Canad. Math. Bull. Vol. **61** (2), 2018 pp. 376–389
http://dx.doi.org/10.4153/CMB-2017-034-7

Elliptic Zeta Functions and Equivariant Functions

Abdellah Sebbar and Isra Al-Shbeil

Abstract. In this paper we establish a close connection between three notions attached to a modular subgroup, namely, the set of weight two meromorphic modular forms, the set of equivariant functions on the upper half-plane commuting with the action of the modular subgroup, and the set of elliptic zeta functions generalizing the Weierstrass zeta functions. In particular, we show that the equivariant functions can be parameterized by modular objects as well as by elliptic objects.

1 Introduction

For a finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, an equivariant function is a meromorphic function on the upper half-plane $\mathbb{H}$ that commutes with the action of Γ on $\mathbb{H}$, namely,

$$f(\gamma\tau) = \gamma f(\tau), \quad \gamma \in \Gamma, \quad \tau \in \mathbb{H},$$

where γ acts by linear fractional transformations on both sides. These were extensively studied in connection with modular forms in [3,4,8] and have important applications to modular forms and vector-valued modular forms [6, 7]. In this paper, we study the equivariant functions from an elliptic point of view. In particular, we will see that they also arise from elliptic objects. To this end we establish correspondences between three distinct notions, the first of which is the set of equivariant functions for Γ. The second is the space of weight 2 meromorphic modular forms $M_2(\Gamma)$. The third set under consideration consists of a generalization of the Weierstrass ζ-function that satisfies $\zeta'(z) = -\wp(z)$, where $\wp$ is the Weierstrass $\wp$-function attached to a rank two lattice of $\mathbb{C}$. In fact, ζ can be viewed as map

$$\zeta\colon \{\text{set of lattices in } \mathbb{C}\} \times \mathbb{C} \longmapsto \mathbb{C} \cup \{\infty\}.$$

For a fixed lattice $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ with $\Im(\omega_2/\omega_1) > 0$, the map $\zeta(\omega_1\mathbb{Z} + \omega_2\mathbb{Z}, \cdot\,)$ is quasi-periodic in the sense that

$$\zeta(\omega_1\mathbb{Z} + \omega_2\mathbb{Z}, z + \omega) = \zeta(\omega_1\mathbb{Z} + \omega_2\mathbb{Z}, z) + H(\omega), \quad z \in \mathbb{C}, \quad \omega \in \omega_1\mathbb{Z} + \omega_2\mathbb{Z}.$$

Here, $H(\omega)$ does not depend on z and is referred to as the *quasi-period map*. It is also $\mathbb{Z}$-linear, and thus it is completely determined by the quasi-periods $H(\omega_1)$ and $H(\omega_2)$. Moreover, ζ is homogeneous in the sense that

$$\zeta\big(\alpha(\omega_1\mathbb{Z} + \omega_2\mathbb{Z}), \alpha z\big) = \alpha^{-1}\zeta(\omega_1\mathbb{Z} + \omega_2\mathbb{Z}, z), \quad \alpha \in \mathbb{C}^*,$$

and so is the quasi-period map $H(\omega)$.

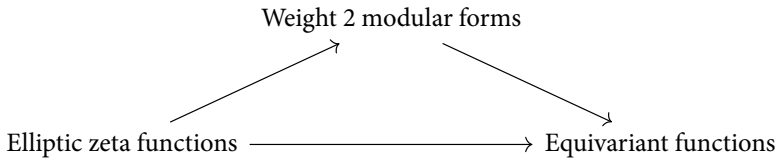
Received by the editors March 1, 2017.
Published electronically July 20, 2017.
AMS subject classification: **11F12**, 35Q15, 32L10.
Keywords: modular form, equivariant function, elliptic zeta function.

In the particular case where the lattice is of the form $\mathbb{Z} + \tau\mathbb{Z}$, $\tau \in \mathbb{H}$, $H(1)$ and $H(\tau)$ are meromorphic as functions of τ. It turns out that the quotient $H(\tau)/H(1)$ is an equivariant function on $\mathbb{H}$ thanks to the linearity and the homogeneity of the quasi-period map H [1].

To generalize the Weierstrass ζ-function, Brady, in *loc. cit.* gave the definition of zeta-type functions that behave like ζ in terms of quasi-periodicity, homogeneity, meromorphic behavior of the quasi-periods $H(1)$ and $H(\tau)$, and additional conditions.

