

## MORE ON THE PRESERVATION OF LARGE CARDINALS UNDER CLASS FORCING

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**Abstract.** We prove two general results about the preservation of extendible and  $C^{(n)}$ -extendible cardinals under a wide class of forcing iterations (Theorems 5.4 and 7.5). As applications we give new proofs of the preservation of Vopěnka's Principle and  $C^{(n)}$ -extendible cardinals under Jensen's iteration for forcing the GCH [17], previously obtained in [8, 27], respectively. We prove that  $C^{(n)}$ -extendible cardinals are preserved by forcing with standard Easton-support iterations for any possible  $\Delta_2$ -definable behaviour of the power-set function on regular cardinals. We show that one can force proper class-many disagreements between the universe and HOD with respect to the calculation of successors of regular cardinals, while preserving  $C^{(n)}$ -extendible cardinals. We also show, assuming the GCH, that the class forcing iteration of Cummings–Foreman–Magidor for forcing  $\diamond_{\kappa^+}$  at every  $\kappa$  [10] preserves  $C^{(n)}$ -extendible cardinals. We give an optimal result on the consistency of weak square principles and  $C^{(n)}$ -extendible cardinals. In the last section we prove another preservation result for  $C^{(n)}$ -extendible cardinals under very general (not necessarily definable or weakly homogeneous) class forcing iterations. As applications we prove the consistency of  $C^{(n)}$ -extendible cardinals with  $V = \text{HOD}$ , and also with GA (the Ground Axiom) plus  $V \neq \text{HOD}$ , the latter being a strengthening of a result from [14].

**§1. Introduction.** The present paper is a contribution to the long-standing program in set theory of studying the robustness of strong large-cardinal notions under forcing extensions. Specifically, we are interested in the section of the large-cardinal hierarchy ranging between extendible cardinals and *Vopěnka's Principle* (VP).

In a groundbreaking work, Laver [19] proved that supercompactness, one of the most prominent large-cardinal properties, can be made indestructible under a wide range of forcing notions. Indeed, given a supercompact cardinal  $\kappa$ , Laver showed that there is a forcing notion (the *Laver preparation*) that preserves the supercompactness of  $\kappa$  and makes it indestructible under further  $<\kappa$ -directed closed forcing.

Inspired by the work of Laver, several authors subsequently obtained similar results for other classical large-cardinal notions. For instance, Gitik and Shelah [13] show that a strong cardinal  $\kappa$  can be made indestructible under so-called  $\kappa^+$ -weakly closed forcing satisfying the Prikry condition; Hamkins [15] uses the *lottery preparation* forcing to make various types of large cardinals indestructible under appropriate forcing notions. More recently, Brooke-Taylor [8] shows that VP is indestructible under reverse Easton forcing iterations of increasingly directed-closed

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forcing notions, without the need for any preparatory forcing. In the present paper we are concerned with the preservation by forcing of  $C^{(n)}$ -extendible cardinals. This family of large cardinals was introduced in [2] (see also [1]) as a strengthening of the classical notion of extendibility and was shown to provide natural milestones on the road from supercompact cardinals up to VP. Extendible cardinals have experienced a renewed interest after Woodin’s proof of the HOD-Dichotomy [29]. Also,  $C^{(n)}$ -extendible cardinals have had remarkable applications in category theory and algebraic topology (see [2]). Thus, the investigation of the preservation of such cardinals under forcing is a worthwhile project, which may lead to further applications.

Recall (see [1]) that, for each  $n < \omega$ , the class  $C^{(n)}$  is the  $\Pi_n$ -definable closed unbounded proper class of all ordinals  $\alpha$  that are  $\Sigma_n$ -correct, i.e., such that  $V_\alpha$  is a  $\Sigma_n$ -elementary substructure of  $V$ . Also, recall that a cardinal  $\kappa$  is  $C^{(n)}$ -extendible if for every  $\lambda > \kappa$  there is an ordinal  $\mu$  and an elementary embedding  $j : V_\lambda \rightarrow V_\mu$  such that  $\text{crit}(j) = \kappa$ ,  $j(\kappa) > \lambda$ , and  $j(\kappa) \in C^{(n)}$ .

It turns out that  $\text{VP}(\Pi_{n+1})$ , namely VP restricted to classes of structures that are  $\Pi_{n+1}$ -definable, is equivalent to the existence of a  $C^{(n)}$ -extendible cardinal. Hence VP is equivalent to the existence of a  $C^{(n)}$ -extendible cardinal for each  $n \geq 1$  (see [1] for details). It is in this sense that  $C^{(n)}$ -extendible cardinals are canonical representatives of the large-cardinal hierarchy in the region between the first supercompact cardinal and VP.

In general, the preservation of very large cardinals by forcing is a delicate issue since it imposes strong forms of agreement between the ground model and the generic forcing extension. For example, suppose  $\kappa \in C^{(n)}$  is inaccessible and  $\mathbb{P}$  is a  $< \kappa$ -distributive forcing notion. If  $\Vdash_{\mathbb{P}} \text{“}\kappa \in C^{(n)}\text{”}$  then  $V_\kappa \prec_{\Sigma_n} V^{\mathbb{P}}$ . This underlines the fact that the more correct a large cardinal is, the harder it is to preserve its correctness under forcing, and therefore the more fragile it becomes. Indeed, one runs into trouble when seeking a result akin to Laver’s indestructibility for supercompact cardinals for stronger large cardinals such as extendible. This phenomenon was first pointed out by Tsaprounis in his Ph.D. thesis [26] and it was afterwards extensively studied in [4], where the following theorem illustrates the fragility we just described.

**THEOREM 1.1** [4]. *Suppose that  $V_\kappa \prec_{\Sigma_2} V_\lambda$  and  $G \subseteq \mathbb{P}$  is a  $V$ -generic filter for nontrivial strategically  $< \kappa$ -closed forcing  $\mathbb{P} \in V_\eta$ , where  $\eta \leq \lambda$ . Then for every  $\theta \geq \eta$ ,*

$$V_\kappa = V[G]_\kappa \not\prec_{\Sigma_3} V[G]_\theta.$$

In particular, if  $\kappa$  is an extendible cardinal and  $\mathbb{P}$  is any nontrivial strategically  $< \kappa$ -closed set forcing notion, then forcing with  $\mathbb{P}$  destroys the extendibility of  $\kappa$ . Moreover, the theorem implies that there is no hope to obtain indestructibility results for  $\Sigma_3$ -correct large cardinals. Thus, if one aims for a general theory of preservation of  $C^{(n)}$ -extendible cardinals one should concentrate on class forcing notions.

The structure of the paper is as follows: In Section 2 we prove that  $C^{(n)}$ -extendible cardinals are uniformly characterizable in a Magidor-like way, i.e., similar to Magidor’s characterization of supercompact cardinals. This reinforces the fact that  $C^{(n)}$ -extendible cardinals are a natural model-theoretic strengthening of supercompactness, first shown in [2] (see also [1]). This characterization of  $C^{(n)}$ -extendibility is used in later sections for carrying out preservation arguments under class forcing. The same characterization has been independently given by Boney in

[6], and also in [3] for the virtual forms of higher-level analogs of supercompact cardinals (i.e.,  $n$ -remarkable cardinals) and virtual  $C^{(n)}$ -extendible cardinals.

Section 3 is devoted to the analysis of some reflection properties of class forcing iterations that will be useful in later sections for the study of the preservation of  $C^{(n)}$ -extendible cardinals. Two key notions are that of *adequate iteration* and  $\mathbb{P}$ - $\Sigma_k$ -*reflecting cardinal*. For the same purpose, in Section 4 we introduce the notion of  $\mathbb{P}$ - $\Sigma_k$ -*supercompact cardinal* and show how it relates to  $C^{(n)}$ -extendible cardinals.

In Section 5 we define the notion of suitable iteration and prove a general result about the preservation of  $C^{(n)}$ -extendible cardinals under a wide class of ORD-length forcing iterations (cf. Theorem 5.4). The prize we pay for considering such general iterations is that we just prove that any  $C^{(n+1)}$ -extendible cardinal retains its  $C^{(n)}$ -extendibility (cf. Corollary 5.14).

Section 6 is focussed on applications of Theorem 5.4. We give a new proof of Brooke-Taylor's theorem on the preservation of VP [8]. The main advantage with respect to the original proof is that our technique allows for a finer control over the amount of Vopěnka's Principle that is preserved.

In Section 7 we concentrate on a more concrete class of forcing iterations we call *fitting* and improve the results obtained in Section 5. Our main result here is Theorem 7.5. In Section 8 we give several applications of this theorem. First, we show that  $C^{(n)}$ -extendible cardinals are preserved by standard Easton class-forcing iterations for any  $\Delta_2$ -definable possible behaviour of the power-set function on regular cardinals. This extends the main result of [28]. Second, with an eye on Woodin's HOD Conjecture, we explore briefly the connections between  $C^{(n)}$ -extendible cardinals (and thus also VP) with the principle  $V = \text{HOD}$ . In particular, we prove that it is possible to force a complete disagreement, and in many possible forms, between  $V$  and HOD with respect to the calculation of successors of regular cardinals, while  $C^{(n)}$ -extendible cardinals are preserved. Third, we show that, assuming the GCH, the class forcing iteration of Cummings–Foreman–Magidor that forces  $\diamond_{\kappa^+}^+$  at every  $\kappa$  ([10]) preserves  $C^{(n)}$ -extendible cardinals. Finally, we prove that  $C^{(n)}$ -extendible cardinals are consistent with  $\square_{\lambda, \text{cof}(\lambda)}$  for a proper class of singular cardinals  $\lambda$ . This result is optimal in the sense explained in Section 8.4.

In Section 9, we address the question of the preservation of  $C^{(n)}$ -extendible cardinals under general (non-weakly homogeneous, non-definable) suitable iterations. As applications of our analysis we prove the consistency of  $C^{(n)}$ -extendible cardinals with  $V = \text{HOD}$ , and also with  $\text{GA} + V \neq \text{HOD}$ . This latter is a strengthening of a result of Hamkins, Reitz, and Woodin [14].

**§2. A Magidor-like characterization of  $C^{(n)}$ -extendibility.** We shall prove that  $C^{(n)}$ -extendible cardinals (defined in page 6 of the Introduction) can be characterized in a way analogous to the following characterization of supercompact cardinals due to Magidor.<sup>1</sup>

**THEOREM 2.1** [21]. *For a cardinal  $\delta$ , the following statements are equivalent:*

- (1)  $\delta$  is a supercompact cardinal.

<sup>1</sup>Magidor's original characterization does not require that  $\lambda$  and  $\bar{\lambda}$  are in  $C^{(1)}$ , or that for every  $a \in V_\lambda$  there is  $\bar{a} \in V_{\bar{\lambda}}$  with  $j(\bar{a}) = a$ .

- (2) For every  $\lambda > \delta$  in  $C^{(1)}$  and for every  $a \in V_\lambda$ , there exist ordinals  $\bar{\delta} < \bar{\lambda} < \delta$  and there exist some  $\bar{a} \in V_{\bar{\lambda}}$  and an elementary embedding  $j : V_{\bar{\lambda}} \rightarrow V_\lambda$  such that:
- $\text{crit}(j) = \bar{\delta}$  and  $j(\bar{\delta}) = \delta$ .
  - $j(\bar{a}) = a$ .
  - $\bar{\lambda} \in C^{(1)}$ .

The existence of a supercompact cardinal is thus characterized by a form of reflection for  $\Sigma_1$ -correct strata of the universe, for it implies that any  $\Sigma_1$ -truth (i.e., any  $\Sigma_1$  sentence, with parameters, true in  $V$ ) is captured (up to some change of parameters) by some level below the supercompact cardinal. The following notion generalizes this reflection property to higher levels of complexity.

**DEFINITION 2.2** ( $\Sigma_n$ -supercompact cardinal). Let  $n \geq 1$ . A cardinal  $\delta$  is said to be  $\Sigma_n$ -supercompact if for every  $\lambda > \delta$  in  $C^{(n)}$  and  $a \in V_\lambda$ , there exist  $\bar{\delta} < \bar{\lambda} < \delta$  and  $\bar{a} \in V_{\bar{\lambda}}$ , and there exists an elementary embedding  $j : V_{\bar{\lambda}} \rightarrow V_\lambda$  such that:

- $\text{crit}(j) = \bar{\delta}$  and  $j(\bar{\delta}) = \delta$ .
- $j(\bar{a}) = a$ .
- $\bar{\lambda} \in C^{(n)}$ .

**LEMMA 2.3.** For  $n \geq 1$ , every  $\Sigma_n$ -supercompact cardinal belongs to the class  $C^{(n+1)}$ .

**PROOF.** We prove the lemma by induction on  $n \geq 1$ . The base case  $n = 1$  follows by combining Magidor’s theorem (cf. Theorem 2.1) with the well-known fact that supercompact cardinals are  $\Sigma_2$ -correct [18, Proposition 22.3]. Thus, we shall assume by induction that the lemma holds for  $n$  and prove it for  $n + 1$ .

Let  $\delta$  be a  $\Sigma_{n+1}$ -supercompact cardinal, and let  $\varphi(x, y)$  be a  $\Pi_{n+1}$  formula with  $a \in V_\delta$ . Suppose first that  $V_\delta$  satisfies the sentence  $\exists x\varphi(x, a)$  and let  $b \in V_\delta$  be a witness for it. Since  $\delta$  is  $\Sigma_n$ -supercompact, the induction hypothesis guarantees that  $\delta \in C^{(n+1)}$ , so that  $\varphi(b, a)$  is true.

Conversely, suppose that  $\exists x\varphi(x, a)$  is true. Let  $\lambda < \delta$  be such that  $a \in V_\lambda$ . Let  $b$  be a witness for this formula and let  $\mu \in C^{(n+2)} \setminus \delta^+$  be such that  $b \in V_\mu$ . Then,  $V_\mu \models \exists x\varphi(x, a)$ . By the  $\Sigma_{n+1}$ -supercompactness of  $\delta$  we may find  $\bar{\lambda}, \bar{\delta} < \bar{\mu} < \delta$ ,  $\bar{\mu} \in C^{(n+2)}$ ,  $\bar{a} \in V_{\bar{\lambda}}$ , and  $j : V_{\bar{\mu}} \rightarrow V_\mu$  such that  $\text{crit}(j) = \bar{\delta}$ ,  $j(\bar{\delta}) = \delta$ ,  $j(\bar{\lambda}) = \lambda$ , and  $j(\bar{a}) = a$ . In particular,  $V_{\bar{\mu}} \models \exists x\varphi(x, \bar{a})$ . Notice that, by elementarity,  $\bar{\lambda} < \bar{\delta}$ , hence  $a = j(\bar{a}) = \bar{a}$ , and thus  $V_{\bar{\mu}} \models \exists x\varphi(x, a)$ . Finally, since  $\bar{\mu} < \delta$  and  $\bar{\mu} \in C^{(n+2)}$ , our induction hypothesis yields  $V_{\bar{\mu}} \prec_{\Sigma_{n+1}} V_\delta$ . From this latter assertion it is clear that  $V_\delta \models \exists x\varphi(x, a)$ , as wanted. ⊖

**THEOREM 2.4.** For  $n \geq 1$ ,  $\delta$  is a  $C^{(n)}$ -extendible cardinal if and only if  $\delta$  is  $\Sigma_{n+1}$ -supercompact.

**PROOF.** Suppose that  $\delta$  is  $C^{(n)}$ -extendible. Fix any  $\lambda > \delta$  in  $C^{(n+1)}$  and  $a \in V_\lambda$ . By  $C^{(n)}$ -extendibility, let  $\mu > \lambda$  in  $C^{(n+1)}$ , and let  $j : V_\mu \rightarrow V_\theta$  be such that  $\text{crit}(j) = \delta$ ,  $j(\delta) > \mu$  and  $j(\delta) \in C^{(n)}$ , for some ordinal  $\theta$ . Notice that  $j \upharpoonright V_\lambda \in V_\theta$ .

**CLAIM 2.4.1.**  $V_\theta$  satisfies the following sentence:

$$\exists \bar{\lambda} < j(\delta) \exists \bar{\delta} < \bar{\lambda} \exists \bar{a} \in V_{\bar{\lambda}} \exists j^* : V_{\bar{\lambda}} \rightarrow V_{j(\lambda)}$$

$$(j^*(\bar{a}) = j(a) \wedge j^*(\bar{\delta}) = j(\delta) \wedge V_{\bar{\lambda}} \prec_{\Sigma_{n+1}} V_{j(\lambda)}).$$

PROOF OF CLAIM. It is sufficient to show that  $V_\lambda \prec_{\Sigma_{n+1}} V_{j(\lambda)}$ , for then the claim follows as witnessed by  $\lambda, \delta, a$ , and  $j \upharpoonright V_\lambda$ .

On the one hand, notice that  $V_\delta \prec_{\Sigma_{n+1}} V_\mu$ , because  $C^{(n)}$ -extendible cardinals are  $\Sigma_{n+2}$ -correct. By elementarity, this implies  $V_{j(\delta)} \prec_{\Sigma_{n+1}} V_\theta$ . On the other hand, since  $j(\delta) > \mu$  and  $j(\delta) \in C^{(n)}$ , it is true that  $V_\mu \prec_{\Sigma_{n+1}} V_{j(\delta)}$  and thus  $V_\mu \prec_{\Sigma_{n+1}} V_\theta$ . In addition, since  $\mu$  and  $\lambda$  were both  $\Sigma_{n+1}$ -correct, it is the case that  $V_\lambda \prec_{\Sigma_{n+1}} V_\mu$ . Hence,  $V_\lambda \prec_{\Sigma_{n+1}} V_\theta$ . Also, by elementarity,  $V_{j(\lambda)} \prec_{\Sigma_{n+1}} V_\theta$ . Combining these two facts, we have that  $V_\lambda \prec_{\Sigma_{n+1}} V_{j(\lambda)}$ .  $\dashv$

By elementarity,  $V_\mu$  satisfies the above sentence with the parameters  $j(\delta)$  and  $j(\lambda)$  replaced by  $\delta$  and  $\lambda$ , respectively. Hence, since  $\mu \in C^{(n+1)}$ , the sentence is true in the universe. Since  $\lambda$  was arbitrarily chosen in  $C^{(n+1)}$ , this implies that  $\delta$  is a  $\Sigma_{n+1}$ -supercompact cardinal.

For the converse implication, let  $\lambda$  be greater than  $\delta$  and let us show that there exists an elementary embedding  $j : V_\lambda \rightarrow V_\theta$ , for some ordinal  $\theta$ , such that  $\text{crit}(j) = \delta$ ,  $j(\delta) > \lambda$ , and  $j(\delta) \in C^{(n)}$ . Take  $\mu > \lambda$  in  $C^{(n+1)}$  and let  $\bar{\delta}, \bar{\lambda} < \bar{\mu}$  and  $j : V_{\bar{\mu}} \rightarrow V_{\bar{\mu}}$  be such that  $\text{crit}(j) = \bar{\delta}$ ,  $j(\bar{\delta}) = \delta$ ,  $j(\bar{\lambda}) = \lambda$ , and  $\bar{\mu} \in C^{(n+1)}$ . Now notice that the sentence

$$\exists \alpha \exists j^* : V_{\bar{\lambda}} \rightarrow V_\alpha (\text{crit}(j^*) = \bar{\delta} \wedge j^*(\bar{\delta}) > \bar{\lambda} \wedge j^*(\bar{\delta}) \in C^{(n)}) \tag{1}$$

is  $\Sigma_{n+1}$ -expressible. Moreover, it is true in  $V$  as witnessed by  $\lambda$  and  $j$  because  $j(\bar{\delta}) = \delta > \bar{\lambda}$  and  $\delta \in C^{(n)}$  (cf. Lemma 2.3). Thus, since  $V_{\bar{\mu}}$  is  $\Sigma_{n+1}$ -correct and contains  $\bar{\delta}$  and  $\bar{\lambda}$ , it is also true in  $V_{\bar{\mu}}$ . By elementarity,  $V_\mu$  thinks that the sentence

$$\exists \alpha \exists j^* : V_\lambda \rightarrow V_\alpha (\text{crit}(j^*) = \delta \wedge j^*(\delta) > \lambda \wedge j^*(\delta) \in C^{(n)})$$

is true. Since  $\mu \in C^{(n+1)}$ , the above displayed sentence is true in  $V$  and so  $\delta$  is  $\lambda$ - $C^{(n)}$ -extendible. As  $\lambda$  was arbitrarily chosen,  $\delta$  is a  $C^{(n)}$ -extendible cardinal.  $\dashv$

REMARK 2.5. For  $\lambda \in C^{(n)}$ , a cardinal  $\delta$  is called  $\lambda$ - $C^{(n)+}$ -extendible if it is  $\lambda$ - $C^{(n)}$ -extendible, witnessed by some elementary embedding  $j : V_\lambda \rightarrow V_\theta$  with  $\theta \in C^{(n)}$ . Likewise,  $\delta$  is called  $C^{(n)+}$ -extendible if it is  $\lambda$ - $C^{(n)+}$ -extendible for every  $\lambda > \delta$  in  $C^{(n)}$  (see [1, Section 4]). A close inspection of the preceding argument reveals that  $\Sigma_{n+1}$ -supercompactness is actually equivalent to  $C^{(n)+}$ -extendibility. This gives an alternative proof of [28, Corollary 3.5].

