10

Summary

10.1 Equilibrium States

Equilibrium states are exact solutions of the equations of motion with all occurrences of the time derivative $\partial/\partial t$ set to zero. Such solutions are normally possible only after extreme simplification of the flow geometry, although for generality this is desirable (i.e., we want to focus on features common to a broad class of flows while ignoring the details that distinguish individual instances). Intrepid analysts often apply normal modes to states that are not exactly in equilibrium – the "frozen flow" hypothesis. The validity of this hypothesis must be checked after the fact to ensure that the instability grows faster than the background flow changes.

10.1.1 Mass Conservation

In most cases we have assumed that, if the equilibrium state involves a nonzero current, that current will be directed in one and only one of the coordinate directions.[1] Such a unidirectional current can be incompressible (see 1.17) only if its speed does not vary in the direction of flow. In Cartesian coordinates, this invariance of the equilibrium flow implies that the nonlinear self-advection term in the momentum equation (1.16, 1.14) must vanish:

$$[\vec{u} \cdot \vec{\nabla}]\vec{u} = 0, \tag{10.1}$$

a major simplification. The single exception to (10.1) is circular flow in cylindrical coordinates (Chapter 7), where the self-advection term is not zero but instead contributes the centrifugal force.

[1] We have found ways to accommodate some other classes of flow, e.g., veering flows (section 4.12) and flows that are not quite in equilibrium (sections 5.2 and 6.1.3).

Table 10.1 *Summary of equilibrium states. In each case the named force balances the pressure gradient force.*

Force	Equilibrium	Chapter(s)
none		3
gravity	hydrostatic	2, 4, 8, 9
viscosity	frictional	5, 6
Coriolis	geostrophic	8
Coriolis+gravity	thermal wind	8
centrifugal	cyclostrophic	7

10.1.2 Force Balances

In equilibrium, the momentum equation (1.19) reduces to a statement that the sum of forces must be everywhere zero. Each of the force terms on the right-hand side of the momentum equation can be neglected under certain plausible assumptions, with the exception of the pressure gradient force. The gravitational term can be zero either in a zero-gravity environment where $g = 0$ or in a fluid with uniform density such that $b = 0$. The viscous term can be zero if the fluid is assumed to be inviscid, $\nu = 0$, or if the flow is such that the Laplacian of the velocity field is everywhere zero. The same is true of the diffusion term in the buoyancy equation. The Coriolis term can be zero in a non-rotating environment, $f = 0$, or in a state of no motion. The centrifugal force vanishes in a parallel flow.

In each of the equilibria considered here, most (or all) of these "optional" force terms are assumed to be zero, and that the pressure field is arranged so as to balance whichever force terms remain. We can therefore classify equilibria in terms of the force that the pressure gradient must balance (see Table 10.1).

10.2 Instabilities

10.2.1 Mechanisms

The mechanism of convective instability is intuitively simple: gravity drives vertical accelerations that must overcome the damping effects of viscosity and diffusion. An analogous mechanism was identified for centrifugal, inertial, and symmetric instabilities.

Shear and baroclinic instabilities are understood in terms of wave resonances: vortical waves in the case of parallel shear flow (section 3.12); Eady waves in the case of baroclinic instability. In a stratified environment, gravity waves can also take part in a resonant interaction. Barotropic instability of a circular vortex also falls into this category.

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Table 10.2 *Rules of thumb. Double asterisks ** indicate that the wavelength pertains to the critical state. Otherwise it is the wavelength of the fastest-growing mode.*

instability	wavelength	growth rate	criterion	chapter
convection, $B_z < 0$	$2.8H^{**}$		$Ra > 657.5$	2
shear layer, $U = u_0 \tanh \frac{z}{h}$	$7 \times 2h$	$0.2\,\frac{u_0}{h}$	inflection point	3
jet (sinuous mode), $U = u_0\,\mathrm{sech}^2 \frac{z}{h}$	$3.5 \times 2h$	$0.16\,\frac{u_0}{h}$	inflection point	3
stratified shear flows (all)			$Ri_{min} < 1/4$	4
vortex (barotropic)			inflection point	7
vortex (axisymmetric)			$\min_r 2\Omega Q < 0$	7
plane Poiseuille flow, $U = 4u_0\frac{z}{h}\left(1 - \frac{z}{h}\right)$	$3h^{**}$		$Re > 11600$	5
inertial, $U = U_y y$	none	$\sqrt{-f(f - U_y)}$	$f(f - U_y) < 0$	8
symmetric, $U = U_y y + U_z z$	none	$\lvert f \rvert \sqrt{\frac{1}{Ri} - \frac{f_a}{f}}$	$Ri < \frac{f_a}{f}$	8
baroclinic (Eady), $U = U_z z$	$\frac{3.9}{P}H$	$0.3\frac{\lvert f \rvert}{\sqrt{Ri}}$		8
salt fingering	$2\pi\sqrt{\frac{B_{Tz}}{\nu\kappa_T}}$		$1 < R_\rho < \tau^{-1}$	9

10.2.2 Rules of Thumb

Table 10.2 includes a (non-exhaustive) list of properties of stability boundaries, critical states and fastest-growing instabilities in the simplest more or less accurate form. See the chapter listed for details.

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