\pi}T_n+O\left(\log T_n\right) \quad \textit{as } n\to\infty.$$

Proof. Let $R(T)$ denote the rectangle with vertexes $1/2-iT$, $\sigma_0'-iT$, $\sigma_0'+iT$, $1/2+iT$, where $\sigma_0'>\sigma_0$, where σ_0 is defined in § 1.3. Therefore, the series (8) converges uniformly and absolutely for Re $s \geqslant \sigma_0'$, and all zeros of ϕ_M with real part greater than $1/2$ lie inside $R(T)$. Recall that the zeros of ϕ_M appear in pairs of the form ρ and $\overline{\rho}$. As a result, the proposition follow by studying the expression

$$2N_{\text{ver}}(T;\phi_M) = \frac{1}{2\pi i}\int_{R(T)} \frac{\phi_M'}{\phi_M}(s)\,ds = \frac{-1}{2\pi}\int_{-T}^{T}\frac{\phi_M'}{\phi_M}\left(\frac{1}{2}+it\right)dt + \frac{1}{2\pi}\int_{-T}^{T}\frac{\phi_M'}{\phi_M}(\sigma_0'+it)\,dt + I_1(T) + I_2(T)$$

where I_1 and I_2 denote the integrals along the horizontal lines which bound $R(T)$. In [18, Theorem 7.1] it is proved that ϕ_M is of regularized product type with order $M=0$. As a result, from [19, Chapter 1], we have the existence of a sequence of real numbers $\{T_n\}$ tending to infinity such that $I_1(T_n) = O(\log T_n)$ and $I_2(T_n) = O(\log T_n)$ when $n \to \infty$, so then

$$2N_{\text{ver}}(T_n;\phi_M) = -\frac{1}{2\pi}\int_{-T_n}^{T_n}\frac{\phi_M'}{\phi_M}\left(\frac{1}{2}+it\right)dt + \frac{1}{2\pi}\int_{-T_n}^{T_n}\frac{\phi_M'}{\phi_M}(\sigma_0'+it)\,dt + O(\log T_n) \quad \text{when } n\to\infty.$$

Using the notation as above, we now write

$$\int_{-T_n}^{T_n}\frac{\phi_M'}{\phi_M}(\sigma_0'+it)\,dt = \sum_{i=1}^{\infty}\frac{b\left(q_i\right)}{q_i^{\sigma_0'}}\int_{-T_n}^{T_n}\frac{dt}{q_i^{it}} - 4T\log\mathfrak{g}_1 + n_1\int_{-T_n}^{T_n}\left(\frac{\Gamma'}{\Gamma}\left(\sigma_0'+it-\frac{1}{2}\right) - \frac{\Gamma'}{\Gamma}(\sigma_0'+it)\right)dt.$$

Interchanging the sum and the integral above is justified by the fact that the series defining $H_M'/H_M(s)$ converges absolutely and uniformly for $\text{Re}(s) > \sigma_0$. Furthermore, we also have that

$$\sum_{i=1}^{\infty}\frac{b\left(q_i\right)}{q_i^{\sigma_0'}}\int_{-T_n}^{T_n}\frac{dt}{q_i^{it}} = O\left(1\right) \quad \text{as } n\to\infty,$$

Using the series representation of the digamma function we get that

$$\int_{-T_n}^{T_n}\left(\frac{\Gamma'}{\Gamma}\left(\sigma_0'+it-\frac{1}{2}\right) - \frac{\Gamma'}{\Gamma}\left(\sigma_0'+it\right)\right)dt = O(1) \quad \text{as } n\to\infty.$$

With all this, the proof of the Proposition is complete. ∎

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REMARK 26. The above proposition shows that the term $-(\log \mathfrak{g}_1/\pi)T$ measures the discrepancy between the number of zeros of ϕ_M with real part greater that $1/2$, meaning $N_{\text{ver}}(T; \phi_M)$, and the quantity $N_{M,\text{con}}(T)$, appearing in the classical version of the Weyl's law.