COROLLARY 2.6. *A cardinal is extendible if and only if it is  $\Sigma_2$ -supercompact.*

A close inspection of the proof of Theorem 2.4 shows that the following holds:

COROLLARY 2.7. *For  $n \geq 1$ , a cardinal  $\delta$  is  $C^{(n)}$ -extendible if and only if for a proper class of  $\lambda$  in  $C^{(n+1)}$ , for every  $\alpha < \lambda$  there exist  $\bar{\delta}, \bar{\alpha} < \bar{\lambda}$  and an elementary embedding  $j : V_{\bar{\lambda}} \rightarrow V_{\bar{\lambda}}$  such that:*

- $\text{crit}(j) = \bar{\delta}$  and  $j(\bar{\delta}) = \delta$ .
- $j(\bar{\alpha}) = \alpha$ .
- $\bar{\lambda} \in C^{(n+1)}$ .

In the light of the above results it is natural to define the class of  $C^{(0)}$ -extendible cardinals as the class of supercompact cardinals. Note that since every  $C^{(n+1)}$ -extendible cardinal is a limit of  $C^{(n)}$ -extendible cardinals (see [1]),

every  $\Sigma_{n+1}$ -supercompact cardinal is a limit of  $\Sigma_n$ -supercompact cardinals. It will become apparent in the following sections that the notion of  $\Sigma_{n+1}$ -supercompactness is a useful reformulation of  $C^{(n)}$ -extendibility in the context of class forcing.

**§3. Some reflection properties for class forcing iterations.** In the sequel we will only work with ORD-length forcing iterations, since extendible cardinals are generally destroyed by set-size ones (see Theorem 1.1 and the related discussion).

If  $\mathbb{P}$  is a forcing iteration, and  $G$  is  $\mathbb{P}$ -generic over  $V$ , then for every ordinal  $\lambda$ , we denote as customary by  $G_\lambda$  the  $\mathbb{P}_\lambda$ -generic filter induced by  $G$ , i.e.,  $G_\lambda := G \cap \mathbb{P}_\lambda$ . Also, as usual we denote by  $i_{G_\lambda}(\tau)$  the interpretation of the  $\mathbb{P}_\lambda$ -name  $\tau$  by the filter  $G_\lambda$ .

For the main preservation results given in the following sections we will need to ensure that there are many cardinals  $\lambda$  that satisfy  $V_\lambda[G_\lambda] = V[G]_\lambda$ .

**DEFINITION 3.1.** Let  $\mathbb{P}$  be a forcing iteration. A cardinal  $\lambda$  is  $\mathbb{P}$ -reflecting if  $\mathbb{P}$  forces that  $V[\dot{G}]_\lambda = V_\lambda[\dot{G}_\lambda]$ .

**REMARK 3.2.** Note that since the rank of  $i_{G_\lambda}(\tau)$  in  $V[G_\lambda]$  is never bigger than the rank of  $\tau$  in  $V$ , for any  $\tau \in V^{\mathbb{P}_\lambda}$ , we clearly have

$$V_\lambda[G_\lambda] \subseteq V[G_\lambda]_\lambda \subseteq V[G]_\lambda.$$

Thus,  $\mathbb{P}$  always forces “ $V_\lambda[\dot{G}_\lambda] \subseteq V[\dot{G}]_\lambda$ .” Hence a cardinal  $\lambda$  is  $\mathbb{P}$ -reflecting if and only if  $\mathbb{P}$  forces “ $V[\dot{G}]_\lambda \subseteq V_\lambda[\dot{G}_\lambda]$ .”

**PROPOSITION 3.3.**  $K := \{\lambda \mid \lambda \text{ is } \mathbb{P}\text{-reflecting}\}$  is closed.

**PROOF.** Let  $\kappa$  be an accumulation point of  $K$ . Let  $p \in \mathbb{P}$  and  $\tau \in V^{\mathbb{P}}$  be such that  $p \Vdash_{\mathbb{P}} \tau \in V[\dot{G}]_\kappa$ . Since  $\kappa$  is a limit cardinal there is  $q \leq_{\mathbb{P}} p$  and  $\lambda < \kappa$  such that  $q \Vdash_{\mathbb{P}} \tau \in V[\dot{G}]_\lambda$ . Extending  $q$  if necessary, we may find  $\theta \in K \cap \kappa$  above  $\lambda$  such that  $q \Vdash_{\mathbb{P}} \tau \in V[\dot{G}]_\lambda \subseteq V_\theta[\dot{G}_\theta]$ . Thus,  $\kappa \in K$ , as wanted.  $\dashv$

The following proposition gives some sufficient conditions for a cardinal to be  $\mathbb{P}$ -reflecting.

**PROPOSITION 3.4.** Suppose  $\lambda$  is an inaccessible cardinal and  $\mathbb{P}$  is a forcing iteration such that :  $\mathbb{P}_\lambda$  is a  $\lambda$ -cc forcing which preserves the inaccessibility of  $\lambda$ ,  $\mathbb{P}_\lambda \subseteq V_\lambda$ , and  $\Vdash_{\mathbb{P}_\lambda}$  “ $\dot{Q}$  is  $\lambda$ -distributive,” where  $\mathbb{P} \cong \mathbb{P}_\lambda * \dot{Q}$ . Then  $\lambda$  is  $\mathbb{P}$ -reflecting.

**PROOF.** On the one hand, by induction on the rank and using the fact that  $\mathbb{P}_\lambda$  is  $\lambda$ -cc and preserves the inaccessibility of  $\lambda$ , one can easily show that  $V[G_\lambda]_\lambda \subseteq V_\lambda[G_\lambda]$ .

On the other hand, as  $\mathbb{P}_\lambda$  preserves the inaccessibility of  $\lambda$ ,  $|V[G_\lambda]_\lambda| = \lambda$ . Hence, since  $\Vdash_{\mathbb{P}_\lambda}$  “ $\dot{Q}$  is  $\lambda$ -distributive,” and so  $i_{G_\lambda}(\dot{Q})$  does not add any new subsets of  $V[G_\lambda]_\lambda$ , we have

$$V[G]_\lambda \subseteq V[G_\lambda]_\lambda.$$

Hence,  $V[G]_\lambda \subseteq V_\lambda[G_\lambda]$ .  $\dashv$

Let  $\mathcal{L}$  denote the language of set theory augmented with an additional unary predicate  $\mathbb{P}$ . We will denote by  $\Sigma_k^{\mathcal{L}}$  (resp.  $\Pi_k^{\mathcal{L}}$ ) the class of  $\Sigma_k$  (resp.  $\Pi_k$ ) formulae of  $\mathcal{L}$ . This choice of language will be useful when working with expressions involving

a given iteration  $\mathbb{P}$ . For instance, we shall need to compute the complexity of the notion

$$\langle V_\alpha, \in, \mathbb{P} \cap V_\alpha \rangle \prec_{\Sigma_k^{\mathcal{L}}} \langle V, \in, \mathbb{P} \rangle$$

as a property of  $\alpha$ , when  $\mathbb{P}$  is a definable ORD-length forcing iteration.

REMARK 3.5. For such  $\mathbb{P}$  and  $\alpha$ , since we are dealing with iterations it would perhaps seem more natural to consider the predicate  $\mathbb{P}_\alpha$  instead of  $\mathbb{P} \cap V_\alpha$ . However, if  $\mathbb{P}_\alpha \subseteq V_\alpha$  (and so  $\alpha$  is a limit ordinal and the direct limit is taken at stage  $\alpha$  of the iteration), then we have  $\mathbb{P} \cap V_\alpha = \mathbb{P}_\alpha$ .

It is a well-known fact (see, e.g., [16, Section 13]) that the truth predicate (in the language of set theory) for  $\Sigma_0$  formulae is  $\Delta_1$  definable (i.e., both  $\Sigma_1$  and  $\Pi_1$ -definable); and in general, the truth predicate for  $\Sigma_k$  (resp.  $\Pi_k$ ) formulae, for  $k \geq 1$ , is  $\Sigma_k$ -definable (resp.  $\Pi_k$ ). However, if  $\mathbb{P}$  is a definable predicate, then the complexity (in the language of set theory) of the truth predicate for formulae in the language  $\mathcal{L}$  depends naturally on the complexity of the definition of  $\mathbb{P}$ .

PROPOSITION 3.6. *If  $\mathbb{P}$  is either  $\Sigma_m$  or  $\Pi_m$ -definable, then the truth predicate  $\models_{\Sigma_0^{\mathcal{L}}}$  for  $\Sigma_0^{\mathcal{L}}$  formulae is  $\Delta_{m+1}$  (i.e., both  $\Sigma_{m+1}$  and  $\Pi_{m+1}$ ). In general, for  $k \geq 1$ , the truth predicate  $\models_{\Sigma_k^{\mathcal{L}}}$  for  $\Sigma_k^{\mathcal{L}}$  formulae (resp.  $\models_{\Pi_k^{\mathcal{L}}}$  for  $\Pi_k^{\mathcal{L}}$  formulae) is  $\Sigma_{m+k}$  (resp.  $\Pi_{m+k}$ ). If  $\mathbb{P}$  is  $\Delta_m$ -definable (with  $m \geq 1$ ), then  $\models_{\Sigma_0^{\mathcal{L}}}$  is  $\Delta_m$ , and  $\models_{\Sigma_k^{\mathcal{L}}}$  (resp.  $\models_{\Pi_k^{\mathcal{L}}}$ ) is  $\Sigma_{m+k-1}$  (resp.  $\Pi_{m+k-1}$ ).*

PROOF. Note first that the only atomic formulas in the language  $\mathcal{L}$  are of the form “ $x \in y$ ,” “ $x = y$ ,” or “ $x \in \mathbb{P}$ ,” where  $x$  and  $y$  are variable symbols. Hence, if  $\mathbb{P}$  is  $\Sigma_m$  (resp.  $\Pi_m$ ) definable, then the formula “ $x \in \mathbb{P}$ ” is equivalent to a  $\Sigma_m$  (resp.  $\Pi_m$ ) formula of the language of set theory. It follows that a Boolean combination of atomic formulas of the language  $\mathcal{L}$  is equivalent to a Boolean combination of  $\Sigma_m$  and  $\Pi_m$  formulas in the language of set theory. Hence, the truth predicate for Boolean combinations of atomic formulas in the language  $\mathcal{L}$  is  $\Delta_{m+1}$  definable. The same applies to formulas with bounded quantifiers. By induction on  $k$  one can now show (as in [16, Section 13]) that  $\models_{\Sigma_k^{\mathcal{L}}}$  (resp.  $\models_{\Pi_k^{\mathcal{L}}}$ ) is  $\Sigma_{m+k}$  (resp.  $\Pi_{m+k}$ ) definable.

If  $\mathbb{P}$  is  $\Delta_m$ -definable, with  $m \geq 1$ , then “ $x \in \mathbb{P}$ ” is both  $\Sigma_m$  and  $\Pi_m$ -expressible. It easily follows that  $\models_{\Sigma_0^{\mathcal{L}}}$  is  $\Delta_m$ -definable, and by induction on  $k$  one readily shows that  $\models_{\Sigma_k^{\mathcal{L}}}$  (resp.  $\models_{\Pi_k^{\mathcal{L}}}$ ) is  $\Sigma_{m+k-1}$  (resp.  $\Pi_{m+k-1}$ ) definable. ⊣

DEFINITION 3.7. For  $k \geq 0$  and a predicate<sup>2</sup>  $\mathbb{P}$ , we shall denote by  $C_{\mathbb{P}}^{(k)}$  the class of all ordinals  $\alpha$  such that

$$\langle V_\alpha, \in, \mathbb{P} \cap V_\alpha \rangle \prec_{\Sigma_k^{\mathcal{L}}} \langle V, \in, \mathbb{P} \rangle.$$

It is easily seen that  $C_{\mathbb{P}}^{(k)}$  is a closed unbounded proper class, and  $C_{\mathbb{P}}^{(k+1)} \subseteq C_{\mathbb{P}}^{(k)}$ , for each  $k \geq 0$ . Notice that  $C_{\mathbb{P}}^{(0)}$  is the class of all ordinals. Let us compute next the complexity of  $C_{\mathbb{P}}^{(k)}$ , in the language of set theory, when  $\mathbb{P}$  is a definable predicate.

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<sup>2</sup>We are interested in the case  $\mathbb{P}$  is a class-forcing iteration, but the definition as well as the next proposition make sense and hold also for any predicate  $\mathbb{P}$ .

NOTATION 3.8. For  $m < \omega$ , we will denote by  $\Gamma_m$  (resp.  $\mathbf{\Gamma}_m$ ) the collection of all the formulas in the language of set theory that are either  $\Sigma_m$  or  $\Pi_m$  (resp. with parameters). In the sequel, expressions such as “ $X$  is  $\Gamma_m$ ” should be read as “ $X$  is definable by a formula in  $\Gamma_m$ .”

PROPOSITION 3.9. *The class  $C_{\mathbb{P}}^{(k)}$  is*

$$\begin{aligned} &\Pi_{m+k}, && \text{if } k \geq 1 \text{ and } \mathbb{P} \text{ is } \Gamma_m, \\ &\Pi_{m+k-1}, && \text{if } k, m \geq 1 \text{ and } \mathbb{P} \text{ is } \Delta_m. \end{aligned}$$

PROOF. The class  $C_{\mathbb{P}}^{(0)}$  is  $\Pi_0$ , being the class of all ordinals. If  $k \geq 1$ , then we have that  $\alpha \in C_{\mathbb{P}}^{(k)}$  if and only if  $\alpha \in C_{\mathbb{P}}^{(k-1)}$  and

$$\forall X, Y (X = V_\alpha \wedge Y = \mathbb{P} \cap X \rightarrow \forall \bar{a} \in X \forall \varphi \in \Sigma_k^{\mathcal{L}} (\models_{\Sigma_k^{\mathcal{L}}} \varphi(\bar{a}) \rightarrow \langle X, \in, Y \rangle \models \varphi(\bar{a}))).$$

Now using induction on  $k$  and Proposition 3.6 the complexity of the class  $C_{\mathbb{P}}^{(k)}$  is easily seen to be as desired. ⊢

It is well-known that if  $\mathbb{P}$  is a set-forcing notion, then for every condition  $p \in \mathbb{P}$ , and  $\mathbb{P}$ -names  $\tau_1, \dots, \tau_n$ , if  $\varphi(x_1, \dots, x_n)$  is a  $\Sigma_k$  (resp.  $\Pi_k$ ) formula of the language of set theory, with  $k \geq 1$ , then the sentence “ $p \Vdash_{\mathbb{P}} \varphi(\tau_1, \dots, \tau_n)$ ” is also  $\Sigma_k$  (resp.  $\Pi_k$ ), in the parameters  $\mathbb{P}$  and  $\tau_1, \dots, \tau_n$ . We shall see next that for definable ORD-length forcing iterations that satisfy some mild conditions the complexity of the forcing relation for  $\mathbb{P}$  depends only on the complexity of the definition of  $\mathbb{P}$ . Thus, if, e.g.,  $\mathbb{P}$  is  $\Gamma_m$ -definable, then the forcing relation for  $\Sigma_k$  formulae is  $\Sigma_{m+k}$ -definable (Proposition 3.13).

DEFINITION 3.10. Let  $\mathbb{P}$  be a definable ORD-length forcing iteration. Then  $\mathbb{P}$  is *adequate* if there is a sufficiently rich finite fragment  $ZFC^*$  of ZFC which allows to define the forcing relation  $\Vdash_{\mathbb{P}}$  for  $\Delta_0$ -formulae of the language of set theory and proving the Forcing Theorem for such formulae, and there is a proper class of ordinals  $\kappa$  such that  $V_\kappa \models ZFC^*$  and  $\mathbb{P}^{V_\kappa} = \mathbb{P}_\kappa$  (hence also  $\mathbb{P}_\kappa = \mathbb{P} \cap V_\kappa$ ).<sup>3</sup>

Although there are examples of definable class forcing notions  $\mathbb{P}$  for which the forcing relation  $\Vdash_{\mathbb{P}}$  is not definable (see [20, Section 7]), if  $\mathbb{P}$  satisfies certain conditions, e.g., is either *pretame* (see [12]), *progressively-closed* [23], or *suitable* (Definition 5.1), then the forcing relation  $\Vdash_{\mathbb{P}}$  is definable and the Forcing Theorem holds for  $\mathbb{P}$ .

PROPOSITION 3.11. *Let  $\mathbb{P}$  be adequate and let  $ZFC^*$  be a finite fragment of ZFC which is sufficient for defining the forcing relation  $\Vdash_{\mathbb{P}}$  for  $\Delta_0$ -formulae of the language of set theory and proving the Forcing Theorem for such formulae. Suppose  $M$  is a transitive set and  $\kappa \in M$  is such that  $M_\kappa \models ZFC^*$  and  $\mathbb{P}^{M_\kappa} = \mathbb{P}_\kappa$ . Then, for every  $\Delta_0$ -formula  $\varphi$  of the language of set theory, every  $p \in \mathbb{P}$ , and every  $\mathbb{P}$ -name  $\tau$  such that  $p, \tau \in M_\kappa$ ,*

$$p \Vdash_{\mathbb{P}} \varphi(\tau) \quad \text{iff} \quad M_\kappa \models “p \Vdash_{\mathbb{P}_\kappa} \varphi(\tau).”$$

<sup>3</sup>See Remark 3.5.



PROOF. Assume  $p \Vdash_{\mathbb{P}} \varphi(\tau)$ . Suppose, for a contradiction, that  $q \in \mathbb{P}_\kappa$  is such that  $q \leq_{\mathbb{P}_\kappa} p$  and  $M_\kappa \models "q \Vdash_{\mathbb{P}_\kappa} \neg\varphi(\tau)." Suppose  $G$  is  $\mathbb{P}$ -generic over  $V$ , with  $q \in G$ . Then$

$$V[G] \models \varphi(i_G(\tau)).$$

Note that, since  $\mathbb{P}^{M_\kappa} = \mathbb{P}_\kappa$ , a direct limit is taken at  $\kappa$  (for otherwise some conditions in  $\mathbb{P}_\kappa$  would be proper classes in  $M_\kappa$ ), and therefore being a dense subset of  $\mathbb{P}_\kappa$  is absolute between  $M$  and  $V$ . Thus,  $G_\kappa := G \cap \mathbb{P}_\kappa$  is  $\mathbb{P}_\kappa$ -generic over  $M_\kappa$ . Also, since  $\tau \in M_\kappa$ , hence  $\tau \in M_\kappa^{\mathbb{P}_\kappa}$ ,  $i_{G_\kappa}(\tau) \in M_\kappa[G_\kappa]$  and  $i_G(\tau) = i_{G_\kappa}(\tau)$ . Since  $\varphi$  is  $\Delta_0$ , hence absolute for transitive classes,  $M_\kappa[G_\kappa] \models \varphi(i_{G_\kappa}(\tau))$ . But as  $q \in G_\kappa$ , this contradicts  $M_\kappa \models "q \Vdash_{\mathbb{P}_\kappa} \neg\varphi(\tau)." \dashv$

A similar argument proves the converse. -

COROLLARY 3.12. *Let  $\mathbb{P}$  be adequate. Then, for every  $\Delta_0$ -formula  $\varphi$ , every  $p \in \mathbb{P}$ , and every  $\mathbb{P}$ -name  $\tau$ , the sentence " $p \Vdash_{\mathbb{P}} \varphi(\tau)$ " is:*

- (1)  $\Delta_1^{\mathcal{L}}$ ,
- (2)  $\Delta_{m+1}$ , if  $\mathbb{P}$  is  $\Gamma_m$ -definable (some  $m \geq 1$ ),
- (3)  $\Delta_m$ , if  $\mathbb{P}$  is  $\Delta_m$ -definable (some  $m \geq 1$ ),

with  $p$  and  $\tau$  as parameters.

PROOF. The proof of Proposition 3.11 shows that " $p \Vdash_{\mathbb{P}} \varphi(\tau)$ " holds if and only if

$$\forall M \forall \kappa (M \text{ is a transitive set} \wedge \kappa \in M \wedge M_\kappa \models ZFC^* \wedge \mathbb{P}^{M_\kappa} = \mathbb{P}_\kappa \wedge p, \tau \in M_\kappa \rightarrow M_\kappa \models "p \Vdash_{\mathbb{P}_\kappa} \varphi(\tau)")$$

and also if and only if

$$\exists M \exists \kappa (M \text{ is a transitive set} \wedge \kappa \in M \wedge M_\kappa \models ZFC^* \wedge \mathbb{P}^{M_\kappa} = \mathbb{P}_\kappa \wedge p, \tau \in M_\kappa \wedge M_\kappa \models "p \Vdash_{\mathbb{P}_\kappa} \varphi(\tau)").$$

Now the two displayed sentences above are easily seen to be  $\Pi_1^{\mathcal{L}}$  and  $\Sigma_1^{\mathcal{L}}$ , respectively, with  $p$  and  $\tau$  as parameters.