In our case, we adapt and simplify these maps, which we call *elliptic zeta functions*. The quasi-periods $H(1)$ and $H(\tau)$ turn out to hold important information, and they are used to construct equivariant functions as well as elements of $M_2(\mathrm{SL}_2(\mathbb{Z}))$.

If Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, we generalize the above construction by defining the notion of Γ-elliptic zeta functions. Here the lattices are replaced by appropriate classes involving Γ, which essentially can be identified with pairs of lattices (L, L'), L' being a sub-lattice of finite index of L. The group $\mathrm{SL}_2(\mathbb{Z})$ acts by automorphisms on L by change of basis, and Γ becomes the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ that leaves L' invariant. We then establish a triangular correspondence between the set of Γ-elliptic zeta functions, $M_2(\Gamma)$, and the set of Γ-equivariant functions summarized in the following commutative diagram in which every arrow is surjective.



This paper is organized as follows. In Section 2 we review the basic notions of periodic and quasi-periodic functions in the context of the Weierstrass $\wp$ and ζ functions. In Section 3, inspired by M. Brady [1], we introduce the notion of elliptic zeta functions and study their structure. In Section 4, we establish the connection between weight 2 modular forms and the elliptic zeta functions. In Section 5, we review the notion of equivariant functions and establish a correspondence with the weight two modular forms. In Section 6 we generalize the constructions of the previous sections to any finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Finally, in Section 7 we provide some interesting examples related to the powers of the Weierstrass $\wp$-function.

2 Quasi-periodic Functions

The main reference in this section is [5]. Let $\Lambda \subset \mathbb{C}$ be a lattice in $\mathbb{C}$, that is, $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ with $\Im(\omega_2/\omega_1) > 0$. Such lattice can be expressed with a different basis (ω_1', ω_2') if $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$ with $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$, that is $(\omega_1', \omega_2') = (\omega_1, \omega_2)\gamma^t$, where γ^t denotes the transpose of the matrix γ. The Weierstrass $\wp$-function is the elliptic function with respect to Λ given by

$$\wp(\Lambda, z) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0}} \Big(\frac{1}{(z-w)^2} - \frac{1}{\omega^2} \Big).$$

https://doi.org/10.4153/CMB-2017-034-7 Published online by Cambridge University Press

It is absolutely and uniformly convergent on compact subsets of $\mathbb{C} \setminus \Lambda$ and defines a meromorphic function on $\mathbb{C}$ with poles of order 2 at the points of Λ and no other poles.

The Weierstrass ζ-function is defined by the series

$$\zeta(\Lambda, z) = \frac{1}{z} + \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0}} \Big(\frac{1}{z - w} + \frac{1}{\omega} + \frac{z}{\omega^2} \Big). \tag{2.1}$$

It is absolutely and uniformly convergent on compact subsets of $\mathbb{C} \setminus \Lambda$. Moreover, it defines a meromorphic function on $\mathbb{C}$ with simple poles at the points of Λ and no other poles. Differentiating the above series we get

$$\frac{d}{dz}\zeta(\Lambda, z) = -\wp(\Lambda, z).$$

for all $z \in \mathbb{C}$. Since $\wp$ is periodic relative to Λ, ζ is quasi-periodic in the sense that for all $\omega \in \Lambda$ and for all $z \in \mathbb{C}$, we have

$$\zeta(\Lambda, z + \omega) = \zeta(\Lambda, z) + \eta_\Lambda(\omega), \tag{2.2}$$

where $\eta_\Lambda(\omega)$ is independent of z. We call $\eta_\Lambda \colon \Lambda \to \mathbb{C}$ the quasi-period map associated with ζ. It is clear that η_Λ is $\mathbb{Z}$-linear, and thus it is completely determined by the values of $\eta_\Lambda(\omega_1)$ and $\eta_\Lambda(\omega_2)$. Also, since ζ is an odd function, it follows that if $\omega \in \Lambda$ and $\omega \notin 2\Lambda$, then $\eta_\Lambda(\omega)$ is given by

$$\eta_\Lambda(\omega) = 2\zeta\Big(\Lambda, \frac{\omega}{2}\Big). \tag{2.3}$$

The periods and the quasi-periods are related by the Legendre relation

$$\omega_1\eta_\Lambda(\omega_2) - \omega_2\eta_\Lambda(\omega_1) = 2\pi i. \tag{2.4}$$

The following homogeneity property of ζ and η will be very useful.