Furthermore, one can restate Proposition 25 as the relation representing Weyl's law

$$N_{\text{ver}}(T_n; Z_M H_M) = \frac{\text{vol}(M)}{4\pi} T_n^2 - \frac{n_1}{\pi} T_n \log T_n + \frac{T_n}{\pi} (n_1(1 - \log 2) - \log \mathfrak{g}_1) + O(T_n / \log T_n), \tag{69}$$

as $n \to \infty$.

A direct consequence of the relation (69), Main Theorem and Theorem 22 is the following reformulation of the Weyl's law:

COROLLARY 27. *There exist a sequence $\{T_n\}$ of positive real numbers tending to infinity such that, for every positive integer k*

$$N_{\text{ver}}(T_n; Z_M H_M) = N_{\text{ver}}(T_n; (Z_M H_M)^{(k)}) - \frac{n_1}{\pi} T_n \log T_n + \frac{T_n}{2\pi}(2n_1 + \log A_M) + o(T_n) \quad \textit{as } n \to \infty.$$

REMARK 28. An interpretation of the constant $\log \mathfrak{g}_1$, similar to the one derived in Proposition 25 is obtained in [7, formula (3.4.15)], where it is shown, in our notation, that

$$\log \mathfrak{g}_1 = \lim_{x\to\infty} \lim_{y\to\infty} \left[2x \int_0^\infty \frac{N_{M,\text{con}}(t) - N_{\text{ver}}(t; \phi_M)}{t} \left(\frac{1}{t^2 + x^2} - \frac{1}{t^2 + y^2} \right) dt \right].$$

A geometric interpretation of the constant $\mathfrak{g}_1$, in the case when the surface has one cusp $\mathfrak{a}$ is derived in [17]. In that case, $\mathfrak{g}_1^{-1}$ is the radius of the largest isometric circle arising in the construction of the standard polygon for the group $\sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{a}}$, where $\sigma_{\mathfrak{a}}$ denotes the scaling matrix of the cusp $\mathfrak{a}$.

As stated in the introduction, we do not know of a spectral or geometric interpretation of the constant $\mathfrak{g}_2$, besides the trivial one which realizes $\mathfrak{g}_2$ as the second largest denominator of the Dirichlet series portion of the scattering determinant. Therefore, we view Corollary 27 as giving rise to a new spectral invariant.

9.2 A comparison of counting functions

In this section, we will prove Theorem 2. In effect, it is necessary to recall results from [14], translate the notation in [14] to the notation in the present paper, then combine the result with (2) and parts (c) of the Main Theorem and Theorem 22.

From [14, Theorem 2.22] we have the asymptotic relation

$$N_{\mathrm{w}}(T;H_M)=\frac{n_1}{2}\cdot\frac{T\log T}{2\pi}+\frac{T}{2}\left(-\frac{n_1}{2\pi}-\frac{1}{\pi}\log|b_2|\right)+O(\log T) \quad \text{as } T\to\infty. \tag{70}$$

Note that in [14], the author counts the zeros of H_M in both the upper and lower half-planes, whereas the counting function $N_{\mathrm{w}}(T;H_M)$ only considers those zeros in the upper half-plane. Recall that the zeros and poles of H appear symmetrically about the real axis. As a result, the relation (70) differs from [14, Theorem 2.22] by a factor of two. Comparing [14, eq. (2.15) on p. 445] with our notation we deduce that $b_2=\pi^{n_1/2}\mathfrak{g}_1^{-1}d(1)$, hence, we are able to rewrite [14, Theorem 2.22] as

$$N_{\mathrm{w}}(T;H_M)=\frac{n_1}{2}\cdot\frac{T\log T}{2\pi}-\frac{T}{2\pi}\left(\frac{n_1}{2}+\frac{n_1}{2}\log\pi+\log|d(1)|-\log\mathfrak{g}_1\right) + O(\log T) \quad \text{as } T\to\infty. \tag{71}$$

Comparing (71) with part (c) of Main Theorem we deduce that

$$N_{\mathrm{w}}(T;(Z_MH_M)')-N_{\mathrm{w}}(T;H_M)=\frac{T\log T}{2\pi}+\frac{T}{2\pi}C+o(T) \quad \text{as } T\to\infty$$

with $C=\frac{1}{2}\log A_M-\log|a_M|+\log\mathrm{vol}(M)-1$.