Items (2) and (3) now follow easily from (1) and Proposition 3.6. -

PROPOSITION 3.13. *Let  $\mathbb{P}$  be adequate. Let  $k \geq 1$ . Then, for every  $\Sigma_k$  (resp.  $\Pi_k$ ) formula  $\varphi(t)$  of the language of set theory, every  $p \in \mathbb{P}$ , and every  $\mathbb{P}$ -name  $\tau$ , the sentence " $p \Vdash_{\mathbb{P}} \varphi(\tau)$ " is:*

- (1)  $\Sigma_k^{\mathcal{L}}$  (resp.  $\Pi_k^{\mathcal{L}}$ ).
- (2)  $\Sigma_{m+k}$  (resp.  $\Pi_{m+k}$ ), if  $\mathbb{P}$  is  $\Gamma_m$ -definable (some  $m \geq 1$ ).
- (3)  $\Sigma_{m+k-1}$  (resp.  $\Pi_{m+k-1}$ ), if  $\mathbb{P}$  is  $\Delta_m$ -definable (some  $m \geq 1$ ).

PROOF. First, note that the class  $V^{\mathbb{P}}$  of  $\mathbb{P}$ -names is  $\Delta_1^{\mathcal{L}}$ . Now given a  $\Sigma_k$  formula  $\exists x \forall y \exists z \dots \psi(x, y, z, \dots, t)$  of the language of set theory, and given a condition  $p \in \mathbb{P}$  and  $\tau \in V^{\mathbb{P}}$ ,

$$p \Vdash_{\mathbb{P}} \exists x \forall y \exists z \dots \psi(x, y, z, \dots, \tau)$$

if and only if

$$\exists \sigma (\sigma \in V^{\mathbb{P}} \wedge \forall \sigma' (\sigma' \in V^{\mathbb{P}} \rightarrow \exists \sigma'' (\sigma'' \in V^{\mathbb{P}} \wedge \dots p \Vdash_{\mathbb{P}} \psi(\sigma, \sigma', \sigma'', \tau))))).$$

Since  $p \Vdash_{\mathbb{P}} \psi(\sigma, \sigma', \sigma'', \tau)$  is  $\Delta_1^{\mathcal{L}}$  (Corollary 3.12), the last displayed sentence is  $\Sigma_k^{\mathcal{L}}$ , as wanted.

Items (2) and (3) follow easily from (1) and Proposition 3.6. ⊢

Let us compute next the complexity of the notion of  $\mathbb{P}$ -reflecting cardinal, for adequate  $\mathbb{P}$ .

**PROPOSITION 3.14.** *Let  $\mathbb{P}$  be adequate. Then the sentence “ $\kappa$  is a  $\mathbb{P}$ -reflecting cardinal” is:*

- (1)  $\Pi_2^{\mathcal{L}}$ , with  $\kappa$  as a parameter.
- (2)  $\Pi_{m+2}$ , if  $\mathbb{P}$  is  $\Gamma_m$ -definable, with  $m \geq 1$ .
- (3)  $\Pi_{m+1}$ , if  $\mathbb{P}$  is  $\Delta_m$ -definable, with  $m \geq 1$ .

**PROOF.** Note that  $\mathbb{P}$  forces  $V[\dot{G}]_{\kappa} \subseteq V_{\kappa}[\dot{G}_{\kappa}]$  if and only if the  $\mathcal{L}$ -sentence

$$\forall p, \tau (\tau \in V^{\mathbb{P}} \wedge p \Vdash_{\mathbb{P}} “rk(\tau) < \kappa” \rightarrow \exists \sigma, q (\sigma \in V^{\mathbb{P}} \wedge q \leq p \wedge rk(\sigma) < \kappa \wedge q \Vdash_{\mathbb{P}} “\sigma = \tau”))$$

holds in  $\langle V, \in, \mathbb{P} \rangle$ . Thus, since  $V^{\mathbb{P}}$  is a  $\Delta_1^{\mathcal{L}}$ -definable class, and the expressions “ $p \Vdash_{\mathbb{P}} rk(\tau) < \kappa$ ” and “ $q \Vdash_{\mathbb{P}} \sigma = \tau$ ” are also  $\Delta_1^{\mathcal{L}}$  (Corollary 3.12), it is easily seen that the sentence “ $\kappa$  is a  $\mathbb{P}$ -reflecting cardinal” is  $\Pi_2^{\mathcal{L}}$ , with  $\kappa$  as an additional parameter.

Items (2) and (3) follow easily from (1) using Proposition 3.6. ⊢

Let us consider, for the sake of conciseness, the following strengthening of the notion of  $\mathbb{P}$ -reflecting cardinal (cf. Definition 3.1).

**DEFINITION 3.15.** If  $k \geq 1$  and  $\mathbb{P}$  is an ORD-length forcing iteration, then a cardinal  $\kappa$  is  $\mathbb{P}$ - $\Sigma_k$ -reflecting if it is  $\mathbb{P}$ -reflecting, belongs to  $C_{\mathbb{P}}^{(k)}$ , and  $\mathbb{P} \cap V_{\kappa} = \mathbb{P}_{\kappa}$ .

The next lemma will be crucial in future arguments.

**LEMMA 3.16.** *Suppose  $\mathbb{P}$  is adequate. Then for every  $k \geq 0$ , if  $\kappa$  is  $\mathbb{P}$ - $\Sigma_k$ -reflecting and  $V_{\kappa} \models ZFC^*$ , then  $\mathbb{P}$  forces  $V[\dot{G}]_{\kappa} \prec_{\Sigma_k} V[\dot{G}]$ .*

**PROOF.** The claim is clear for  $k = 0$ . So, assume inductively that  $\mathbb{P}$  forces  $V[\dot{G}]_{\kappa} \prec_{\Sigma_{k-1}} V[\dot{G}]$ . Notice that, as  $\kappa$  is  $\mathbb{P}$ -reflecting, any member of  $V[\dot{G}]_{\kappa}$  is given by a  $\mathbb{P}$ -name in  $V_{\kappa}$ . Let  $\varphi(x)$  be a  $\Sigma_k$  formula in the language of set theory and let  $\tau \in V_{\kappa}$  be a  $\mathbb{P}_{\kappa}$ -name such that  $p \Vdash_{\mathbb{P}} \varphi(\tau)$ , for some  $p \in \mathbb{P}$ . We only need to show that  $p \Vdash_{\mathbb{P}} “V[\dot{G}]_{\kappa} \models \varphi(\tau).”$

**CLAIM 3.16.1.** The set of conditions  $q \in \mathbb{P}_{\kappa}$  such that  $q \Vdash_{\mathbb{P}} \varphi(\tau)$  is dense below  $p \restriction \kappa$ .

**PROOF OF CLAIM.** Suppose, aiming for a contradiction, that  $p' \leq_{\mathbb{P}_{\kappa}} p \restriction \kappa$  is such that  $q \not\Vdash_{\mathbb{P}} \varphi(\tau)$ , for all  $q \leq_{\mathbb{P}_{\kappa}} p'$ . Since  $\langle V_{\kappa}, \in, \mathbb{P}_{\kappa} \rangle \prec_{\Sigma_k} \langle V, \in, \mathbb{P} \rangle$ , and for every  $q \in \mathbb{P}_{\kappa}$  the sentence “ $q \not\Vdash_{\mathbb{P}} \varphi(\tau)$ ” is  $\Pi_k^{\mathcal{L}}$  (Proposition 3.13), we have that “ $q \not\Vdash_{\mathbb{P}_{\kappa}} \varphi(\tau)$ ” holds in  $\langle V_{\kappa}, \in, \mathbb{P}_{\kappa} \rangle$ . Therefore,

$$\langle V_{\kappa}, \in, \mathbb{P}_{\kappa} \rangle \models “p' \Vdash_{\mathbb{P}_{\kappa}} \neg \varphi(\tau).”$$

Again, since  $\langle V_{\kappa}, \in, \mathbb{P}_{\kappa} \rangle \prec_{\Sigma_k} \langle V, \in, \mathbb{P} \rangle$ , and the quoted displayed sentence is  $\Pi_k^{\mathcal{L}}$ ,

$$\langle V, \in, \mathbb{P} \rangle \models “p' \Vdash_{\mathbb{P}} \neg \varphi(\tau),”$$

which yields the desired contradiction to the fact that  $p \Vdash_{\mathbb{P}} \varphi(\tau)$ . ⊢

By the claim,  $p \upharpoonright \kappa \Vdash_{\mathbb{P}} \varphi(\tau)$ , and since  $\langle V_\kappa, \in, \mathbb{P}_\kappa \rangle \prec_{\Sigma_k} \langle V, \in, \mathbb{P} \rangle$  and the above sentence is  $\Sigma_k^{\mathcal{L}}$  we have

$$\langle V_\kappa, \in, \mathbb{P}_\kappa \rangle \models \text{“} p \upharpoonright \kappa \Vdash_{\mathbb{P}_\kappa} \varphi(\tau)\text{.”}$$

Thus, if  $G$  is  $\mathbb{P}$ -generic over  $V$ , with  $p \in G$ , then  $V_\kappa[G_\kappa] \models \varphi(i_{G_\kappa}(\tau))$ . Hence, since  $i_{G_\kappa}(\tau) = i_G(\tau)$ , and  $\kappa$  is  $\mathbb{P}$ -reflecting,  $V[G]_\kappa \models \varphi(i_G(\tau))$ , as wanted.  $\dashv$

For future use, let us calculate the complexity of the predicate “ $\kappa$  is a  $\mathbb{P}$ - $\Sigma_k$ -reflecting cardinal” for definable forcing iterations.

**PROPOSITION 3.17.** *Let  $\mathbb{P}$  be adequate. Then the predicate “ $\kappa$  is  $\mathbb{P}$ - $\Sigma_k$ -reflecting” is:*

$$\begin{aligned} \Pi_{m+2}, & \quad \text{if } k = 1 \text{ and } \mathbb{P} \text{ is } \Gamma_m. \\ \Pi_{m+k}, & \quad \text{if } k > 1 \text{ and } \mathbb{P} \text{ is } \Gamma_m. \\ \Pi_{m+1}, & \quad \text{if } k = 1 \text{ and } \mathbb{P} \text{ is } \Delta_m. \\ \Pi_{m+k-1}, & \quad \text{if } k > 1 \text{ and } \mathbb{P} \text{ is } \Delta_m. \end{aligned}$$

**PROOF.** First, the assertion “ $\mathbb{P} \cap V_\kappa = \mathbb{P}_\kappa$ ” is  $\Pi_{m+1}$  (in the parameter  $\kappa$ ) if  $\mathbb{P}$  is  $\Pi_m$ -definable, and  $\Pi_m$  if  $\mathbb{P}$  is  $\Sigma_m$ -definable. For the quoted sentence holds if and only if:

$$\forall x(x \in \mathbb{P} \rightarrow (\text{rk}(x) < \kappa \leftrightarrow \text{lh}(x) < \kappa)).$$

Second, by Proposition 3.14, the sentence “ $\kappa$  is a  $\mathbb{P}$ -reflecting cardinal” is  $\Pi_{m+2}$  if  $\mathbb{P}$  is  $\Gamma_m$ -definable; and it is  $\Pi_{m+1}$  if  $\mathbb{P}$  is  $\Delta_m$ -definable. Finally, as shown in Proposition 3.9, for  $k \geq 0$  the class  $C_{\mathbb{P}}^{(k)}$  is  $\Pi_{m+k}$  if  $\mathbb{P}$  is  $\Pi_m$  or  $\Sigma_m$ -definable; and it is  $\Pi_{m+k-1}$  if  $\mathbb{P}$  is  $\Delta_m$ -definable.

An easy calculation now shows that the predicate “ $\kappa$  is a  $\mathbb{P}$ - $\Sigma_k$ -reflecting cardinal” has the claimed complexity.  $\dashv$

**§4.  $\mathbb{P}$ - $\Sigma_n$ -supercompactness.** The following definition gives a refinement of the notion of  $\Sigma_n$ -supercompact cardinal, relative to definable iterations.

**DEFINITION 4.1** ( $\mathbb{P}$ - $\Sigma_n$ -supercompactness). If  $n \geq 1$  and  $\mathbb{P}$  is an ORD-length forcing iteration, then a cardinal  $\delta$  is  $\mathbb{P}$ - $\Sigma_n$ -supercompact if there exists a proper class of  $\mathbb{P}$ - $\Sigma_n$ -reflecting cardinals, and for every such cardinal  $\lambda > \delta$  and every  $a \in V_\lambda$  there exist  $\bar{\delta} < \lambda < \delta$  and  $\bar{a} \in V_{\bar{\lambda}}$ , together with an elementary embedding  $j : V_{\bar{\lambda}} \rightarrow V_\lambda$  such that:

- (i)  $\text{crit}(j) = \bar{\delta}$  and  $j(\bar{\delta}) = \delta$ .
- (ii)  $j(\bar{a}) = a$ .
- (iii)  $\bar{\lambda}$  is  $\mathbb{P}$ - $\Sigma_n$ -reflecting.

The next proposition and corollary unveil the connection between the notions of  $\Sigma_n$ -supercompact and  $\mathbb{P}$ - $\Sigma_n$ -supercompact cardinals. For conciseness, let us denote by  $\mathcal{S}^{\Sigma_n}$ ,  $\mathcal{S}_{\mathbb{P}}^{\Sigma_n}$ , and  $\mathcal{E}^{(n)}$  the classes of  $\Sigma_n$ -supercompact,  $\mathbb{P}$ - $\Sigma_n$ -supercompact, and  $C^{(n)}$ -extendible cardinals, respectively.

PROPOSITION 4.2. *Let  $\mathbb{P}$  be an ORD-length forcing iteration, and suppose there is a proper class of  $\mathbb{P}$ - $\Sigma_n$ -reflecting cardinals. Then,*

- (1)  $\mathcal{S}_{\mathbb{P}}^{\Sigma_n} \subseteq \mathcal{S}^{\Sigma_n}$ .
- (2) *If  $\mathbb{P}$  is adequate and  $\Delta_2$ -definable, then  $\mathcal{S}^{\Sigma_3} \subseteq \mathcal{S}_{\mathbb{P}}^{\Sigma_1}$ .*
- (3) *If  $\mathbb{P}$  is adequate and  $\Delta_{m+1}$ -definable, some  $m \geq 1$ , and either  $m$  or  $n$  are greater than 1, then  $\mathcal{S}^{\Sigma_{m+n}} \subseteq \mathcal{S}_{\mathbb{P}}^{\Sigma_n}$ .*

*In particular, if  $\mathbb{P}$  is adequate and  $\Delta_2$ , then every  $\Sigma_2$ -supercompact cardinal is  $\mathbb{P}$ - $\Sigma_1$ -supercompact; and for every  $n > 1$ , a cardinal is  $\Sigma_n$ -supercompact if and only if it is  $\mathbb{P}$ - $\Sigma_n$ -supercompact.*

PROOF. (1): Assume  $\delta$  is  $\mathbb{P}$ - $\Sigma_n$ -supercompact. Let  $\lambda > \delta$  be a  $\Sigma_n$ -correct cardinal and let  $a \in V_\lambda$ . Let  $\kappa > \lambda$  be a  $\mathbb{P}$ - $\Sigma_n$ -reflecting cardinal. Notice that  $V_\kappa \models \text{“}\lambda \in C^{(n)}\text{”}$ . By  $\mathbb{P}$ - $\Sigma_n$ -supercompactness, there are  $\bar{\delta} < \bar{\kappa} < \delta$ , with  $\bar{\kappa}$  being  $\mathbb{P}$ - $\Sigma_n$ -reflecting, and there are  $\bar{a} \in V_{\bar{\kappa}}$  and  $\bar{\lambda} < \bar{\kappa}$ , together with an elementary embedding  $j : V_{\bar{\kappa}} \rightarrow V_\kappa$  such that  $\text{crit}(j) = \bar{\delta}$ ,  $j(\bar{\delta}) = \delta$ ,  $j(\bar{a}) = a$ , and  $j(\bar{\lambda}) = \lambda$ . By elementarity,  $V_{\bar{\kappa}}$  thinks that  $\bar{\lambda}$  is a  $\Sigma_n$ -correct cardinal. Since  $\bar{\kappa} \in C^{(n)}$ , it follows that  $\bar{\lambda} \in C^{(n)}$ .

(2) and (3): Assume  $\delta$  is  $\Sigma_3$ -supercompact (in case (2), i.e.,  $m = n = 1$ ) or  $\Sigma_{m+n}$ -supercompact, in case (3). Let  $\lambda > \delta$  be  $\mathbb{P}$ - $\Sigma_n$ -reflecting, and let  $\kappa > \lambda$  be a  $\Sigma_{m+n}$ -correct cardinal. Since being a  $\mathbb{P}$ - $\Sigma_n$ -reflecting cardinal is a  $\Pi_3$  property (case (2)), and a  $\Pi_{m+n}$  property otherwise (case (3)),  $V_\kappa$  thinks that  $\lambda$  is  $\mathbb{P}$ - $\Sigma_n$ -reflecting (cf. Proposition 3.17). By our assumption, there exist  $\bar{\delta} < \bar{\lambda} < \bar{\kappa} < \delta$  with  $\bar{\kappa} \in C^{(3)}$  (case (2)), or  $\bar{\kappa} \in C^{(m+n)}$ , otherwise, and there exists an elementary embedding  $j : V_{\bar{\kappa}} \rightarrow V_\kappa$  such that  $\text{crit}(j) = \bar{\delta}$ ,  $j(\bar{\delta}) = \delta$ , and  $j(\bar{\lambda}) = \lambda$ . By elementarity,  $V_{\bar{\kappa}}$  thinks that  $\bar{\lambda}$  is  $\mathbb{P}$ - $\Sigma_n$ -reflecting. Since  $V_{\bar{\kappa}} \prec_{\Sigma_3} V$  (case (2)), or  $V_{\bar{\kappa}} \prec_{\Sigma_{m+n}} V$ , otherwise,  $\bar{\lambda}$  is  $\mathbb{P}$ - $\Sigma_n$ -reflecting in  $V$ . Thus, the restriction  $j \upharpoonright V_{\bar{\lambda}}$  witnesses the  $\mathbb{P}$ - $\Sigma_n$ -supercompactness of  $\delta$ . ⊣

The proposition above together with Theorem 2.4 yields the following:

COROLLARY 4.3. *Suppose  $\mathbb{P}$  is an ORD-length forcing iteration and there is a proper class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinals. Then,*

- (1)  $\mathcal{S}_{\mathbb{P}}^{\Sigma_{n+1}} \subseteq \mathcal{E}^{(n)}$ .
- (2) *If  $\mathbb{P}$  is adequate and  $\Delta_{m+1}$ -definable, some  $m \geq 1$ , then  $\mathcal{E}^{(m+n)} \subseteq \mathcal{S}_{\mathbb{P}}^{\Sigma_{n+1}}$ .*

*In particular, if  $\mathbb{P}$  is adequate and  $\Delta_2$ -definable, then for every  $n \geq 1$ , every  $C^{(n+1)}$ -extendible is  $\mathbb{P}$ - $\Sigma_{n+1}$ -supercompact.*

**§5. Suitable iterations.** The following is a property enjoyed by many well-known ORD-length forcing iterations, such as Jensen’s canonical class forcing for obtaining the global GCH [17], or the McAloon class-forcing iteration for forcing  $V = \text{HOD}$  [22]. The property will be needed to prove a general result (Theorem 5.4) about the preservation of  $C^{(n)}$ -extendibility.

DEFINITION 5.1 (Suitable iterations). An ORD-length forcing iteration  $\mathbb{P}$  is *suitable* if it is the direct limit of an Easton support iteration<sup>4</sup>  $\langle\langle \mathbb{P}_\alpha : \alpha \in$

<sup>4</sup>Recall that an Easton support iteration is a forcing iteration where direct limits are taken at inaccessible stages and inverse limits elsewhere.

ORD),  $\langle \dot{Q}_\alpha : \alpha \in \text{ORD} \rangle$  with the property that for each  $\lambda \in \text{ORD}$  there is some  $\theta \in \text{ORD}$  greater than  $\lambda$  such that

$$\Vdash_{\mathbb{P}_\nu} \text{“} \dot{Q}_\nu \text{ is } \lambda\text{-directed closed”}$$

for all  $\nu \geq \theta$ .