Proposition 2.1 *If Λ is a lattice and $\alpha \in \mathbb{C}$, then*

$$\zeta(\alpha\Lambda, \alpha z) = \alpha^{-1}\zeta(\Lambda, z) \quad \textit{and} \quad \eta_{\alpha\Lambda}(\alpha\omega) = \alpha^{-1}\eta_\Lambda(\omega). \tag{2.5}$$

Proof The first relation follows from the expansion (2.1), and the second relation follows from (2.2) ∎

We refer to (2.5) by saying that ζ and η are homogeneous of weight -1.

We now focus on lattices of the form $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ where τ is in the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$. From (2.3) we can readily see that the quasi-periods $\eta_{\Lambda_\tau}(1)$ and $\eta_{\Lambda_\tau}(\tau)$ are meromorphic functions on $\mathbb{H}$ and so is the function defined by

$$h(\tau) = \frac{\eta_{\Lambda_\tau}(\tau)}{\eta_{\Lambda_\tau}(1)}. \tag{2.6}$$

For the remainder of this paper, if $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbb{C})$ and $z \in \mathbb{C}$, we define the action γz by

$$\gamma z = \frac{az + b}{cz + d}.$$

https://doi.org/10.4153/CMB-2017-034-7 Published online by Cambridge University Press

When $\gamma \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$, this is the usual action on $\mathbb{H}$ by linear fractional transformation providing all the automorphisms of $\mathbb{H}$.

Proposition 2.2 ([1]) *The function h defined by (2.6) satisfies*

$$h(\gamma\tau) = \gamma h(\tau), \quad \gamma \in SL_2(\mathbb{Z}), \quad \tau \in \mathbb{H}. \tag{2.7}$$

Proof Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$. We have

$$\begin{aligned}
\eta_{\Lambda_{\gamma\tau}}(\gamma\tau) &= \eta_{\mathbb{Z}+\frac{a\tau+b}{c\tau+d}\mathbb{Z}}\Big(\frac{a\tau+b}{c\tau+d}\Big) \\
&= (c\tau+d)\eta_{(c\tau+d)\mathbb{Z}+(a\tau+b)\mathbb{Z}}(a\tau+b) \quad \text{(by homogeneity of } \eta) \\
&= (c\tau+d)\eta_{\Lambda_\tau}(a\tau+b) \\
&= (c\tau+d)(a\eta_{\Lambda_\tau}(\tau) + b\eta_{\Lambda_\tau}(1))
\end{aligned}$$

where we have used the linearity of η and the fact that the lattices Λ_τ and $(c\tau+d)\mathbb{Z} + (a\tau+b)\mathbb{Z}$ are the same. Also,

$$\begin{aligned}
\eta_{\Lambda_{\gamma\tau}}(1) &= (c\tau+d)\eta_{(c\tau+d)\mathbb{Z}+(a\tau+b)\mathbb{Z}}(c\tau+d) \quad \text{(by homogeneity of } \eta) \\
&= (c\tau+d)\eta_{\Lambda_\tau}(c\tau+d) \\
&= (c\tau+d)(c\eta_{\Lambda_\tau}(\tau) + d\eta_{\Lambda_\tau}(1)).
\end{aligned}$$

Therefore,

$$h(\gamma\tau) = \frac{a\eta_{\Lambda_\tau}(\tau) + b\eta_{\Lambda_\tau}(1)}{c\eta_{\Lambda_\tau}(\tau) + d\eta_{\Lambda_\tau}(1)} = \frac{ah(\tau)+b}{ch(\tau)+d} = \gamma h(\tau). \qquad \blacksquare$$

A meromorphic function on $\mathbb{H}$ that satisfies (2.7) will be called *equivariant* with respect to $\mathrm{SL}_2(\mathbb{Z})$. We will expand more on these functions in later sections.

3 Elliptic Zeta Functions

Following [1], we will generalize the notion of Weierstrass zeta function and its quasi-periods. Let $\mathcal{L}$ be the set of lattices $\Lambda_{(\omega_1,\omega_2)} = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ with $\Im(\omega_2/\omega_1 > 0)$. We define an elliptic zeta function of weight $k \in \mathbb{Z}$ as a map

$$\mathcal{Z}\colon \mathcal{L} \times \mathbb{C} \longrightarrow \mathbb{C} \cup \{\infty\}$$

satisfying the following properties.