Assume that $e^{\ell_{M,0}}<(\mathfrak{g}_2/\mathfrak{g}_1)^2$, so then

$$C=\log\left(\frac{2\mathrm{vol}(M)\sinh(\ell_{M,0}/2)}{e\cdot m_{M,0}\ell_{M,0}}\right).$$

Let $\widetilde{M}$ be any co-compact hyperbolic Riemann surface such that $\mathrm{vol}(\widetilde{M})=\mathrm{vol}(M)$. Assume that M and $\widetilde{M}$ have systoles of equal length, and the same number of inconjugate classes of systoles. Then, using (2), we arrive at the conclusion that

$$N_{\mathrm{w}}(T;(Z_MH_M)')-N_{\mathrm{w}}(T;H_M)=N_{\mathrm{w}}(T;Z'_{\widetilde{M}})+o(T) \quad \text{as } T\to\infty. \tag{72}$$

Furthermore, when $e^{\ell_{M,0}}<(\mathfrak{g}_2/\mathfrak{g}_1)^2$, comparing (72) with part (c) of Theorem 22, for $k\geqslant 2$ we arrive at

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$$\begin{aligned} N_{\mathrm{w}}(T;(Z_M H_M)^{(k)}) - N_{\mathrm{w}}(T;H_M) = N_{\mathrm{w}}(T;Z'_{\widetilde{M}}) \\ + \frac{(k-1)T}{2\pi}[\log(T\mathrm{vol}(M)) - 1] \\ - \frac{T}{2\pi}\log((k-1)\ell_{M,0}) + o(T) \\ \text{as } T \to \infty. \end{aligned}$$

Then, from part (c) of Theorem 22 applied to the zeta function $Z_{\widetilde{M}}$ we deduce

$$N_{\mathrm{w}}(T;(Z_M H_M)^{(k)}) - N_{\mathrm{w}}(T;H_M) = N_{\mathrm{w}}(T;(Z_{\widetilde{M}})^{(k)}) + o(T) \quad \text{as } T \to \infty.$$

This proves Theorem 2.

We find the comparison of counting functions, as summarized in (13) very interesting, especially since the coefficients in the asymptotic expansions in (70) and part (c) of the Main Theorem are somewhat involved and dissimilar from other known asymptotic expansions.

9.3 Concluding remarks

In [5] the authors defined 213 genus zero subgroups of which 171 are associated to "Moonshine". It would be interesting to compute the invariant A_M for each of these groups to see if further information regarding the groups, possibly related to "moonshine", is uncovered.

Is it possible to explicitly determine an example of a surface where $e^{\ell_{M,0}} = (\mathfrak{g}_2/\mathfrak{g}_1)^2$? More generally, one could study the set of such surfaces, as a subset of moduli space. Is the set of surfaces where $e^{\ell_{M,0}} > (\mathfrak{g}_2/\mathfrak{g}_1)^2$ a connected subset of moduli space, or are there several components? Is there another characterization of surfaces where $e^{\ell_{M,0}} = (\mathfrak{g}_2/\mathfrak{g}_1)^2$? Many other basic questions can be easily posed, and we find these problems very interesting.

In [2], the authors determined the asymptotic behavior of Selberg's zeta function through degeneration up to the critical line. It would be interesting to study the asymptotic behavior of the zeros of the derivative of Selberg's zeta function through degeneration, either in moduli space or through elliptic degeneration.

To come full circle, we return to the setting of the Riemann zeta function and speculate if one can attempt to mimic results which follow from the Levinson–Montgomery article [21]. Specifically, we recall, that Levinson used results from the distribution of zeros of $\zeta'_{\mathbb{Q}}$ to prove that more than

1/3 of the zeros of the Riemann zeta function lie on the critical line. Can one follow a similar investigation in the setting of the Selberg zeta function associated to a noncompact, finite volume surface? To do so, we note that a starting point would be to establish an analogue of the approximate functional equation for the Selberg zeta function. Results in this direction would be very significant, and we plan to undertake the project in the near future.

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