It is well-known that for suitable ORD-length forcing iterations  $\mathbb{P}$ , the forcing relation  $\Vdash_{\mathbb{P}}$  is definable, the forcing theorem holds, and forcing with  $\mathbb{P}$  preserves ZFC (see [12] or [23]). The condition of eventual  $\lambda$ -directed closedness in the definition above can be strengthened on a club proper class. Namely,

PROPOSITION 5.2. *Let  $\mathbb{P}$  be a suitable iteration. The class*

$$C(\mathbb{P}) := \{ \lambda : \forall \eta \geq \lambda, \Vdash_{\mathbb{P}_\eta} \text{“} \dot{Q}_\eta \text{ is } \lambda\text{-directed closed”} \}$$

*is a club class.*

PROOF. Closedness is obvious. As for unboundedness, fix any  $\lambda$  and build inductively a sequence  $\{\theta_n\}_{n \in \omega}$  of ordinals greater than  $\lambda$  such that for all  $\eta \geq \theta_{n+1}$ ,

$$\Vdash_{\mathbb{P}_\eta} \text{“} \dot{Q}_\eta \text{ is } \theta_n\text{-directed closed.”}$$

Notice now that  $\theta^* := \sup_n \theta_n$  is an element of  $C$ . ◻

The next theorem establishes some sufficient conditions for the preservation of  $C^{(n)}$ -extendible cardinals under definable iterations. Recall that a partial ordering  $\mathbb{P}$  is *weakly homogeneous* if for any  $p, q \in \mathbb{P}$  there is an automorphism  $\pi$  of  $\mathbb{P}$  such that  $\pi(p)$  and  $q$  are compatible.

In the case of definable ORD-length forcing iterations we define weak homogeneity as follows:

DEFINITION 5.3. A  $\Gamma_m$ -definable ORD-length forcing iteration  $\mathbb{P}$  is *weakly homogeneous* if there exists a  $\Gamma_m$  formula  $\varphi(x, y, z_1, z_2, z_3)$  such that for every  $\alpha$ ,  $\mathbb{P}_\alpha$  forces that for every  $\dot{p}, \dot{q} \in \dot{Q}_\alpha$ ,  $\varphi(x, y, \dot{p}, \dot{q}, \alpha)$  defines an automorphism  $\pi$  of  $\dot{Q}_\alpha$  such that  $\pi(\dot{p})$  and  $\dot{q}$  are compatible.

Let us note that if  $\mathbb{P}$  is a weakly homogeneous  $\Gamma_m$ -definable iteration, and  $\lambda$  is  $\mathbb{P}$ - $\Gamma_m$ -reflecting, then  $\mathbb{P}_\lambda$  is also a weakly homogeneous iteration in  $V_\lambda$ , with the same formula witnessing it. Also notice that, for every  $\alpha$ ,  $\mathbb{P}_\alpha$  forces that the remaining part of the iteration is weakly homogeneous. Indeed, in  $V^{\mathbb{P}_\alpha}$ , for every  $p, q \in \mathbb{P}_{[\alpha, \text{ORD}]}$ , the map  $\pi$  given by:  $\pi(x) = y$  iff for all  $\beta > 0$ ,

$$\Vdash_{\mathbb{P}_{[\alpha, \alpha + \beta]}} \text{“} \varphi(x(\beta), y(\beta), p(\beta), q(\beta), \beta)\text{”}$$

is a definable automorphism of  $\dot{\mathbb{P}}_{[\alpha, \text{ORD}]}$  (with parameters  $\dot{p}$  and  $\dot{q}$ ) such that  $\mathbb{1}_{\mathbb{P}_\alpha}$  forces  $\pi(\dot{p})$  and  $\dot{q}$  to be compatible.

THEOREM 5.4. *Let  $m, n \geq 1$  and  $m \leq n$ . Let  $\mathbb{P}$  be an adequate  $\Delta_{m+1}$ -definable and weakly homogeneous suitable iteration. Suppose there is a proper class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinals. If  $\delta$  is a  $\mathbb{P}$ - $\Sigma_{n+1}$ -supercompact cardinal, then*

$$\Vdash_{\mathbb{P}} \text{“} \delta \text{ is } C^{(n)}\text{-extendible.”}$$

PROOF. Suppose  $G$  is  $\mathbb{P}$ -generic over  $V$ . By Corollary 2.7 and Lemma 3.16, it is sufficient to take an arbitrary  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinal  $\lambda > \delta$  and any  $\alpha < \lambda$ , and find a  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinal  $\bar{\lambda}$ , ordinals  $\bar{\delta}, \bar{\alpha} < \bar{\lambda}$ , and an elementary embedding  $j : V[G]_{\bar{\lambda}} \rightarrow V[G]_{\lambda}$  such that  $\text{crit}(j) = \bar{\delta}$ ,  $j(\bar{\delta}) = \delta$ , and  $j(\bar{\alpha}) = \alpha$ .

So pick a  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinal  $\lambda > \delta$  and any  $\alpha < \lambda$ . Since  $\delta$  is  $\mathbb{P}$ - $\Sigma_{n+1}$ -supercompact there exist  $\bar{\delta} < \bar{\lambda} < \delta$  and  $\bar{\alpha} < \bar{\lambda}$ , together with an elementary embedding  $j : V_{\bar{\lambda}} \rightarrow V_{\lambda}$  such that:

- (i)  $\text{crit}(j) = \bar{\delta}$  and  $j(\bar{\delta}) = \delta$ .
- (ii)  $j(\bar{\alpha}) = \alpha$ .
- (iii)  $\bar{\lambda}$  is  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting.

It will then suffice to show that  $j$  can be lifted to an elementary embedding  $j : V_{\bar{\lambda}}[G_{\bar{\lambda}}] \rightarrow V_{\lambda}[G_{\lambda}]$ , for then, since both  $\lambda$  and  $\bar{\lambda}$  are  $\mathbb{P}$ -reflecting, we have that  $V_{\lambda}[G_{\lambda}] = V[G]_{\lambda}$  and  $V_{\bar{\lambda}}[G_{\bar{\lambda}}] = V[G]_{\bar{\lambda}}$ .

The iterations  $\mathbb{P}_{\bar{\lambda}}$  and  $\mathbb{P}_{\lambda}$  factorize as follows:

- (i)  $\mathbb{P}_{\bar{\lambda}} \cong \mathbb{P}_{\bar{\delta}} * \mathbb{Q}$  with  $|\mathbb{Q}| \leq \bar{\lambda}$ .
- (ii)  $\mathbb{P}_{\lambda} \cong \mathbb{P}_{\delta} * \mathbb{Q}^*$  with

$$\Vdash_{\mathbb{P}_{\delta}} \text{“}\mathbb{Q}^* \text{ is weakly homogeneous and } \delta\text{-directed closed.} \text{”}$$

Indeed, (i) is clear since  $\bar{\lambda}$  is a  $\beth$ -fixed point. For (ii), since  $\mathbb{P}$  is weakly homogeneous and  $\lambda$  is  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting,  $\mathbb{P}_{\lambda}$  is weakly homogeneous in  $V_{\lambda}$ , and therefore  $\mathbb{P}_{\delta}$  forces that  $\mathbb{Q}^*$  is weakly homogeneous. Thus, we only need to see that  $\Vdash_{\mathbb{P}_{\delta}} \text{“}\mathbb{Q}^* \text{ is } \delta\text{-directed closed.} \text{”}$

Recall from Proposition 5.2 that the class

$$C(\mathbb{P}) := \{ \mu : \forall \eta \geq \mu, \Vdash_{\mathbb{P}_{\eta}} \text{“}\dot{\mathbb{Q}}_{\eta} \text{ is } \mu\text{-directed closed} \text{”} \}$$

is a club class. Thus, it will be sufficient to show that  $\delta$  is a limit point of  $C(\mathbb{P})$ , and therefore it belongs to  $C(\mathbb{P})$ . So, let  $\mu < \delta$  and notice that since  $\mathbb{P}$  is a suitable iteration, the sentence  $\varphi(\mu)$  asserting:

$$\exists \theta > \mu \forall \eta \geq \theta (\Vdash_{\mathbb{P}_{\eta}} \text{“}\dot{\mathbb{Q}}_{\eta} \text{ is } \mu\text{-directed closed} \text{”})$$

holds in  $V$ . Since  $\mathbb{P}$  is  $\Delta_{m+1}$ -definable,  $\varphi(\mu)$  is easily seen to be equivalent to the  $\Sigma_{m+2}$  sentence

$$\begin{aligned} &\exists \theta > \mu \forall \eta \geq \theta \forall \alpha > \eta (\alpha \in C^{(m)} \rightarrow \\ &V_{\alpha} \models \text{“}\Vdash_{\mathbb{P}_{\eta}} \text{“}\dot{\mathbb{Q}}_{\eta} \text{ is } \mu\text{-directed closed} \text{”} \text{”}). \end{aligned}$$

Since  $\delta$  is a  $\Sigma_{n+2}$ -correct cardinal (by Lemma 2.3 and Proposition 4.2), and  $m \leq n$ , there must be a witness for  $\varphi(\mu)$  below  $\delta$ . Arguing inductively, we define an  $\omega$ -sequence of ordinals above  $\mu$  with limit in  $C(\mathbb{P})$ . This shows that  $C(\mathbb{P})$  is unbounded in  $\delta$ , as wanted.

Since  $\delta \in C^{(n+2)}$ , and  $j$  is elementary with  $j(\bar{\delta}) = \delta$ , we have that  $j(\mathbb{P}_{\bar{\delta}}) = \mathbb{P}_{\delta}$ . Also, since  $\bar{\delta}$  is the critical point of  $j$ , we have that  $j \text{“}G_{\bar{\delta}} = G_{\bar{\delta}} \subseteq G_{\delta}$ , and so  $j \upharpoonright V_{\bar{\lambda}}$  can be lifted to an elementary embedding

$$j : V_{\bar{\lambda}}[G_{\bar{\delta}}] \rightarrow V_{\lambda}[G_{\delta}].$$

Let us denote by  $G_{[\delta, \bar{\lambda}]}$  and  $G_{[\delta, \lambda]}$  the filters  $G \cap \mathbb{Q}$  and  $G \cap \mathbb{Q}^*$ , respectively. Notice that these filters are generic for  $\mathbb{Q}$  and  $\mathbb{Q}^*$  over  $V_{\bar{\lambda}}[G_{\delta}]$  and  $V_{\lambda}[G_{\delta}]$ , respectively. In order to lift the embedding  $j$  to the further generic extension  $V_{\bar{\lambda}}[G_{\bar{\lambda}}] = V_{\bar{\lambda}}[G_{\delta}][G_{[\delta, \bar{\lambda}]}]$ , notice first that  $j^{\ast}G_{[\delta, \bar{\lambda}]}$  is a directed subset of  $\mathbb{Q}^*$  of cardinality  $\leq \bar{\lambda}$ . Also, since  $j \upharpoonright V_{\bar{\lambda}} \in V_{\lambda}[G_{\delta}]$ , and  $j^{\ast}G_{[\delta, \bar{\lambda}]}$  can be computed from  $j \upharpoonright V_{\bar{\lambda}}$  and  $G_{[\delta, \bar{\lambda}]}$ , we have that  $j^{\ast}G_{[\delta, \bar{\lambda}]} \in V_{\lambda}[G_{\delta}]$ . Therefore, since  $\mathbb{Q}^*$  is a  $\delta$ -directed closed forcing notion in  $V_{\lambda}[G_{\delta}]$ , there is some condition  $p \in \mathbb{Q}^*$  such that  $p \leq q$ , for every  $q \in j^{\ast}G_{[\delta, \bar{\lambda}]}$ . Thus,  $p$  is a master condition in  $\mathbb{Q}^*$  for the embedding  $j$  and the generic filter  $G_{[\delta, \bar{\lambda}]}$ . So, if  $H \subseteq \mathbb{Q}^*$  is a generic filter over  $V_{\lambda}[G_{\delta}]$  containing  $p$ , then  $j$  can be lifted to an elementary embedding

$$j : V_{\bar{\lambda}}[G_{\bar{\lambda}}] \longrightarrow V_{\lambda}[G_{\delta} * H].$$

CLAIM 5.4.1. In  $V[G]$  there exists some generic filter  $H \subseteq \mathbb{Q}^*$  over  $V_{\lambda}[G_{\delta}]$  containing  $p$  such that  $V_{\lambda}[G_{\delta} * H] = V_{\lambda}[G_{\lambda}]$ .

PROOF OF CLAIM. By (ii) above,  $\mathbb{Q}^*$  is a weakly homogeneous class forcing in  $V_{\lambda}[G_{\delta}]$ . Thus, the set of conditions  $r \in \mathbb{Q}^*$  for which there is an automorphism  $\pi$  of  $\mathbb{Q}^*$  that is definable in  $V_{\lambda}[G_{\delta}]$  and such that  $\pi(r) \leq p$  is dense. Pick such an  $r$  in  $G_{[\delta, \lambda]}$  and such an automorphism  $\pi$ . Now, notice that the filter  $H$  generated by the set  $\pi^{\ast}G_{[\delta, \lambda]}$  contains  $\pi(r)$  and therefore it contains  $p$ . Since  $H$  is definable by means of  $\pi$  and  $G_{[\delta, \lambda]}$ , and also  $\pi$  is definable in  $V_{\lambda}[G_{\delta}]$ , we conclude that  $V_{\lambda}[G_{\delta} * H] = V_{\lambda}[G_{\lambda}]$ .  $\dashv$

By taking  $H \subseteq \mathbb{Q}^*$  as in the claim above, we thus obtain a lifting

$$j : V_{\bar{\lambda}}[G_{\bar{\lambda}}] \longrightarrow V_{\lambda}[G_{\lambda}],$$

as wanted.  $\dashv$

REMARK 5.5. Note that the above proof shows more than what is stated in Theorem 5.4. Specifically, what is proved is the following local result: every elementary embedding (in  $V$ ) witnessing some partial degree of  $\mathbb{P}$ - $\Sigma_{n+1}$ -supercompactness lifts to an elementary embedding (in  $V^{\mathbb{P}}$ ) witnessing the same partial degree of  $C^{(n)}$ -extendibility.

COROLLARY 5.6. Suppose  $n \geq 1$ . Let  $\mathbb{P}$  be an adequate  $\Delta_2$ -definable and weakly homogeneous suitable iteration. Suppose there is a proper class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinals. If  $\delta$  is a  $C^{(n+1)}$ -extendible cardinal, then

$$\Vdash_{\mathbb{P}} \text{“} \delta \text{ is } C^{(n)}\text{-extendible.”}$$

PROOF. Since  $\mathbb{P}$  is adequate,  $\Delta_2$ -definable, and there exists a proper class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinals, Corollary 4.3 implies that  $\delta$  is  $\mathbb{P}$ - $\Sigma_{n+1}$ -supercompact. Now, Theorem 5.4 applies to get the desired result.  $\dashv$

Let  $\mathbb{P}$  be an adequate  $\Delta_{m+1}$ -definable suitable iteration. Let us look next into the conditions under which, for some  $n \geq 1$ , there exists a proper class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinals (this was one of the assumptions of Theorem 5.4).

First, the class  $C_{\mathbb{P}}^{(n)}$  is closed, unbounded, and  $\Pi_{m+n}$ -definable (Proposition 3.9). Also, the class  $D$  of ordinals  $\alpha$  such that  $\mathbb{P}_{\alpha} = \mathbb{P} \cap V_{\alpha}$  is  $\Pi_{m+1}$ -definable (cf. the

proof of Proposition 3.17). Observe that  $D$  is a proper class by assumption, hence  $\overline{D}$  (the closure of  $D$ ) is a club proper class. Further, the following holds:

LEMMA 5.7. *The class  $K$  of  $\mathbb{P}$ -reflecting cardinals is unbounded.*

PROOF. We begin with the following auxiliary claim:

CLAIM 5.7.1. For each  $\alpha \in \text{ORD}$  there is a  $\mathbb{P}$ -name  $\tau_\alpha$  such that

$$\Vdash_{\mathbb{P}} \text{“}\tau_\alpha = V[\dot{G}]_\alpha\text{.”}$$

PROOF. By induction on  $\alpha$ . If  $\alpha = 0$ , then the result is clear. So, assume that  $\langle \tau_\beta \mid \beta < \alpha \rangle$  has already been found, and let us find  $\tau_\alpha$ . If  $\alpha$  is a limit ordinal, then let  $\tau_\alpha := \bigcup_{\beta < \alpha} \tau_\beta$ . Clearly,  $\tau_\alpha$  is as desired.

Now suppose that  $\alpha$  is a successor ordinal; say,  $\alpha = \bar{\alpha} + 1$ . Let  $\varphi : \delta \rightarrow \tau_{\bar{\alpha}}$  be a bijection, some cardinal  $\delta$ . Thus,  $\Vdash_{\mathbb{P}} \text{“}\check{\varphi} : \check{\delta} \rightarrow \tau_{\bar{\alpha}} \text{ is onto.”}$  Since by the induction hypothesis,  $\Vdash_{\mathbb{P}} \text{“}\tau_{\bar{\alpha}} = V[\dot{G}]_{\bar{\alpha}}\text{,”}$  we have that  $\Vdash_{\mathbb{P}} \text{“}\check{\varphi} : \check{\delta} \rightarrow V[\dot{G}]_{\bar{\alpha}} \text{ is onto.”}$  As  $\mathbb{P}$  is suitable we can find a cardinal  $\theta$  such that  $\tau_{\bar{\alpha}}$  is a  $\mathbb{P}_\theta$ -name and, moreover,

$$\Vdash_{\mathbb{P}_\theta} \text{“}\dot{\mathbb{P}}_{[\theta, \text{ORD}]} \text{ is } \delta^+ \text{-closed.”}$$

As the remaining part  $\mathbb{P}_{[\theta, \text{ORD}]}$  of the iteration does not add any new subsets of  $\delta$ , hence no new elements of  $V[\dot{G}]_\alpha$ , we have that

$$\Vdash_{\mathbb{P}} \text{“}V[\dot{G}]_\alpha = V[\dot{G}_\theta]_\alpha\text{.”}$$

So we can just take  $\tau_\alpha$  to be a  $\mathbb{P}_\theta$ -name for  $V[\dot{G}_\theta]_\alpha$ , where  $\dot{G}_\theta$  is the standard  $\mathbb{P}_\theta$ -name for the generic. ⊣

We are now in conditions to show that the class  $K$  of  $\mathbb{P}$ -reflecting cardinals is unbounded. To this aim fix any ordinal  $\alpha$  and define inductively a sequence of ordinals  $\langle \alpha_n : n < \omega \rangle$  in  $D$ , with  $\alpha_0 = \alpha$  and  $\Vdash_{\mathbb{P}} \text{“}V[\dot{G}]_{\alpha_n} \subseteq V_{\alpha_{n+1}}[\dot{G}_{\alpha_{n+1}}]\text{,”}$  as follows: Suppose that  $\alpha_n$  has already been defined. By the above claim there is a  $\mathbb{P}$ -name  $\tau$  such that  $\mathbb{P}$  forces  $\text{“}\tau = V[\dot{G}]_{\alpha_n}\text{.”}$  Let  $\beta$  be an ordinal such that  $p \in \mathbb{P}_\beta$  and  $\mathbb{P}_\beta = \mathbb{P} \cap V_\beta$ . Then,  $\Vdash_{\mathbb{P}} \text{“}\tau \in V_\beta[\dot{G}_\beta]\text{”}$  and thus  $\Vdash_{\mathbb{P}} \text{“}V[\dot{G}]_{\alpha_n} \subseteq V_{\alpha_{n+1}}[\dot{G}_{\alpha_{n+1}}]\text{,”}$  where  $\beta := \alpha_{n+1}$ . Finally, let  $\lambda$  be the supremum of the  $\alpha_n$ 's. In this case we have that  $\Vdash_{\mathbb{P}} \text{“}V[\dot{G}]_\lambda \subseteq V_\lambda[\dot{G}_\lambda]\text{,”}$  and so  $\lambda$  is  $\mathbb{P}$ -reflecting. ⊣

Thus,  $K$  is a  $\Pi_{m+2}$ -definable club proper class (see Propositions 3.3 and 3.14). For each  $n \geq 1$ , set  $\mathcal{K}_n := C_{\mathbb{P}}^{(n)} \cap K \cap \overline{D}$ .

PROPOSITION 5.8. *Let  $\mathbb{P}$  be an adequate  $\Delta_{m+1}$ -definable suitable iteration. Then  $\mathcal{K}_{n+1}$  (resp.  $\mathcal{K}_1$ ) is a  $\Pi_{m+n+1}$ -definable (resp.  $\Pi_{m+2}$ ) club class such that  $\mathcal{K}_{n+1} \cap \text{Reg}$  (resp.  $\mathcal{K}_1 \cap \text{Reg}$ ) is contained in the class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinals (resp.  $\mathbb{P}$ - $\Sigma_1$ -reflecting cardinals).*

PROOF. Clearly,  $\mathcal{K}_{n+1}$  is a club class. Also, by the above discussion,  $\mathcal{K}_{n+1}$  is  $\Pi_{m+n+1}$ , and  $\mathcal{K}_1$  is  $\Pi_{m+2}$ . Let  $\kappa \in \mathcal{K}_{n+1} \cap \text{Reg}$ . To show that  $\kappa$  is  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting it is enough to verify that  $\kappa \in D$ .