(a) For each $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, the map

$$\mathcal{Z}(\Lambda, \cdot)\colon \mathbb{C} \longrightarrow \mathbb{C} \cup \{\infty\}$$

is quasi-periodic, that is,

$$\mathcal{Z}(\Lambda, z+\omega) = \mathcal{Z}(\Lambda, z) + H_\Lambda(\omega), \quad z \in \mathbb{C}, \quad \omega \in \Lambda,$$

where the quasi-period function $H_\Lambda(\omega)$ does not depend on z.

(b) $\mathcal{Z}$ is homogeneous of weight k in the sense that

$$\mathcal{Z}(\alpha\Lambda, \alpha z) = \alpha^k\, \mathcal{Z}(\Lambda, z), \quad \alpha \in \mathbb{C}^*, \quad z \in \mathbb{C}.$$

(c) If $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$, $\tau \in \mathbb{H}$, then the quasi-periods $H_{\Lambda_\tau}(\tau)$ and $H_{\Lambda_\tau}(1)$ as functions of τ are meromorphic on $\mathbb{H}$.

https://doi.org/10.4153/CMB-2017-034-7 Published online by Cambridge University Press

It follows from (a) that for each Λ, the quasi-period function H_Λ is $\mathbb{Z}$-linear, and therefore, it is completely determined by $H_\Lambda(\omega_1)$ and $H_\Lambda(\omega_2)$. Moreover, we have the following result generalizing Proposition 2.1.

Proposition 3.1 *Let $\mathcal{Z}$ be an elliptic zeta function of weight k and let H_Λ be the quasi-period function for each lattice Λ. Then for all $\alpha \in \mathbb{C}^\times$ and $\omega \in \Lambda$, we have*

$$H_{\alpha\Lambda}(\alpha\omega) = \alpha^k\, H_\Lambda(\omega).$$

Proof On one hand we have

$$\mathcal{Z}\big(\alpha\Lambda, \alpha(z+\omega)\big) = \mathcal{Z}(\alpha\Lambda, \alpha z) + H_{\alpha\Lambda}(\alpha\omega) = \alpha^k \mathcal{Z}(\Lambda, z) + H_{\alpha\Lambda}(\alpha\omega).$$

On the other hand, we have

$$\mathcal{Z}\big(\alpha\Lambda, \alpha(z+\omega)\big) = \alpha^k \mathcal{Z}(\Lambda, z+\omega) = \alpha^k\big(\mathcal{Z}(\Lambda, z) + H_\Lambda(\omega)\big),$$

and the proposition follows. ∎

Notice that two elliptic zeta functions having the same quasi-period function must differ by an elliptic function. Simple examples are given by the identity map z, or the Weierstrass zeta function $\zeta(\Lambda, z)$. We will see below that these two examples will, in a certain sense, generate all the other elliptic zeta functions. Also, since for a fixed lattice the derivative of an elliptic zeta function with respect to z is an elliptic function for the lattice, this provides a way to construct infinitely many of them by taking integrals of elliptic functions.

Let ω_1 and ω_2 be such that $\Im(\omega_2/\omega_1) > 0$ and set

$$M_{(\omega_1,\omega_2)} = \begin{bmatrix} \omega_2 & \eta(\omega_2) \\ \omega_1 & \eta(\omega_1) \end{bmatrix},$$

where η is the quasi-period map of the Weierstrass zeta function $\zeta(\omega_1\mathbb{Z} + \omega_2\mathbb{Z}, z)$. Using the Legendre relation (2.4), we have

$$\det M_{(\omega_1,\omega_2)} = -2\pi i.$$

Let $\mathcal{Z}$ be an elliptic zeta function of weight k with the two quasi-periods $H(\omega_1)$ and $H(\omega_2)$. Set

$$\begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = M^{-1}_{(\omega_1,\omega_2)} \begin{bmatrix} H(\omega_2) \\ H(\omega_1) \end{bmatrix}.$$

In other words,

$$2\pi i\Phi = \eta(\omega_2)H(\omega_1) - \eta(\omega_1)H(\omega_2), \tag{3.1}$$

$$2\pi i\Psi = \omega_1 H(\omega_2) - \omega_2 H(\omega_1). \tag{3.2}$$

Proposition 3.2 *The quantities Φ and Ψ do not depend on the choice of the basis (ω_1, ω_2) and, as functions of the lattice $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, they are homogeneous of respective weights $k-1$ and $k+1$.*

https://doi.org/10.4153/CMB-2017-034-7 Published online by Cambridge University Press