Since  $\kappa$  is an accumulation point of  $D$ ,  $\mathbb{P} \cap V_\kappa \subseteq \mathbb{P}_\kappa$ . Conversely,  $\kappa$  is inaccessible, hence  $\mathbb{P}_\kappa$  is the direct limit of the previous stages, and so  $\mathbb{P}_\kappa \subseteq \mathbb{P} \cap V_\kappa$ . Hence  $\kappa \in D$ . The proof for  $\mathcal{K}_1$  is the same. ⊣



Recall that if  $\kappa$  is a cardinal and  $\alpha$  an ordinal,

- $\kappa$  is 0-Mahlo if  $\kappa$  is inaccessible;
- $\kappa$  is  $(\alpha + 1)$ -Mahlo iff  $\{\lambda < \kappa \mid \lambda \text{ is } \alpha\text{-Mahlo}\}$  is stationary;
- In case  $\alpha > 0$  is a limit ordinal,  $\kappa$  is  $\alpha$ -Mahlo iff  $\kappa$  is  $\beta$ -Mahlo for every  $\beta < \alpha$ .

DEFINITION 5.9. For  $n < \omega$  and  $\alpha \in \text{ORD}$ , the class  $\text{ORD}$  is  $\Gamma_n$ - $(\alpha + 1)$ -Mahlo if every  $\Gamma_n$ -definable<sup>5</sup> club proper class of ordinals contains an  $\alpha$ -Mahlo cardinal.

Note that a cardinal is Mahlo if and only if it is 1-Mahlo, hence  $\text{ORD}$  is  $\Gamma_n$ -Mahlo if and only if  $\text{ORD}$  is  $\Gamma_n$ -1-Mahlo.

DEFINITION 5.10. A proper class  $S$  of ordinals is  $\Gamma_n$ -stationary if  $S$  intersects every  $\Gamma_n$ -definable club proper class of ordinals.

PROPOSITION 5.11. Let  $\mathbb{P}$  be an adequate  $\Delta_{m+1}$ -definable suitable iteration. If  $\text{ORD}$  is  $\Pi_{m+n+1}$ -Mahlo, then the class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinals is  $\Pi_{m+n+1}$ -stationary, so it is a proper class.

PROOF. Let  $\mathcal{C}$  be a  $\Pi_{m+n+1}$  club proper class of ordinals. By Proposition 5.8,  $\mathcal{C} \cap \mathcal{K}_{n+1}$  is a  $\Pi_{m+n+1}$  club proper class such that  $\mathcal{C} \cap \mathcal{K}_{n+1} \cap \text{Reg}$  is contained in the class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinals. Thus, the class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinals is  $\Pi_{m+n+1}$ -stationary. To show the class is unbounded, given any cardinal  $\kappa$ , let  $\mathcal{C} := C^{(n+m)} \setminus \kappa^+$ . Then  $\mathcal{C}$  is a  $\Pi_{m+n+1}$ -definable club proper class which, as before, contains a  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinal.  $\dashv$

LEMMA 5.12. Let  $\mathbb{P}$  be a  $\Delta_{m+1}$ -definable suitable iteration. If  $\text{ORD}$  is  $\Pi_{m+1}$ -Mahlo, then  $\mathbb{P}$  is adequate.

PROOF. Since  $\mathbb{P}$  is  $\Delta_{m+1}$ -definable,  $\mathbb{P}^{V_\kappa} = \mathbb{P} \cap V_\kappa$  for every  $\kappa \in C^{(m+1)}$ . Since  $\text{ORD}$  is  $\Pi_{m+1}$ -Mahlo, and direct limits are taken at inaccessible stages of the iteration, the class of inaccessible cardinals  $\kappa$  such that  $\mathbb{P}^{V_\kappa} = \mathbb{P}_\kappa$  is a proper class. Thus,  $\mathbb{P}$  is adequate.  $\dashv$

The following corollaries now follow immediately from Theorem 5.4, Corollary 5.6, Proposition 5.11, and Lemma 5.12:

COROLLARY 5.13. Let  $m, n \geq 1$  and  $m \leq n$ . Let  $\mathbb{P}$  be  $\Delta_{m+1}$ -definable weakly homogeneous suitable iteration. Suppose  $\text{ORD}$  is  $\Pi_{m+n+1}$ -Mahlo. If  $\delta$  is a  $\mathbb{P}$ - $\Sigma_{n+1}$ -supercompact cardinal, then

$$\Vdash_{\mathbb{P}} \text{“}\delta \text{ is } C^{(n)}\text{-extendible.”}$$

COROLLARY 5.14. Let  $n \geq 1$  and let  $\mathbb{P}$  be  $\Delta_2$ -definable weakly homogeneous suitable iteration. Suppose  $\text{ORD}$  is  $\Pi_{n+2}$ -Mahlo. If  $\delta$  is a  $C^{(n+1)}$ -extendible cardinal, then

$$\Vdash_{\mathbb{P}} \text{“}\delta \text{ is } C^{(n)}\text{-extendible.”}$$

**§6. Vopěnka’s principle and suitable iterations.** Let us recall the following characterizations of Vopěnka’s Principle, as well as of its restriction to definable classes of structures of a given complexity, in terms of  $C^{(n)}$ -extendible cardinals.

<sup>5</sup>That is,  $\Gamma_n$ -definable with parameters (see Notation 3.8).

THEOREM 6.1 [2]. *The following are equivalent:*

- (1) VP.
- (2) For every  $n \geq 1$  there exists a  $C^{(n)}$ -extendible cardinal.

THEOREM 6.2 [1]. *Let  $n \geq 1$ . The following are equivalent:*

- (1)  $VP(\Pi_{n+1})$ , i.e., VP restricted to proper classes of structures that are  $\Pi_{n+1}$ -definable with parameters.
- (2) There exists a proper class of  $C^{(n)}$ -extendible cardinals.

For  $n = 0$ , the equivalence is between  $VP(\Pi_1)$  and the existence of a proper class of supercompact cardinal. The *lightface* version also holds. Namely,  $VP(\Pi_{n+1})$  is equivalent to the existence of a  $C^{(n)}$ -extendible cardinal (see [1]).

Vopěnka’s Principle can be also characterized in terms of the existence of  $\mathbb{P}$ - $\Sigma_n$ -supercompact cardinals (cf. Theorems 6.4 and 6.5). The following lemma will be useful for this purpose.

LEMMA 6.3. *Let  $n \geq 1$ . Then  $VP(\Pi_n)$  implies that ORD is  $\Sigma_{n+1}$ - $(\kappa + 1)$ -Mahlo, for every cardinal  $\kappa$ .*<sup>6</sup>

PROOF. Let us prove the lemma for  $n > 1$ . The case  $n = 1$  is similar, using the fact that  $VP(\Pi_1)$  is equivalent to the existence of a proper class of supercompact cardinal, and that every supercompact cardinal belongs to  $C^{(2)}$ . So, let  $n > 1$  and assume that  $VP(\Pi_n)$  holds.

Let  $\mathcal{C}$  be a  $\Sigma_{n+1}$ -definable club proper class of ordinals and let  $\varphi(x, \vec{a})$  be some  $\Sigma_{n+1}$ -formula defining it. Let  $\kappa$  be a cardinal and  $\lambda$  be a  $C^{(n-1)}$ -extendible cardinal with  $\vec{a} \in V_\lambda$  and  $\kappa < \lambda$ . Note that this  $\lambda$  exists by virtue of Theorem 6.2. We claim that  $\mathcal{C} \cap \lambda$  is unbounded in  $\lambda$ . For if  $\alpha < \lambda$ , then the sentence “ $\exists \beta > \alpha (\beta \in \mathcal{C})$ ” is  $\Sigma_{n+1}$  (with  $\vec{a}$  as parameters), hence it is true in  $V_\lambda$  because  $\lambda$  is  $C^{(n-1)}$ -extendible and so it belongs to  $C^{(n+1)}$  [1, Proposition 3.4]. Since  $\mathcal{C}$  is closed,  $\lambda \in \mathcal{C}$ . Since  $\lambda$  is  $\lambda$ -Mahlo, hence also  $\kappa$ -Mahlo, the result follows.  $\dashv$

THEOREM 6.4. *The following are equivalent:*

- (1) VP holds.
- (2) For every  $n \geq 1$  and every definable weakly homogeneous suitable iteration  $\mathbb{P}$ , there is a proper class of  $\mathbb{P}$ - $\Sigma_n$ -reflecting cardinals and there exists a proper class of  $\mathbb{P}$ - $\Sigma_n$ -supercompact cardinal.

PROOF. (1)  $\Rightarrow$  (2): Let  $n \geq 1$  and let  $\mathbb{P}$  be  $\Delta_{m+1}$ -definable weakly homogeneous suitable iteration, some  $m \geq 1$ . By Lemma 5.12,  $\mathbb{P}$  is adequate. By Lemma 6.3, ORD is  $\Pi_{m+n+1}$ -Mahlo and so Proposition 5.11 yields a proper class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinals. Also, by Corollary 4.3 and Theorem 6.2 we infer that there is a proper class of  $\mathbb{P}$ - $\Sigma_n$ -supercompact cardinal.

(2)  $\Rightarrow$  (1): Since (2) holds, by Corollary 4.3 there exists a proper class of  $C^{(n)}$ -extendible cardinals, for every  $n \geq 1$ , hence by Theorem 6.2, VP holds.  $\dashv$

The following gives more precise information about the relationship between  $\mathbb{P}$ - $\Sigma_n$ -supercompact cardinals and fragments of VP.

<sup>6</sup>The lightface version of the statement also holds.

**THEOREM 6.5.** *Let  $m, n \geq 1$  and let  $\mathbb{P}$  be an ORD-length forcing iteration. Then,*

- (1) *Assume there is a proper class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinals. If there is a  $\mathbb{P}$ - $\Sigma_{n+1}$ -supercompact cardinal, then  $\text{VP}(\Pi_{n+1})$  holds; and if there is a proper class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -supercompact cardinals, then  $\text{VP}(\Pi_{n+1})$  holds.*
- (2) *If  $\mathbb{P}$  is  $\Delta_{m+1}$ -definable, some  $m \geq 1$ , suitable, and weakly homogeneous, then  $\text{VP}(\Pi_{m+n+1})$  implies the existence of a proper class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinals and a proper class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -supercompact cardinals.*

**PROOF.** Item (1) is a direct consequence of Corollary 4.3 and our remarks following Theorem 6.2. As for (2), Theorem 6.2 shows that  $\text{VP}(\Pi_{m+n+1})$  is equivalent to the existence of a proper class of  $C^{(m+n)}$ -extendible cardinals. Also, Lemma 6.3 shows that  $\text{VP}(\Pi_{m+n+1})$  implies that ORD is  $\Sigma_{m+n+2}$ -Mahlo, hence Lemma 5.12 implies that  $\mathbb{P}$  is adequate. Thus, by Proposition 5.11, there exists a proper class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinals. By Corollary 4.3 there exists also a proper class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -supercompact cardinals. −

We end this section by proving a level-by-level version of Brooke-Taylor’s result on the preservation of Vopěnka’s Principle under definable suitable iterations.

**THEOREM 6.6.** *Let  $m, n \geq 1$  and  $m \leq n$ . Let  $\mathbb{P}$  be a  $\Delta_{m+1}$ -definable weakly homogeneous suitable iteration. If  $\text{VP}(\Pi_{m+n+1})$  holds, then*

$$\Vdash_{\mathbb{P}} \text{“VP}(\Pi_{n+1}) \text{ holds.”}$$

**PROOF.** By Theorem 6.5(2), there is a proper class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinals, as well as a proper class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -supercompact cardinals. Also, by Lemma 5.12,  $\mathbb{P}$  is adequate. Now the result follows combining Theorem 5.4 and the remarks just after Theorem 6.2. −

**COROLLARY 6.7 [8].** *Let  $\mathbb{P}$  be a definable weakly homogeneous suitable iteration. If VP holds in  $V$ , then VP holds in  $V^{\mathbb{P}}$ .*

Our version of Brooke-Taylor’s result differs from the original one in that we require the weak homogeneity of  $\mathbb{P}$ . However, our proof shows more than Brooke-Taylor’s, for it shows that every relevant elementary embedding from the ground model lifts to an elementary embedding in the forcing extension (recall Remark 5.5). Even though weak homogeneity holds for a wide family of forcing notions, it puts some restrictions on the sort of statements that can be forced. One example is “ $V = \text{HOD}$ .” In Section 9.1 we will address this problem and will prove Theorem 6.6 without the weak homogeneity assumption (Theorem 9.6). We are very grateful to Brooke-Taylor for his valuable comments on this matter.

**§7.  $C^{(n)}$ -extendible cardinals and fitting iterations.** While our main corollary from Section 5 (cf. Corollary 5.14) applies to a wide class of forcing iterations, it requires that the cardinal we begin with is  $C^{(n+1)}$ -extendible. In this section we argue that this extra assumption can be avoided by restricting to a certain subclass of forcing iterations, which nevertheless are still general enough.

**DEFINITION 7.1.** An ORD-length forcing iteration  $\mathbb{P}$  is *fitting* if it is  $\Delta_2$ -definable, suitable, and weakly homogeneous, and there is a  $\Delta_2$ -definable  $\Sigma_2$ -stationary class  $K_{\mathbb{P}}$  of regular  $\mathbb{P}$ -reflecting cardinals  $\kappa$  such that  $\mathbb{P} \cap V_{\kappa} = \mathbb{P}_{\kappa}$ .

REMARK 7.2. Note that every fitting iteration is adequate. Also notice that  $K_{\mathbb{P}} \cap C^{(1)}$  is a  $\Delta_2$ -definable proper class of inaccessible cardinals.

THEOREM 7.3. *Suppose that  $\mathbb{P}$  is a fitting iteration. If  $\delta$  is a supercompact cardinal in  $C(\mathbb{P})$ , then  $\Vdash_{\mathbb{P}}$  “ $\delta$  is supercompact.”<sup>7</sup>*

PROOF. Let  $\delta$  be a supercompact cardinal in  $C(\mathbb{P})$  and suppose  $G$  is a  $\mathbb{P}$ -generic filter over  $V$ . It will be sufficient to verify that for each  $\lambda \in K_{\mathbb{P}} \cap C^{(1)}$  there are  $\bar{\lambda} \in K_{\mathbb{P}} \cap C^{(1)}$  and  $\bar{\delta}$  with  $\bar{\delta} < \bar{\lambda} < \delta$ , and an elementary embedding

$$j : V_{\bar{\lambda}}[G_{\bar{\lambda}}] \rightarrow V_{\lambda}[G_{\lambda}]$$

with  $\text{crit}(j) = \bar{\delta}$  and  $j(\bar{\delta}) = \delta$ . For since  $\lambda$  and  $\bar{\lambda}$  are  $\mathbb{P}$ -reflecting we then have  $V_{\bar{\lambda}}[G_{\bar{\lambda}}] = V[G]_{\bar{\lambda}}$  and  $V_{\lambda}[G_{\lambda}] = V[G]_{\lambda}$ .

So let  $\lambda$  be as above. Since  $\delta$  is  $\Sigma_1$ -supercompact, given  $\mu \in C^{(1)}$  with  $\mu > \lambda$ , there is  $\bar{\mu} \in C^{(1)}$  and  $\bar{\delta} < \bar{\lambda} < \bar{\mu}$ , and an elementary embedding  $j : V_{\bar{\mu}} \rightarrow V_{\mu}$  with  $\text{crit}(j) = \bar{\delta}$ ,  $j(\bar{\delta}) = \delta$ , and  $j(\bar{\lambda}) = \lambda$ .

Since  $K_{\mathbb{P}}$  is  $\Delta_2$ -definable,  $V_{\mu} \models \lambda \in K_{\mathbb{P}} \cap C^{(1)}$ . Hence, by elementarity and  $\Sigma_1$ -correctness of  $\bar{\mu}$ ,  $\bar{\lambda} \in K_{\mathbb{P}} \cap C^{(1)}$ .

We only need to show how to lift  $j \upharpoonright V_{\bar{\lambda}} : V_{\bar{\lambda}} \rightarrow V_{\lambda}$  to an elementary embedding  $V_{\bar{\lambda}}[G_{\bar{\lambda}}] \rightarrow V_{\lambda}[G_{\lambda}]$ . Since  $\mathbb{P}$  is  $\Delta_2$ -definable,  $\mathbb{P}^{V_{\bar{\lambda}}} = \mathbb{P} \cap V_{\bar{\lambda}} = \mathbb{P}_{\bar{\lambda}}$ , where the rightmost equality follows from  $\lambda \in K_{\mathbb{P}}$ . Analogously, the same holds for  $\mathbb{P}^{V_{\bar{\lambda}}}$ . Thus, we have:

- (i)  $\mathbb{P}_{\bar{\lambda}} \cong \mathbb{P}_{\bar{\delta}} * \dot{Q}$ , with  $|\dot{Q}| = \bar{\lambda}$ .
- (ii)  $\mathbb{P}_{\lambda} \cong \mathbb{P}_{\delta} * \dot{Q}^*$  and  $\Vdash_{\mathbb{P}_{\delta}}$  “ $\dot{Q}^*$  is weakly homogeneous.”

Since  $\delta \in C(\mathbb{P})$ ,  $\Vdash_{\mathbb{P}_{\delta}}$  “ $\dot{Q}^*$  is  $\delta$ -directed closed.” We may now proceed as in the proof of Theorem 5.4 to get the desired elementary embedding  $j : V_{\bar{\lambda}}[G_{\bar{\lambda}}] \rightarrow V_{\lambda}[G_{\lambda}]$ .  $\dashv$

LEMMA 7.4. *Suppose that  $\mathbb{P}$  is a fitting iteration and  $\delta$  is an extendible cardinal. Then,  $\Vdash_{\mathbb{P}}$  “ $\delta$  is extendible.”*

PROOF. Suppose  $G$  is a  $\mathbb{P}$ -generic filter over  $V$ . It suffices to verify that for each  $\lambda \in K_{\mathbb{P}}$  greater than  $\delta$ , there is  $\theta \in K_{\mathbb{P}}$  and an elementary embedding

$$j : V_{\lambda}[G_{\lambda}] \rightarrow V_{\theta}[G_{\theta}]$$

with  $\text{crit}(j) = \delta$  and  $j(\delta) > \lambda$ . For since  $\lambda$  and  $\theta$  are  $\mathbb{P}$ -reflecting we then have  $V_{\lambda}[G_{\lambda}] = V[G]_{\lambda}$  and  $V_{\theta}[G_{\theta}] = V[G]_{\theta}$ .

So let  $\lambda$  be as above. Since  $\delta$  is extendible, hence  $C^{(1)+}$ -extendible, given  $\mu \in C^{(1)}$ ,  $\mu > \lambda$ , there is  $\eta \in C^{(1)}$  and an elementary embedding  $j : V_{\mu} \rightarrow V_{\eta}$  with  $\text{crit}(j) = \delta$  and  $j(\delta) > \mu$ . Since  $K_{\mathbb{P}}$  is  $\Delta_2$ -definable,  $V_{\mu} \models \lambda \in K_{\mathbb{P}}$ , hence by elementarity and since  $\eta \in C^{(1)}$ ,  $j(\lambda) \in K_{\mathbb{P}}$ .

Let  $\theta := j(\lambda)$ . We will show how to lift  $j \upharpoonright V_{\lambda} : V_{\lambda} \rightarrow V_{\theta}$  to an elementary embedding  $V_{\lambda}[G_{\lambda}] \rightarrow V_{\theta}[G_{\theta}]$ . Since  $\mathbb{P}$  is  $\Delta_2$ -definable,  $\mathbb{P}^{V_{\lambda}} = \mathbb{P} \cap V_{\lambda} = \mathbb{P}_{\lambda}$ , where the rightmost equality follows from  $\lambda \in K_{\mathbb{P}}$ . Analogously, the same holds for  $\mathbb{P}^{V_{\theta}}$ . Thus, we have:

- (i)  $\mathbb{P}_{\lambda} \cong \mathbb{P}_{\delta} * \dot{Q}$ , with  $|\dot{Q}| = \lambda$ .
- (ii)  $\mathbb{P}_{\theta} \cong \mathbb{P}_{i(\delta)} * \dot{Q}^*$  and  $\Vdash_{\mathbb{P}_{i(\delta)}}$  “ $\dot{Q}^*$  is weakly homogeneous.”

<sup>7</sup>For the definition of  $C(\mathbb{P})$ , see Proposition 5.2.

CLAIM 7.4.1.  $\Vdash_{\mathbb{P}_{i(\delta)}} \dot{Q}^*$  is  $i(\delta)$ -directed closed.”

PROOF OF CLAIM. We first show that  $\delta$  is an accumulation point of  $C(\mathbb{P})$ , hence it belongs to  $C(\mathbb{P})$ . So, fix an ordinal  $\sigma_0 < \delta$ . Since  $\mathbb{P}$  is suitable the  $\Sigma_3$ -formula  $\varphi(\sigma_0)$

$$\exists\sigma\forall\rho\forall\alpha\forall X(\sigma_0 < \sigma \leq \rho < \alpha \wedge \alpha \in \overline{K_{\mathbb{P}}^{(1)}} \wedge X = V_\alpha \rightarrow X \Vdash \Vdash_{\mathbb{P}_\rho} \dot{Q}_\rho \text{ is } \sigma_0\text{-directed closed}^{\ast})$$

holds. Since  $\delta \in C^{(3)}$  and  $\sigma_0 < \delta$ , there is a witness  $\sigma_0 < \sigma_1 < \delta$  for  $\varphi(\sigma_0)$ . Arguing inductively we define a sequence  $\langle \sigma_n \mid n < \omega \rangle$  of ordinals  $< \delta$  such that  $\sigma_{n+1}$  is a witness for  $\varphi(\sigma_n)$ . Setting  $\sigma_\omega := \sup_{n < \omega} \sigma_n$  one can easily see that  $\sigma_\omega \in C(\mathbb{P}) \cap \delta$ .

Since  $\delta \in C(\mathbb{P})$ , for each  $\sigma \in [\delta, \lambda)$ ,  $\Vdash_{\mathbb{P}_\delta} \dot{\mathbb{P}}_{[\delta, \sigma]}$  is  $\delta$ -directed closed.”<sup>8</sup> Since  $\lambda$  is inaccessible,  ${}^{<\delta}V_\lambda \subseteq V_\lambda$ , and therefore the same is true in  $V_\lambda$ , namely  $V_\lambda \Vdash \Vdash_{\mathbb{P}_\delta} \dot{Q}$  is  $\delta$ -directed closed.” By elementarity,

$$V_\theta \Vdash \Vdash_{\mathbb{P}_{i(\delta)}} \dot{Q}^* \text{ is } i(\delta)\text{-directed closed.}^{\ast}$$

Since  $\theta$  is inaccessible,  ${}^{<i(\delta)}V_\theta \subseteq V_\theta$ , and therefore the quoted sentence displayed above also holds in  $V$ , as desired.  $\dashv$

From this point on the argument for the lifting of  $j \upharpoonright V_\lambda$  to an elementary embedding  $V_\lambda[G_\lambda] \rightarrow V_\theta[G_\theta]$  is essentially the same as in Theorem 5.4.  $\dashv$

THEOREM 7.5. *Suppose that  $\mathbb{P}$  is a fitting iteration such that  $C(\mathbb{P})$  contains all supercompact cardinals, in case there are any. Then forcing with  $\mathbb{P}$  preserves  $C^{(n)}$ -extendible cardinals, for all  $n < \omega$ .*<sup>9</sup>

PROOF. We prove the theorem by induction over  $n$ . The case  $n = 1$  is covered by Lemma 7.4. So, suppose that  $n \geq 2$  and for each  $0 \leq k < n$  forcing with  $\mathbb{P}$  preserves  $C^{(k)}$ -extendible cardinals.

We argue similarly as in the proof of Lemma 7.4. So, let  $\delta$  be a  $C^{(n)}$ -extendible cardinal and suppose  $G$  is a  $\mathbb{P}$ -generic filter over  $V$ . Let  $\lambda \in K_{\mathbb{P}}^{(1)}$  be greater than  $\delta$ , and let  $\mu \in C^{(n)}$  be greater than  $\lambda$ . Then, there is  $\eta \in C^{(n)}$  and an elementary embedding

$$j : V_\mu \rightarrow V_\eta$$

with  $\text{crit}(j) = \delta$ ,  $j(\delta) > \mu$ , and  $j(\delta) \in C^{(n)}$ . Actually,  $j(\delta)$  is  $C^{(n-2)}$ -extendible. Since  $n \geq 2$  and  $K_{\mathbb{P}}$  is  $\Delta_2$ -definable,  $V_\mu \Vdash \lambda \in K_{\mathbb{P}}$ . Hence, by elementarity and since  $\eta \in C^{(n)}$ ,  $j(\lambda) \in K_{\mathbb{P}}$ .

Arguing exactly as in the proof of Theorem 7.4, we can lift  $j \upharpoonright V_\lambda$  to an elementary embedding  $V[G]_\lambda \rightarrow V[G]_\theta$ , where  $\theta := j(\lambda)$ .

It only remains to show that  $j(\delta) \in C^{(n)}$ . Suppose first  $n = 2$ . Since  $j(\delta)$  is supercompact and every supercompact belongs to  $C^{(2)}$ , it is enough to show that  $j(\delta)$  is supercompact in  $V[G]$ . By our assumption,  $j(\delta) \in C(\mathbb{P})$ , hence  $j(\delta)$  is supercompact in  $V[G]$  (cf. Theorem 7.3).

<sup>8</sup>Here  $\dot{\mathbb{P}}_{[\delta, \sigma]}$  is a  $\mathbb{P}_\delta$ -name for the iteration  $\mathbb{Q}$  up to  $\sigma$ .

<sup>9</sup>By convention, a cardinal is  $C^{(0)}$ -extendible iff it is supercompact (cf. page 20).

For  $1 \leq n - 2$  and  $j(\delta)$  being  $C^{(n-2)}$ -extendible, our induction hypothesis implies that  $i(\delta)$  is  $C^{(n-2)}$ -extendible in  $V[G]$ . In particular,  $V[G]_{j(\delta)} \prec_{\Sigma_n} V[G]$ . Thus,  $\delta$  is  $C^{(n)}$ -extendible in  $V[G]$ .  $\dashv$

In the next section we provide several applications of this theorem.

**§8. Some applications.**

**8.1. Forcing the GCH and related combinatorial principles.** Let  $\mathbb{P} = \langle \mathbb{P}_\alpha; \dot{Q}_\alpha : \alpha \in \text{ORD} \rangle$  be the standard Jensen’s proper class iteration for forcing the global GCH. Namely,  $\mathbb{P}$  is the direct limit of the iteration with Easton support where  $\mathbb{P}_0$  is the trivial forcing and for each ordinal  $\alpha$ , if  $\Vdash_{\mathbb{P}_\alpha}$  “ $\alpha$  is an uncountable cardinal,” then  $\Vdash_{\mathbb{P}_\alpha}$  “ $\dot{Q}_\alpha = \text{Add}$ ,” and  $\Vdash_{\mathbb{P}_\alpha}$  “ $\dot{Q}_\alpha$  is trivial” otherwise.

LEMMA 8.1. *Assume ORD is  $\Sigma_2$ -2-Mahlo. Then  $\mathbb{P}$  is fitting and  $C(\mathbb{P})$  contains all supercompact cardinals, in case there are any.*

PROOF. Clearly,  $\mathbb{P}$  is a suitable and weakly homogeneous iteration. Also,  $\mathbb{P}$  is  $\Delta_2$ -definable, as “ $p \in \mathbb{P}$ ” if and only if  $V_\alpha \models$  “ $p \in \mathbb{P}$ ,” for (some) every  $\alpha \in C^{(1)}$  such that  $p \in V_\alpha$ .

Let  $K_{\mathbb{P}}$  denote the class of all Mahlo cardinals. Clearly,  $K_{\mathbb{P}}$  is  $\Delta_2$ -definable. Also, since ORD is  $\Sigma_2$ -2-Mahlo,  $K_{\mathbb{P}}$  is a  $\Sigma_2$ -stationary proper class. Let  $\kappa \in K_{\mathbb{P}}$ . Thus,  $\kappa$  is regular and, clearly,  $\mathbb{P} \cap V_\kappa = \mathbb{P}_\kappa$ .

CLAIM 8.1.1.  $\kappa$  is  $\mathbb{P}$ -reflecting.

PROOF OF CLAIM. Clearly,  $\kappa$  is inaccessible and  $\Vdash_{\mathbb{P}_\kappa}$  “ $\dot{Q}$  is  $\kappa$ -distributive,” where  $\mathbb{P} \cong \mathbb{P}_\kappa * \dot{Q}$ . Also,  $\mathbb{P}_\kappa \subseteq V_\kappa$ . Let  $\lambda < \kappa$  be forced by  $\mathbb{P}_\kappa$  to be a cardinal. For every inaccessible cardinal  $\theta < \kappa$ ,  $\theta \in C(\mathbb{P})$ , hence

$$\Vdash_{\mathbb{P}_\theta} \text{“}\dot{Q}_{[\theta, \kappa]} \text{ is } \theta\text{-distributive”}$$

and  $\mathbb{P}_\theta = \mathbb{P} \cap V_\theta$ . Since  $\kappa$  is Mahlo and  $\{\theta < \kappa \mid \mathbb{P}_\theta = \mathbb{P} \cap V_\theta\}$  is unbounded,  $\mathbb{P}_\kappa$  is  $\kappa$ -cc. Also, standard arguments show that  $\mathbb{P}_\kappa$  forces “ $|\dot{\mathcal{P}}(\lambda)| < \check{\kappa}$ .” Altogether,  $\mathbb{P}$  forces  $\kappa$  to be inaccessible. Finally, we appeal to Proposition 3.4 to get that  $\kappa$  is  $\mathbb{P}$ -reflecting, as wanted.  $\dashv$

The claim about  $C(\mathbb{P})$  containing all supercompacts is obvious.  $\dashv$

In [28, Section 5], Tsaprounis shows that  $\mathbb{P}$  preserves  $C^{(n)}$ -extendible cardinals. The following theorem gives an improvement of his result:

THEOREM 8.2. (1) *If ORD is  $\Sigma_2$ -2-Mahlo, and  $\delta$  is supercompact then*

$$\Vdash_{\mathbb{P}} \text{“}\delta \text{ is supercompact.”}$$

(2) *If for some  $n \geq 1$ ,  $\delta$  is  $C^{(n)}$ -extendible then*

$$\Vdash_{\mathbb{P}} \text{“}\delta \text{ is } C^{(n)}\text{-extendible.”}$$

*In<sup>10</sup> particular, if there is a proper class of supercompact cardinals (equivalently, if  $\text{VP}(\Pi_1)$  holds), then forcing with  $\mathbb{P}$  preserves  $C^{(n)}$ -extendible cardinals, for all  $n < \omega$ .*

<sup>10</sup>An ad hoc argument for Jensen’s iteration can be composed to show that it preserves supercompact cardinals without the assumption, used in Lemma 8.1, of ORD being  $\Sigma_2$ -2-Mahlo. Nevertheless, the

PROOF. This follows from Theorem 7.5 and Lemma 8.1. The particular case can be proved using Lemma 6.3.  $\dashv$

Recall that a class function  $E$  from the class  $\text{Reg}$  of infinite regular cardinals to the class of cardinals is an *Easton function* if it satisfies König’s theorem (i.e.,  $\text{cf}(E(\kappa)) > \kappa$ , for all  $\kappa \in \text{Reg}$ ) and is increasingly monotone. Let  $\mathbb{P}_E$  be the direct limit of the iteration  $\langle \mathbb{P}_\alpha, \dot{Q}_\alpha : \alpha \in \text{ORD} \rangle$  with Easton support where  $\mathbb{P}_0$  is the trivial forcing and for each ordinal  $\alpha$ , if  $\Vdash_{\mathbb{P}_\alpha}$  “ $\alpha$  is a regular cardinal,” then  $\Vdash_{\mathbb{P}_\alpha}$  “ $\dot{Q}_\alpha = \text{Add}(\alpha, E(\alpha))$ ,” and  $\Vdash_{\mathbb{P}_\alpha}$  “ $\dot{Q}_\alpha$  is trivial” otherwise. Standard arguments [16, Section 15] show that if the GCH holds in the ground model, then  $\mathbb{P}_E$  preserves all cardinals and cofinalities and forces that  $2^\kappa = E(\kappa)$ , for each  $\kappa \in \text{Reg}$ .

LEMMA 8.3. *Let  $E$  be a  $\Delta_2$ -definable Easton function and assume that  $\text{ORD}$  is  $\Sigma_2$ -2-Mahlo. Then  $\mathbb{P}_E$  is fitting and  $C(\mathbb{P})$  contains all supercompact cardinals, in case there are any.*

PROOF. Arguing as in Lemma 8.1,  $\mathbb{P}_E$  is a  $\Delta_2$ -definable, weakly homogeneous suitable iteration. Set  $\mathbb{P} := \mathbb{P}_E$  and let

$$K_{\mathbb{P}} := \{ \kappa \mid \kappa \text{ is Mahlo and } E[\kappa] \subseteq \kappa \}.$$

Clearly, every  $\kappa \in K_{\mathbb{P}}$  is regular and witnesses  $\mathbb{P} \cap V_\kappa = \mathbb{P}_\kappa$ . The argument for the verification that each  $\kappa \in K_{\mathbb{P}}$  is  $\mathbb{P}$ -reflecting is analogous to the one given in Lemma 8.1. The claim about  $C(\mathbb{P})$  is obvious. Also, since  $E$  is  $\Delta_2$ -definable, so is  $K_{\mathbb{P}}$ . Moreover,  $K_{\mathbb{P}}$  is  $\Sigma_2$ -stationary, for if  $\mathcal{C}$  is a  $\Sigma_2$ -definable club class of ordinals, then  $\mathcal{C}_E := \{ \alpha \in \mathcal{C} \mid E[\alpha] \subseteq \alpha \}$  is a  $\Sigma_2$ -definable proper club class, and since  $\text{ORD}$  is  $\Sigma_2$ -2-Mahlo,  $K_{\mathbb{P}} \cap \mathcal{C} \neq \emptyset$ .  $\dashv$

Similarly as in Theorem 8.2, we now obtain the following:

THEOREM 8.4. *Assume the GCH holds and let  $E$  be a  $\Delta_2$ -definable Easton function.<sup>11</sup> Then the following hold:*

- (1) *If  $\text{ORD}$  is  $\Sigma_2$ -2-Mahlo and  $\delta$  is supercompact then*

$$\Vdash_{\mathbb{P}_E} \text{“}\delta \text{ is supercompact.”}$$

- (2) *If  $n \geq 1$  and  $\delta$  is  $C^{(n)}$ -extendible then*

$$\Vdash_{\mathbb{P}_E} \text{“}\delta \text{ is } C^{(n)}\text{-extendible.”}$$

*In particular, if the GCH holds and there is a proper class of supercompact cardinals (equivalently,  $\text{VP}(\Pi_1)$  holds), then forcing with  $\mathbb{P}_E$  preserves  $C^{(n)}$ -extendible cardinals, for all  $n < \omega$ .*

COROLLARY 8.5. *Let  $E$  be a  $\Delta_2$ -definable Easton function and assume the GCH holds. Then the following are true:*

- (1) *If  $\text{VP}(\Pi_1)$  holds, then  $\Vdash_{\mathbb{P}_E}$  “ $\text{VP}(\Pi_1)$ .”*

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arguments given in the lemma also apply to other class forcing iterations considered in the subsections below.

<sup>11</sup>The GCH assumption is superfluous if we are only interested in preserving  $C^{(n)}$ -extendible cardinals,  $n \geq 1$ , for in that case we may first force the GCH while preserving  $C^{(n)}$ -extendible cardinals (cf. Theorem 8.2).

- (2) If  $\text{VP}(\Pi_{n+1})$  holds, then  $\Vdash_{\mathbb{P}_E}$  “ $\text{VP}(\Pi_{n+1})$ .” Also, if  $\text{VP}(\Pi_{n+1})$  holds then  $\Vdash_{\mathbb{P}_E}$  “ $\text{VP}(\Pi_{n+1})$ .”

The next corollary shows that VP is also consistent with any possible behaviour of the power-set function given by an arbitrary definable Easton function.

**COROLLARY 8.6.** *If VP holds, then in some class forcing extension, for every definable Easton function  $E$  there is a further class forcing extension that preserves VP and where  $2^\kappa = E(\kappa)$ , for every  $\kappa \in \text{Reg}$ .*

**PROOF.** First force with the standard Jensen’s iteration for forcing the GCH and call the resulting generic extension  $V$ . By Corollary 6.7,  $V \models \text{VP}$ . Then, given a  $V$ -definable Easton function  $E$ , force over  $V$  with  $\mathbb{P}_E$ . Once again by Corollary 6.7,  $V^{\mathbb{P}_E} \models \text{VP}$ . Finally, since  $V \models \text{GCH}$ , it follows that  $\mathbb{P}_E$  forces  $2^\kappa = E(\kappa)$ , for every  $\kappa \in \text{Reg}$ .  $\dashv$

**8.2. A remark on Woodin’s HOD Conjecture.** The HOD Dichotomy theorem of Woodin states that if there exists an extendible cardinal, then either  $V$  is close to HOD or is far from it. Specifically, if  $\kappa$  is an extendible cardinal, then either (1): for every singular cardinal  $\lambda > \delta$ ,  $\lambda$  is singular in HOD and  $(\lambda^+)^{\text{HOD}} = \lambda^+$ , or (2): every regular cardinal  $\lambda > \kappa$  is  $\omega$ -strongly measurable in HOD (see [29]). Woodin’s HOD Hypothesis asserts that there is a proper class of regular cardinals that are not  $\omega$ -strongly measurable in HOD, and therefore that the first option of the HOD Dichotomy is the true one. Woodin’s HOD Conjecture asserts that the HOD Hypothesis is provable in the theory  $\text{ZFC} +$  “There exists an extendible cardinal.” Our arguments may be used to show that if the HOD Conjecture holds, and therefore it is provable in  $\text{ZFC} +$  “There exists an extendible cardinal” that above the first extendible cardinal every singular cardinal  $\lambda$  is singular in HOD and  $(\lambda^+)^{\text{HOD}} = \lambda^+$ , there may still be no agreement at all between  $V$  and HOD about successors of regular cardinals. Moreover, many singular cardinals in HOD need not be cardinals in  $V$ . Let us give some examples.

Let  $\mathbb{P}$  be the direct limit of the iteration  $\langle \mathbb{P}_\alpha; \dot{\mathbb{Q}}_\alpha : \alpha \in \text{ORD} \rangle$  with Easton support, where  $\mathbb{P}_0$  is the trivial forcing and for each ordinal  $\alpha$ , if  $\Vdash_{\mathbb{P}_\alpha}$  “ $\alpha$  is regular” then  $\Vdash_{\mathbb{P}_\alpha}$  “ $\dot{\mathbb{Q}}_\alpha = \text{Coll}(\alpha, \alpha^+)$ ,” and  $\Vdash_{\mathbb{P}_\alpha}$  “ $\dot{\mathbb{Q}}_\alpha$  is trivial” otherwise. Arguing essentially as in Lemma 8.1 we obtain the following:

**LEMMA 8.7.** *Assume ORD is  $\Sigma_2$ -2-Mahlo. Then,  $\mathbb{P}$  is fitting and  $C(\mathbb{P})$  contains all supercompact cardinals, in case there are any.*

**THEOREM 8.8.** (1) *If ORD is  $\Sigma_2$ -2-Mahlo, and  $\delta$  is supercompact then*

$$\Vdash_{\mathbb{P}} \text{“}\delta \text{ is supercompact.”}$$

(2) *If  $n \geq 1$  and  $\delta$  is  $C^{(n)}$ -extendible, then*

$$\Vdash_{\mathbb{P}} \text{“}\delta \text{ is } C^{(n)}\text{-extendible.”}$$

*Moreover,  $\mathbb{P}$  forces “ $\forall \lambda \in \text{Reg} ((\lambda^+)^{\text{HOD}} < \lambda^+)$ .”*

**PROOF.** For the preservation of  $C^{(n)}$ -extendible cardinals we combine Lemma 8.7 and Theorem 7.5. To prove the claim about successors of regular cardinals, note that if  $\lambda$  is a regular cardinal in  $V^{\mathbb{P}}$ , then it was also a regular cardinal at



stage  $\lambda$  of the iteration, hence its successor was collapsed at stage  $\lambda + 1$ . Thus, on the one hand,  $(\lambda^+)^V < (\lambda^+)^{V^{\mathbb{P}}}$ . On the other hand,  $\mathbb{P}$  is weakly homogeneous and ordinal definable, hence  $\text{HOD}^{V^{\mathbb{P}}} \subseteq \text{HOD}^V$  (see, e.g., [16] for details). Hence, in  $V^{\mathbb{P}}$ ,  $(\lambda^+)^{\text{HOD}} < \lambda^+$ , as wanted.  $\dashv$

**COROLLARY 8.9.** *Forcing with  $\mathbb{P}$  preserves VP and forces  $(\lambda^+)^{\text{HOD}} < \lambda^+$  for every regular cardinal  $\lambda$ .*

Theorem 8.8 yields the parallel of the main theorem from [11], at the level of  $C^{(n)}$ -extendible cardinals.

Suppose now that  $K$  is a function on the class of infinite cardinals such that  $K(\lambda) > \lambda$ , and  $K$  is increasingly monotone, for every  $\lambda$ . Let  $\mathbb{P}_K$  be the direct limit of an iteration  $\langle \mathbb{P}_\alpha; \dot{Q}_\alpha : \alpha \in \text{ORD} \rangle$  with Easton support, where  $\mathbb{P}_0$  is the trivial forcing and for each ordinal  $\alpha$ , if  $\Vdash_{\mathbb{P}_\alpha}$  “ $\alpha$  is regular” then  $\Vdash_{\mathbb{P}_\alpha}$  “ $\dot{Q}_\alpha = \text{Coll}(\alpha, K(\alpha))$ ,” and  $\Vdash_{\mathbb{P}_\alpha}$  “ $\dot{Q}_\alpha$  is trivial” otherwise. Notice that for each  $\alpha$  such that  $\Vdash_{\mathbb{P}_\alpha}$  “ $\alpha$  is regular,” the remaining part of the iteration after stage  $\alpha$  is  $\alpha$ -closed, hence it preserves  $\alpha$ . Also note that if  $K$  is  $\Delta_m$ -definable ( $m \geq 1$ ), then  $\mathbb{P}_K$  is also  $\Delta_m$ -definable. Clearly,  $\mathbb{P}_K$  is suitable and weakly homogeneous.

**LEMMA 8.10.** *Assume ORD is  $\Sigma_2$ -2-Mahlo. Let  $K$  be a  $\Delta_2$ -definable class function as above. Then,  $\mathbb{P}_K$  is a fitting iteration and  $C(\mathbb{P})$  contains all supercompact cardinals, in case there are any.*

**PROOF.** Let  $K_{\mathbb{P}} := \{\kappa \mid \kappa \text{ is Mahlo and } K[\kappa] \subseteq \kappa\}$ . The proof that  $K_{\mathbb{P}}$  is a witness for the fittingness of  $\mathbb{P}$  is similar to those given in Lemmas 8.3 and 8.7. The claim about  $C(\mathbb{P})$  is obvious.  $\dashv$

**THEOREM 8.11.** *Let  $K$  be a  $\Delta_2$ -definable class function as above.*

(1) *If ORD is  $\Sigma_2$ -2-Mahlo, and  $\delta$  is supercompact then*

$$\Vdash_{\mathbb{P}_K} \text{“}\delta \text{ is supercompact.”}$$

(2) *If  $\delta$  is  $C^{(n)}$ -extendible, for some  $n \geq 1$ , then*

$$\Vdash_{\mathbb{P}_K} \text{“}\delta \text{ is } C^{(n)}\text{-extendible.”}$$

Moreover,  $\mathbb{P}_K$  forces

$$(\lambda^+)^{\text{HOD}} \leq K(\lambda) < \lambda^+$$

for all infinite regular cardinals  $\lambda$ .

**PROOF.** The preservation of  $C^{(n)}$ -extendible cardinals,  $n < \omega$ , follows from Lemma 8.10 and Theorem 7.5.

If  $G$  is  $\mathbb{P}_K$ -generic over  $V$  and  $\lambda$  is regular in  $V[G]$ , then it is also regular at the  $\lambda$ -stage of the iteration. Hence,  $\mathbb{Q}_\lambda = \text{Coll}(\lambda, K(\lambda))$ , and therefore  $K(\lambda) < \lambda^+$  holds in  $V[G]$ . The other inequality (i.e.,  $(\lambda^+)^{\text{HOD}} \leq K(\lambda)$ ) follows from the fact that  $\mathbb{P}_K$  is weakly homogeneous and ordinal definable, and thus that  $\text{HOD}^{V[G]} \subseteq \text{HOD}^V$ .  $\dashv$

The theorem above implies that many kinds of disagreement between successors of regulars in HOD and in  $V$  may be forced while preserving  $C^{(n)}$ -extendible cardinals. It also implies that one can destroy many singular cardinals in HOD while preserving  $C^{(n)}$ -extendible cardinals. For example, let  $K$  be such that  $K(\lambda)$  is the least singular

cardinal in HOD greater than  $\lambda$ , i.e.,  $K(\lambda) = (\lambda^{+\omega})^{\text{HOD}}$ . It is easily seen that  $K$ , and therefore also  $\mathbb{P}_K$  as defined above, are  $\Delta_2$ -definable. Then we have the following.

**COROLLARY 8.12.** *For each  $n \geq 1$ ,  $\mathbb{P}_K$  preserves  $C^{(n)}$ -extendible cardinals and forces “ $\forall \lambda \in \text{Reg}((\lambda^{+\omega})^{\text{HOD}} < \lambda^+)$ .”*

**8.3. On diamonds.** Other combinatorial statements that we can force while preserving  $C^{(n)}$ -extendible cardinals are the diamond principles  $\diamond_S$ . Namely, given a stationary set  $S \subseteq \kappa$ , a sequence  $\langle A_\alpha : \alpha \in S \rangle$  is a  $\diamond_S$ -sequence if  $A_\alpha \subseteq \alpha$  and for every  $A \subseteq \kappa$  the set  $\{\alpha \in S : A \cap \alpha = A_\alpha\}$  is stationary. We say that  $\diamond_S$  holds if there is a  $\diamond_S$ -sequence.

It is well-known that  $\text{Add}(\kappa^+, 1)$  automatically forces  $\diamond_S$ , for every stationary  $S \subseteq \kappa^+$  in  $V^{\text{Add}(\kappa^+, 1)}$ . Thus, from Theorem 8.2, we obtain:

- COROLLARY 8.13.** (1) *If ORD is  $\Sigma_2$ -2-Mahlo and  $\delta$  is a supercompact cardinal, then there is a generic extension where  $\delta$  is still supercompact and  $\diamond_S$  holds for every cardinal  $\kappa$  and every stationary  $S \subseteq \kappa^+$ .*  
 (2) *If  $n \geq 1$  and  $\delta$  is a  $C^{(n)}$ -extendible cardinal, there is a generic extension where  $\delta$  is  $C^{(n)}$ -extendible and  $\diamond_S$  holds for every cardinal  $\kappa$  and every stationary  $S \subseteq \kappa^+$ .*

Hence, if VP holds in  $V$ , there is a generic extension where VP holds together with  $\diamond_S$ , for every  $\kappa$  and every stationary  $S \subseteq \kappa^+$ .

Another relevant diamond principle is the so-called  $\diamond_{\kappa^+}^+$ -principle.

A sequence  $\langle \mathcal{A}_\alpha : \alpha \in \kappa^+ \rangle$  is a  $\diamond_{\kappa^+}^+$ -sequence if  $\mathcal{A}_\alpha \in [\mathcal{P}(\alpha)]^{\leq \kappa}$  and for every  $A \subseteq \kappa^+$  there is a club  $C \subseteq \kappa^+$  such that

$$C \subseteq \{\alpha \in \kappa^+ \mid A \cap \alpha \in \mathcal{A}_\alpha \wedge C \cap \alpha \in \mathcal{A}_\alpha\}.$$

We say that  $\diamond_{\kappa^+}^+$  holds if there is a  $\diamond_{\kappa^+}^+$ -sequence.

In [10, Theorem 12.2] it is shown that, assuming  $2^\kappa = \kappa^+$  and  $2^{\kappa^+} = \kappa^{++}$ , there is a  $\kappa^+$ -closed and  $\kappa^{++}$ -cc forcing notion that forces  $\diamond_{\kappa^+}^+$ . The forcing is an iteration  $\mathbb{D}_{\kappa^{++}}^+ = \langle \mathbb{P}_\alpha, \dot{Q}_\beta : \beta < \alpha \leq \kappa^{++} \rangle$  with supports of size  $\leq \kappa$ , where  $\mathbb{P}_0$  is the natural forcing notion that introduces a sequence  $\vec{A}$  of the right form whereas the rest of the iterates forces the club sets  $C \subseteq \kappa^+$  witnessing that  $\vec{A}$  is a  $\diamond_{\kappa^+}^+$ -sequence.

Arguing as in Lemma 8.1 one has the following:

**LEMMA 8.14.** *Assume the GCH holds and that ORD is  $\Sigma_2$ -2-Mahlo. Let  $\mathbb{D}$  be the standard Easton support iteration of the forcings  $\mathbb{D}_{\kappa^{++}}^+$ , for  $\kappa$  a cardinal. Then  $\mathbb{D}$  is fitting and  $C(\mathbb{P})$  contains all supercompact cardinals, in case there are any.*

**THEOREM 8.15.** *Assume the GCH holds.*

- (1) *If ORD is  $\Sigma_2$ -2-Mahlo and  $\delta$  is a supercompact cardinal, then forcing with  $\mathbb{D}$  preserves the supercompactness of  $\delta$  and yields a generic extension where  $\diamond_{\kappa^+}^+$  holds, for every cardinal  $\kappa$ .*  
 (2) *If  $n \geq 1$  and  $\delta$  is a  $C^{(n)}$ -extendible cardinal, then forcing with  $\mathbb{D}$  preserves the  $C^{(n)}$ -extendibility of  $\delta$  and yields a generic extension where  $\diamond_{\kappa^+}^+$  holds, for every cardinal  $\kappa$ .*

Hence, if VP and the GCH hold in  $V$ , then VP also holds in  $V^{\mathbb{D}}$ , together with  $\diamond_{\kappa^+}^+$ , for every cardinal  $\kappa$ .

The claim of the theorem above referring to VP was first proved by Brooke-Taylor in [8, Corollary 26]

**8.4. On weak square sequences.** A classical result due to Solovay is that Jensen’s square principle  $\square_\lambda$  must fail for every cardinal  $\lambda$  greater than or equal to the first strongly compact cardinal [25]. This result was subsequently refined by Jensen, who proved that  $\square_\lambda$  fails for any subcompact cardinal  $\lambda$ , a much weaker notion than supercompactness. Further sharper results are due to Brooke-Taylor and Friedman [9] and to Bagaria and Magidor [5]. In our context, namely with the existence of extendible cardinals, and therefore with the failure of  $\square_\lambda$  for a tail of  $\lambda$ ’s, we shall consider  $\square_{\lambda,\mu}$ -principles, a weak form of the square principle introduced by Schimmerling in [24].

Following up on Solovay’s work, Shelah proved that if  $\kappa$  is supercompact and  $\text{cof}(\lambda) < \kappa < \lambda$ , then  $\square_{\lambda,\lambda}$  (also known as  $\square_\lambda^*$ ) fails [10, Section 2]. Also, Burke and Kanamori showed that if  $\kappa$  is  $\lambda^+$ -strongly compact, then  $\square_{\lambda, < \text{cof}(\lambda)}$  fails [10].<sup>12</sup> The remaining cases, namely  $\square_{\lambda,\mu}$  with  $\kappa \leq \text{cof}(\lambda) \leq \mu \leq \lambda$  turned out to be consistent with the existence of a supercompact cardinal  $\kappa$  [10, Section 9]. Specifically, in [10, Theorem 9.2], the authors define, for each  $\kappa \leq \text{cof}(\lambda) < \lambda$ , a forcing notion  $\mathbb{S}_\lambda$  which forces  $\square_{\lambda, \text{cof}(\lambda)}$  and is  $\text{cof}(\lambda)$ -directed closed and  $\lambda$ -strategically closed. Thus, if  $\kappa$  is Laver-indestructible, then forcing with  $\mathbb{S}_\lambda$  preserves the supercompactness of  $\kappa$ . In this section we shall extend this result to  $C^{(n)}$ -extendible cardinals.

Let  $K$  be the class function with  $\text{dom}(K) = \text{Card} \setminus \aleph_1$ , such that  $K(\lambda) :=$  “The first singular cardinal of cofinality  $\lambda^+$ .” Observe that  $K$  is  $\Delta_2$ -definable. Now let  $\mathbb{P}_K$  be denote the direct limit of  $\langle \mathbb{P}_\alpha; \dot{Q}_\alpha : \alpha \in \text{ORD} \rangle$ , the iteration with Easton support where  $\mathbb{P}_0$  is the trivial forcing and for every  $\alpha$ , if  $\Vdash_{\mathbb{P}_\alpha} “\alpha \in \text{Card} \setminus \aleph_1,”$  then  $\Vdash_{\mathbb{P}_\alpha} “\dot{Q}_\alpha = \dot{S}_{K(\alpha)},”$  and  $\Vdash_{\mathbb{P}_\alpha} “\dot{Q}_\alpha = \{1\},”$  otherwise.

REMARK 8.16. For every  $\alpha$ , if  $\Vdash_{\mathbb{P}_\alpha} “\alpha \in \text{Card} \setminus \aleph_1,”$  then  $\mathbb{P}_{[\alpha, \text{ORD}]}$  is  $\alpha^+$ -directed closed and  $K(\alpha)$ -strategically closed.

We now prove that  $\mathbb{P}_K$  forces class many instances of  $\square_{K(\lambda), \lambda^+}$ .

LEMMA 8.17.  $\Vdash_{\mathbb{P}} “\forall \lambda \geq \aleph_1 (\square_{K(\lambda), \lambda^+} \text{ holds}).”$

PROOF. Suppose  $\lambda$  is an uncountable cardinal in  $V^{\mathbb{P}}$ . Then, by the above remark,  $\lambda$  is also an uncountable cardinal in  $V^{\mathbb{P}^\lambda}$ , hence  $\Vdash_{\mathbb{P}^\lambda} “\dot{Q}_\lambda = \dot{S}_{K(\lambda)}.”$  Thus,  $\square_{K(\lambda), \lambda^+}$  holds in  $V^{\mathbb{P}^{\lambda+1}}$ . Again, by the above remark,  $\mathbb{P}_{[\lambda+1, \text{ORD}]}$  is distributive enough to preserve  $\square_{K(\lambda), \lambda^+}$ . ⊣

Arguing as in Lemma 8.3 we can show that  $\mathbb{P}_K$  is a fitting iteration. Precisely, we have the following:

LEMMA 8.18. *Assume ORD is  $\Sigma_2$ -2-Mahlo. Then  $\mathbb{P}_K$  is fitting and  $C(\mathbb{P})$  contains all supercompact cardinals, in case there are any.*

THEOREM 8.19. (1) *If ORD is  $\Sigma_2$ -2-Mahlo, then  $\mathbb{P}_K$  preserves supercompact cardinals.*

(2) *For each  $n \geq 1$ , forcing with  $\mathbb{P}_K$  preserves  $C^{(n)}$ -extendible cardinals.*

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<sup>12</sup>Note that this is interesting only for  $\text{cof}(\lambda) \geq \kappa$ .

Moreover, in any generic extension by  $\mathbb{P}_K$  the following holds: for every uncountable cardinal  $\lambda$ , if  $K(\lambda)$  is the first singular cardinal of cofinality  $\lambda^+$ , then  $\square_{K(\lambda), \lambda^+}$  holds.

**COROLLARY 8.20.** *If VP holds, then there is a class forcing iteration that preserves VP and forces  $\square_{\lambda, \text{cof}(\lambda)}$ , for a proper class of singular cardinals  $\lambda$ .*

**§9. General class forcing iterations.** In this section we follow up the discussion at the end of Section 6 about nonweakly homogeneous suitable iterations. One prominent example is the iteration  $\mathbb{P}$  that forces  $V = \text{HOD}$  by coding the universe into the power-set function pattern. This iteration is suitable but not weakly homogenous. One may also want to consider class forcing iterations  $\mathbb{P}$  over some model  $M$  such that  $\mathbb{P}$  is not definable in  $M$ . To deal with such general class forcing notions we shall work within the theory  $\text{ZFC}_{\mathbb{P}}$ , namely ZFC with the axiom schemata of Separation and Replacement allowing for formulas in the language of set theory with the additional predicate symbol  $\mathbb{P}$ .

**DEFINITION 9.1** ( $\mathbb{P}$ -  $C^{(n)}$ -extendible cardinal). For  $n \geq 1$ , we say that a cardinal  $\delta$  is  $\mathbb{P}$ -  $C^{(n)}$ -extendible if for every cardinal  $\lambda \in C_{\mathbb{P}}^{(n)}$ ,  $\lambda > \delta$ , there is an ordinal  $\theta$  and an elementary embedding

$$j : \langle V_\lambda, \in, \mathbb{P} \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, \mathbb{P} \cap V_\theta \rangle$$

with  $\text{crit}(j) = \delta$ ,  $j(\delta) > \lambda$ , and  $j(\delta) \in C^{(n)}$ . If, moreover, we can pick  $\theta \in C_{\mathbb{P}}^{(n)}$ , then we say that  $\delta$  is  $\mathbb{P}$ -  $C^{(n)+}$ -extendible.

Similarly, we may also consider the notion of  $\mathbb{P}$ -  $\Sigma_n$ -supercompactness, for a general class  $\mathbb{P}$  which is not necessarily definable.

**DEFINITION 9.2** ( $\mathbb{P}$ -  $\Sigma_n$ -supercompactness). If  $n \geq 1$ , then a cardinal  $\delta$  is  $\mathbb{P}$ -  $\Sigma_n$ -supercompact if for every  $\lambda \in C_{\mathbb{P}}^{(n)}$  greater than  $\delta$ , and every  $a \in V_\lambda$  there exist  $\bar{\delta} < \bar{\lambda} < \delta$  and  $\bar{a} \in V_{\bar{\lambda}}$ , and there exists an elementary embedding  $j : V_{\bar{\lambda}} \rightarrow V_\lambda$  such that:

- $\text{crit}(j) = \bar{\delta}$  and  $j(\bar{\delta}) = \delta$ .
- $j(\bar{a}) = a$ .
- $\bar{\lambda} \in C_{\mathbb{P}}^{(n)}$ .

The same arguments as in the proof of Theorem 2.4 yield the following equivalence.

**THEOREM 9.3.** *For every  $n \geq 1$ , every class  $\mathbb{P}$ , and every cardinal  $\kappa$ , the following are equivalent:*

- (1)  $\kappa$  is  $\mathbb{P}$ -  $C^{(n)}$ -extendible.
- (2)  $\kappa$  is  $\mathbb{P}$ -  $\Sigma_{n+1}$ -supercompact.
- (3)  $\kappa$  is  $\mathbb{P}$ -  $C^{(n)+}$ -extendible.<sup>13</sup>

We will say that a cardinal  $\delta$  is  $\mathbb{P}$ -  $C^{(n)}$ -extendible with  $\mathbb{P}$ -  $\Sigma_n$ -reflecting target if in Definition 9.1 we may moreover require that  $j(\delta)$  is  $\mathbb{P}$ -  $\Sigma_n$ -reflecting (cf. Definition

<sup>13</sup>See Remark 2.5.

3.15). Likewise, one defines the notion of  $\mathbb{P}$ - $C^{(n)+}$ -extendible with  $\mathbb{P}$ - $\Sigma_n$ -reflecting target. Arguing as usual one can check that both notions are equivalent.

Clearly, any  $\mathbb{P}$ - $C^{(n)+}$ -extendible with  $\mathbb{P}$ - $\Sigma_n$ -reflecting target is  $C^{(n)}$ -extendible. Moreover, if the predicate  $\mathbb{P}$  is definable with low complexity and satisfies some minor requirements, then, as the next proposition shows, a cardinal is  $\mathbb{P}$ - $C^{(n)+}$ -extendible with  $\mathbb{P}$ - $\Sigma_n$ -reflecting target if and only if is  $C^{(n)}$ -extendible.

**PROPOSITION 9.4.** *Let  $n, m \geq 1$  with  $m \leq n$ . Let  $\mathbb{P}$  be a  $\Delta_{m+1}$ -definable ORD-length forcing iteration and assume that*

$$\{\kappa \mid \mathbb{P} \cap V_\kappa = \mathbb{P}_\kappa \wedge \kappa \text{ is } \mathbb{P}\text{-reflecting}\}$$

*is a proper class containing all  $C^{(n-2)}$ -extendible cardinals.<sup>14</sup> Then, every  $C^{(n)}$ -extendible cardinal belonging to  $C^{(m+n+1)}$  is  $\mathbb{P}$ - $C^{(n)+}$ -extendible with  $\mathbb{P}$ - $\Sigma_n$ -reflecting target.*

*In particular, if  $\mathbb{P}$  is  $\Delta_2$ -definable, a cardinal is  $C^{(n)}$ -extendible if and only if it is  $\mathbb{P}$ - $C^{(n)+}$ -extendible with  $\mathbb{P}$ - $\Sigma_n$ -reflecting target.*

**PROOF.** Let  $\delta$  be a  $C^{(n)}$ -extendible cardinal in  $C^{(m+n+1)}$ .

**CLAIM 9.4.1.**  $\delta$  is  $\mathbb{P}$ - $C^{(n)}$ -extendible.

**PROOF OF CLAIM.** Let  $\mu > \lambda > \delta$  be with  $\lambda \in C_{\mathbb{P}}^{(n)}$  and  $\mu \in C^{(n)}$ . Since  $\delta$  is actually  $C^{(n)+}$ -extendible (cf. Remark 2.5) we may pick  $\eta \in C^{(n)}$  together with an elementary embedding  $j: V_\mu \rightarrow V_\eta$  with  $\text{crit}(j) = \delta$ ,  $j(\delta) > \mu$ , and  $j(\delta) \in C^{(n)}$ . Observe that  $V_{j(\lambda)} \prec_{\Sigma_n} V_\eta \prec_{\Sigma_n} V$ , hence

$$j(\langle V_\lambda, \in, \mathbb{P}^{V_\lambda} \rangle) = \langle V_{j(\lambda)}, \in, \mathbb{P}^{V_{j(\lambda)}} \rangle = \langle V_{j(\lambda)}, \in, \mathbb{P} \cap V_{j(\lambda)} \rangle,$$

where the right-most equality follows from  $\Delta_{m+1}$ -definability of  $\mathbb{P}$  and  $m \leq n$ . Altogether,  $j \upharpoonright \langle V_\lambda, \in, \mathbb{P} \cap V_\lambda \rangle$  yields the desired embedding.  $\dashv$

By Theorem 9.3,  $\delta$  is actually  $\mathbb{P}$ - $C^{(n)+}$ -extendible, so we concentrate on proving the other assertion. Let  $\lambda \in C_{\mathbb{P}}^{(n)}$  with  $\lambda > \delta$ , together with an elementary embedding

$$j: \langle V_\lambda, \in, \mathbb{P} \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, \mathbb{P} \cap V_\theta \rangle,$$

with  $\text{crit}(j) = \delta$ ,  $\theta \in C_{\mathbb{P}}^{(n)}$ ,  $j(\delta) > \lambda$ , and  $j(\delta) \in C^{(n)}$ . Note that  $j(\delta)$  is  $C^{(n-2)}$ -extendible hence  $\mathbb{P}$ -reflecting and a witness for  $\mathbb{P} \cap V_{j(\delta)} = \mathbb{P}^{V_{j(\delta)}}$ .

**CLAIM 9.4.2.**  $j(\delta) \in C_{\mathbb{P}}^{(n)}$ .

**PROOF OF CLAIM.** Since  $\mathbb{P}$  is  $\Delta_{m+1}$ -definable,  $C_{\mathbb{P}}^{(n)}$  is a  $\Pi_{n+m}$ -definable club class (cf. Proposition 3.9). In particular, as  $\delta \in C^{(n+m+1)}$ ,  $\delta$  is an accumulation point of  $C_{\mathbb{P}}^{(n)}$  and thus a member of  $C_{\mathbb{P}}^{(n)}$ .

Since  $\mathbb{P}$  is  $\Delta_{m+1}$ ,  $m \leq n$  and  $\delta, \lambda \in C_{\mathbb{P}}^{(n)}$ ,  $\langle V_\delta, \in, \mathbb{P}^{V_\delta} \rangle \prec_{\Sigma_n} \langle V_\lambda, \in, \mathbb{P}^{V_\lambda} \rangle$ . By elementarity,  $\langle V_{j(\delta)}, \in, \mathbb{P}^{V_{j(\delta)}} \rangle \prec_{\Sigma_n} \langle V_\theta, \in, \mathbb{P}^{V_\theta} \rangle$  and so, since  $\theta \in C_{\mathbb{P}}^{(n)}$ ,  $\langle V_{j(\delta)}, \in, \mathbb{P}^{V_{j(\delta)}} \rangle \prec_{\Sigma_n} \langle V_\delta, \in, \mathbb{P} \rangle$ . Once again, since  $\mathbb{P}$  is  $\Delta_{m+1}$ -definable,  $m \leq n$  and  $j(\delta) \in C^{(n)}$ ,  $\mathbb{P}^{V_{j(\delta)}} = \mathbb{P} \cap V_{j(\delta)}$ , hence  $j(\delta) \in C_{\mathbb{P}}^{(n)}$ .  $\dashv$

<sup>14</sup>By convention, a  $C^{(-1)}$ -extendible cardinal is a Mahlo cardinal.

The above claims combined give the proof of the proposition. ⊣

The main result of the section is the following:

**THEOREM 9.5.** *Let  $\mathbb{P}$  be a (not necessarily definable) suitable iteration. Assume that there is a proper class of  $\mathbb{P}$ -reflecting cardinals  $\lambda$  such that  $\mathbb{P}_\lambda = \mathbb{P} \cap V_\lambda$ .<sup>15</sup> For each  $n \geq 1$ , if  $\delta$  is  $\mathbb{P}$ - $C^{(n)}$ -extendible with  $\mathbb{P}$ - $\Sigma_n$ -reflecting target then*

$$\Vdash_{\mathbb{P}} \text{“}\delta \text{ is } C^{(n)\text{-extendible.”}$$

Actually for  $n = 1$  the above is true by just assuming that  $\delta$  is  $\mathbb{P}$ - $C^{(1)}$ -extendible.

**PROOF.** Let  $\lambda > \delta$  be  $\mathbb{P}$ -reflecting and such that  $\mathbb{P}_\lambda = \mathbb{P} \cap V_\lambda$ . It will be sufficient to prove that if  $G_\lambda$  is  $\mathbb{P}_\lambda$ -generic over  $V$ , then in the generic extension  $V[G_\lambda]$ , the set  $D$  of conditions  $r \in \mathbb{P}_{[\lambda, Ord]}$  that force the existence of an elementary embedding

$$j : V[G_\lambda][\dot{G}_{[\lambda, Ord]}]_\lambda \rightarrow V[G_\lambda][\dot{G}_{[\lambda, Ord]}]_\theta,$$

some  $\theta$ , with  $\text{crit}(j) = \delta$ ,  $j(\delta) > \lambda$ , and  $j(\delta) \in C^{(n)}$ , is dense in  $\mathbb{P}_{[\lambda, Ord]}$ .

In  $V[G_\lambda]$ , let  $r$  be a condition in  $\mathbb{P}_{[\lambda, Ord]}$ . Back in  $V$ , let  $\mu \in C_{\mathbb{P}}^{(n)}$  be greater than  $\lambda$  and such that

$$\Vdash_{\mathbb{P}_\mu} \text{“}\mathbb{P}_{[\mu, Ord]} \text{ is } |\mathbb{P}_\lambda|^+\text{-directed closed.”}$$

Since  $\delta$  is  $\mathbb{P}$ - $C^{(n)+}$ -extendible with  $\mathbb{P}$ - $\Sigma_n$ -reflecting target, in the ground model  $V$  there exists an elementary embedding

$$j : \langle V_\mu, \in, \mathbb{P} \cap V_\mu \rangle \rightarrow \langle V_\theta, \in, \mathbb{P} \cap V_\theta \rangle$$

with  $\text{crit}(j) = \delta$  such that  $j(\delta) > \mu$ ,  $\theta \in C_{\mathbb{P}}^{(n)}$ , and  $j(\delta)$  being  $\mathbb{P}$ - $\Sigma_n$ -reflecting.

For each  $q \in \mathbb{P}_\lambda$  there is an ordinal  $\alpha < \delta$  such that  $\text{supp}(q) \cap \delta \subseteq \alpha$ . Hence,  $\text{supp}(j(q)) \cap j(\delta) \subseteq \alpha$ , and so  $j(q)$  is a  $\mathbb{P}_{j(\lambda)}$ -condition such that

$$j(q)(\beta) = \begin{cases} q(\beta) & \text{if } \beta < \alpha, \\ \mathbb{1} & \text{if } \beta \in [\alpha, j(\delta)). \end{cases}$$

Since  $\mu < j(\delta)$  we have that  $\text{supp}(j(q)) \cap [\lambda, \mu] = \emptyset$ . So, by our choice of the ordinal  $\mu$ , in  $V[G_\lambda]$  we can take  $r^* \in \mathbb{P}_{[\mu, Ord]}$  such that

$$\Vdash_{\mathbb{P}_{[\lambda, \mu]}} \text{“}r^* \leq j(q) \upharpoonright [\mu, j(\lambda)]\text{”}$$

for all  $q \in G_\lambda$ . Then, the condition  $r \wedge r^*$  such that

$$(r \wedge r^*)(\beta) := \begin{cases} r(\beta) & \text{if } \beta \in [\lambda, \mu), \\ r^*(\beta) & \text{if } \beta \in [\mu, j(\lambda)) \end{cases}$$

is well-defined and works as a master condition for  $j$  and the forcing  $\mathbb{P}_{j(\lambda)}/G_\lambda$ , because

$$r \wedge r^* \Vdash_{\mathbb{P}_{j(\lambda)}/G_\lambda} j \text{“}G_\lambda \subseteq \dot{G}_{j(\lambda)}\text{”}$$

<sup>15</sup>In particular,  $\mathbb{P}$  is adequate (cf. Definition 3.10).

Thus, for any  $\mathbb{P}_{j(\lambda)}$ -generic filter  $G_{j(\lambda)}$  over  $V$  extending  $G_\lambda$  and containing  $r \wedge r^*$ , the elementary embedding

$$j \upharpoonright V_\lambda : \langle V_\lambda, \in, \mathbb{P} \cap V_\lambda \rangle \rightarrow \langle V_{j(\lambda)}, \in, \mathbb{P} \cap V_{j(\lambda)} \rangle$$

lifts to an elementary embedding

$$j^* : \langle V_\lambda[G_\lambda], \in, \mathbb{P} \cap V_\lambda[G_\lambda] \rangle \rightarrow \langle V_{j(\lambda)}[G_{j(\lambda)}], \in, \mathbb{P} \cap V_{j(\lambda)}[G_{j(\lambda)}] \rangle.$$

Now, since  $\lambda$  is  $\mathbb{P}$ -reflecting,  $\mathbb{P}$  forces that  $V_\lambda[\dot{G}_\lambda] = V[\dot{G}]_\lambda$ . Hence, by the choice of  $\mu$ , the same is forced by  $\mathbb{P}_\mu$ . By the elementarity of  $j$ , the structure  $\langle V_\theta, \in, \mathbb{P} \cap V_\theta \rangle$  thinks that the forcing  $\mathbb{P} \cap V_\theta$  forces  $V_{j(\lambda)}[\dot{G}_{j(\lambda)}] = V[\dot{G}]_{j(\lambda)}$ . So, since  $\theta \in C_{\mathbb{P}}^{(n)}$ ,  $\mathbb{P}$  forces the same. Also, since  $j(\delta)$  is  $\mathbb{P}$ - $\Sigma_n$ -reflecting in  $V$  and  $V_{j(\delta)} \models \text{ZFC}$ , Lemma 3.16 yields

$$\Vdash_{\mathbb{P}} \text{“} j^*(\delta) \in \dot{C}^{(n)} \text{.”}$$

We have thus found a condition below  $r$ , namely  $r \wedge r^*$ , forcing the existence of an elementary embedding

$$j^* : \langle V[\dot{G}]_\lambda, \in, \mathbb{P} \cap V[\dot{G}]_\lambda \rangle \rightarrow \langle V[\dot{G}]_{j(\lambda)}, \in, \mathbb{P} \cap V[\dot{G}]_{j(\lambda)} \rangle$$

with  $\text{crit}(j^*) = \delta$ ,  $j^*(\delta) > \lambda$ , and  $j^*(\delta) \in C^{(n)}$ , as wanted. ←

**9.1. VP and non-homogeneous suitable iterations.** We now use Theorem 9.5 to prove Theorem 6.6 without the homogeneity assumption on  $\mathbb{P}$ , hence yielding the desired refinement of Brooke-Taylor’s main theorem of [8].

**THEOREM 9.6.** *Let  $n, m \geq 1$  with  $m \leq n$ . Let  $\mathbb{P}$  be a  $\Delta_{m+1}$ -definable suitable iteration. If  $\text{VP}(\mathbf{\Pi}_{m+n+1})$  holds then  $V^{\mathbb{P}} \models \text{VP}(\mathbf{\Pi}_{n+1})$ .*

*In particular, if VP holds and  $\mathbb{P}$  is a definable suitable iteration, then*

$$V^{\mathbb{P}} \models \text{VP}.$$

**PROOF.** By Theorem 6.2,  $\text{VP}(\mathbf{\Pi}_{m+n+1})$  yields the existence of a proper class of  $C^{(m+n)}$ -extendible cardinals. Also, by Theorem 6.5, it entails the existence of a proper class of  $\mathbb{P}$ - $\Sigma_{n+1}$ -reflecting cardinals, hence a proper class of  $\mathbb{P}$ -reflecting cardinals  $\lambda$  such that  $\mathbb{P}_\lambda = \mathbb{P} \cap V_\lambda$ . In particular, the assumptions of Theorem 9.6 are met.

**CLAIM 9.6.1.** For each  $n \geq 2$ , every  $C^{(m+n)}$ -extendible is  $\mathbb{P}$ - $C^{(n)}$ -extendible with  $\mathbb{P}$ - $\Sigma_n$ -reflecting target.

**PROOF OF CLAIM.** If  $\delta$  is  $C^{(m+n)}$ -extendible then it is  $C^{(n)}$ -extendible and  $\Sigma_{m+n+2}$ -correct. Thus, Claim 9.4.2 implies that  $\delta$  is  $\mathbb{P}$ - $C^{(n)}$ -extendible.

Let  $\lambda \in C_{\mathbb{P}}^{(n)}$  be with  $\lambda > \theta$ . By  $\mathbb{P}$ - $C^{(n)+}$ -extendibility of  $\delta$ , there is  $\theta \in C_{\mathbb{P}}^{(n)}$  and an elementary embedding

$$j : \langle V_\lambda, \in, \mathbb{P} \cap V_\lambda \rangle \rightarrow \langle V_\theta, \in, \mathbb{P} \cap V_\theta \rangle,$$

with  $\text{crit}(j) = \delta$ ,  $j(\delta) > \lambda$ , and  $j(\delta) \in C^{(n)}$ .

**SUBCLAIM 9.6.1.1.**  $j(\delta)$  is  $\mathbb{P}$ - $\Sigma_n$ -reflecting.

**PROOF OF SUBCLAIM PROOF OF SUBCLAIM.** Since  $\delta$  is  $C^{(n+m)}$ -extendible, hence  $C^{(n)}$ -extendible and  $\delta \in C^{(n+m+1)}$ , Claim 9.4.2 yields  $j(\delta) \in C_{\mathbb{P}}^{(n)}$ . Let us now check that  $\mathbb{P}_{j(\delta)} = \mathbb{P} \cap V_{j(\delta)}$  and that  $j(\delta)$  is  $\mathbb{P}$ -reflecting.

For each  $\alpha < j(\delta)$  the formula

$$\exists \beta \exists X (\beta > \alpha \wedge X = V_\beta \wedge \mathbb{P}_\alpha \subseteq V_\beta)$$

is  $\Sigma_{m+1}$ , with parameter  $\alpha$ . Since  $j(\delta) \in C^{(n+m)}$ , for each  $\alpha < j(\delta)$  there is  $\alpha < \beta < j(\delta)$  witnessing the above. Combining this with the  $\Delta_{m+1}$ -definability of  $\mathbb{P}$  and with  $1 \leq m \leq n$  it follows that

$$\mathbb{P}_{j(\delta)} = \mathbb{P} \cap V_{j(\delta)} = \mathbb{P}^{V_{j(\delta)}}.$$

Let us now prove that  $j(\delta)$  is  $\mathbb{P}$ -reflecting by showing that all the assumptions of Proposition 3.4 are met. First,  $j(\delta)$  is inaccessible (actually, Mahlo) and by the above displayed expression  $\mathbb{P}$  is a forcing iteration such that  $\mathbb{P}_{j(\delta)} \subseteq V_{j(\delta)}$ . Second, thanks to the Mahloness of  $j(\delta)$ , the iteration  $\mathbb{P}_{j(\delta)}$  is  $j(\delta)$ -cc and so it is not hard to show that  $\mathbb{P}_{j(\delta)}$  preserves the inaccessibility of  $j(\delta)$ . Finally, we claim that  $j(\delta) \in C(\mathbb{P})$ , and so that  $\Vdash_{\mathbb{P}_{j(\delta)}} \text{“}\mathbb{P}_{[j(\delta), \text{ORD}]}$  is  $j(\delta)$ -distributive.”

Indeed, arguing as in the proof of Theorem 5.4 and using that the cardinal  $j(\delta)$  is  $\Sigma_{m+n}$ -correct, hence  $\Sigma_{m+2}$ -correct, one infers that  $j(\delta)$  is an accumulation point of  $C(\mathbb{P})$ , hence  $j(\delta) \in C(\mathbb{P})$ .<sup>16</sup> ⊣

This shows that  $\delta$  is  $\mathbb{P}$ -  $C^{(n)}$ -extendible with  $\mathbb{P}$ -  $\Sigma_n$ -reflecting target. ⊣

For simplicity let us assume that  $n \geq 2$ , as the argument for  $n = 1$  is the same. The above claim alongside the previous comments implies the existence of a proper class of  $\mathbb{P}$ -  $C^{(n)}$ -extendible cardinals with  $\mathbb{P}$ -  $\Sigma_n$ -reflecting target, hence Theorem 9.5 yields the existence of a proper class of  $C^{(n)}$ -extendible cardinals in  $V^{\mathbb{P}}$  and so  $V^{\mathbb{P}} \models \text{VP}(\Pi_{n+1})$ .

**9.2. On  $V = \text{HOD}$  and the Ground Axiom.** The first forcing iteration producing a generic extension where  $V = \text{HOD}$  holds was defined by McAloon [22]. The idea is to code the universe into the power-set function pattern so that all sets become definable using ordinals as parameters. For more sophisticated codings see [7].

For the purposes of the current section we may also assume that the GCH holds, for otherwise we can force it while preserving  $C^{(n)}$ -extendible cardinals (cf. Theorem 8.4).

Let  $\mathbb{P}$  be the class forcing notion from [23, Theorem 3.3]. It is easy to see that  $\mathbb{P}$  is  $\Delta_2$ -definable and that the class of  $\mathbb{P}$ -reflecting cardinals  $\kappa$  such that  $\mathbb{P} \cap V_\kappa = \mathbb{P}_\kappa$  contains all Mahlo cardinals, if there are any. Moreover, if there is a  $C^{(n)}$ -extendible cardinal, there are class many Mahlo cardinals, hence the above class is proper, and every  $C^{(n)}$ -extendible cardinal is also  $\mathbb{P}$ -  $C^{(n)}$ -extendible with  $\mathbb{P}$ -  $\Sigma_n$ -reflecting target (see Proposition 9.4). Thus  $\mathbb{P}$  fulfils the assumptions of Theorem 9.5. As the GCH holds,  $\mathbb{P}$  yields a cardinal-preserving generic extension in which the *Continuum Coding Axiom* (CCA) holds, hence where  $V = \text{HOD}$  holds [23, Theorem 3.3]. Altogether, we obtain the following, which extends [23, Theorem 3.9].

<sup>16</sup>Note that here we have used that  $n \geq 2$ .



**COROLLARY 9.7.** *Forcing with  $\mathbb{P}$  produces a generic extension of  $\text{CCA} + \neg\text{GCH}$  and preserves  $C^{(n)}$ -extendible cardinals, for  $n \geq 1$ . In particular,  $C^{(n)}$ -extendible cardinals are consistent with  $V = \text{HOD}$ .*

Likewise, the following extends [14, Corollary 4]:

**COROLLARY 9.8.** *There is a class iteration forcing “ $V \neq \text{HOD} + \text{GA}$ ” and preserving  $C^{(n)}$ -extendible cardinals, for  $n \geq 1$ .*

**PROOF.** Let  $\mathbb{P}$  be the forcing iteration of Corollary 9.7 and let  $\dot{Q}$  be a  $\mathbb{P}$ -name for the iteration with Easton support that forces with  $\text{Add}(\kappa, 1)$  at each regular cardinal  $\kappa$  such that  $2^{<\kappa} = \kappa$ . Set  $\mathbb{R} := \mathbb{P} * \dot{Q}$ . By the argument in [14, Theorem 3],  $V^{\mathbb{R}} \models$  “ $V \neq \text{HOD} + \text{GA}$ .” Since  $\mathbb{R}$  is  $\Delta_2$ -definable and the class of  $\mathbb{R}$ -reflecting cardinals  $\kappa$  such that  $\mathbb{R} \cap V_\kappa = \mathbb{R}_\kappa$  contains all Mahlo cardinals, every  $C^{(n)}$ -extendible cardinal is  $\mathbb{P}$ - $C^{(n)}$ -extendible with  $\mathbb{P}$ - $\Sigma_n$ -reflecting target. As  $\mathbb{R}$  satisfies the hypotheses of Theorem 9.5, the result follows.  $\dashv$

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