Proof Let a, b, c, and d be integers such that $ad-bc=1$. The expressions in (3.1) and (3.2) are invariant if we change the basis (ω_1, ω_2) to the basis $(a\omega_1+b\omega_2, c\omega_1+d\omega_2)$. Indeed, using the linearity of η and H, we have for the expression of Φ:

$$\begin{aligned}\eta(c\omega_1+d\omega_2)H(a\omega_1+b\omega_2)&-\eta(a\omega_1+b\omega_2)H(c\omega_1+d\omega_2)\\ &=[c\eta(\omega_1)+d\eta(\omega_2)][aH(\omega_1)+bH(\omega_2)]\\ &\quad-[a\eta(\omega_1)+b\eta_(\omega_2)][cH(\omega_1)+dH(\omega_2)]\\ &=H(\omega_1)\eta(\omega_2)-H(\omega_2)\eta(\omega_1).\end{aligned}$$

Similar calculations hold for the expression of Ψ. The values of the weights are straightforward knowing that the weight is k for H, -1 for η and 1 for both ω_1 and ω_2. ∎

We can therefore denote Φ and Ψ by Φ_Λ and Ψ_Λ, as they depend only on the lattice Λ.

Proposition 3.3 ([1]) *Let $\mathcal{Z}$ be an elliptic zeta function of weight k and quasi-period function H, and let Φ_Λ and Ψ_Λ be as above. Then for each lattice Λ, there exists an elliptic function E_Λ such that*

$$\mathcal{Z}(\Lambda, z)=\Phi_\Lambda\, z\ +\ \Psi_\Lambda\zeta(z)\ +\ E_\Lambda(z). \tag{3.3}$$

Proof It is clear by construction of Φ and Ψ that the map $\Phi_\Lambda z+\Psi_\Lambda\zeta$ satisfies the conditions of a weight k elliptic zeta function. Moreover, for each $\Lambda=\omega_1\mathbb{Z}+\omega_2\mathbb{Z}$, the quasi-periods of $\Phi_\Lambda\, z+\Psi_\Lambda\,\zeta(z)$ are $\Phi_\Lambda\,\omega_i+\Psi_\Lambda\,\eta(\omega_i)$, $i=1,2$, which coincide with the quasi-periods $H(\omega_1)$ and $H(\omega_2)$ of $\mathcal{Z}$ as we have $\left[\begin{smallmatrix}H(\omega_2)\\ H(\omega_1)\end{smallmatrix}\right]=\left[\begin{smallmatrix}\omega_2 & \eta(\omega_2)\\ \omega_1 & \eta(\omega_1)\end{smallmatrix}\right]\left[\begin{smallmatrix}\Phi_\Lambda\\ \Psi_\Lambda\end{smallmatrix}\right]$, and therefore the two elliptic zeta functions differ by an elliptic function for the lattice Λ. ∎

It is clear that expression (3.3) for an elliptic zeta function is unique up to the elliptic function $E_\Lambda(z)$, since Φ_Λ and Ψ_Λ are uniquely determined. Moreover, we view relations (3.1) and (3.2) as the generalization for an elliptic zeta function of the Legendre relation (2.4) for the Weierstrass zeta function. Finally, using a similar proof to that of Proposition 2.7, we have the following proposition.

Proposition 3.4 *Let $\mathcal{Z}$ be an elliptic zeta function with a quasi-period map H. For $\tau\in\mathbb{H}$ and $\Lambda_\tau=\mathbb{Z}+\tau\mathbb{Z}$, suppose that $H(1)$ is not identically zero; then the meromorphic function $h(\tau)=H(\tau)/H(1)$ is equivariant with respect to $SL_2(\mathbb{Z})$.*

4 Modular Forms

In this section we will investigate the connection between elliptic zeta functions and modular forms for $\mathrm{SL}_2(\mathbb{Z})$. In the following theorem, we will show that each elliptic zeta functions gives rise to a weight 2 (meromorphic) modular form for $\mathrm{SL}_2(\mathbb{Z})$, and conversely, each weight 2 modular form yields an elliptic zeta function.

https://doi.org/10.4153/CMB-2017-034-7 Published online by Cambridge University Press

Theorem 4.1 *Let $\mathcal{Z}$ be an elliptic zeta function with Φ_Λ and Ψ_Λ as in (3.3) and suppose Ψ_{Λ_τ} is not identically zero as a function of τ. Then the map*

$$\mathcal{Z} \longmapsto \frac{\Phi_{\Lambda_\tau}}{\Psi_{\Lambda_\tau}} \tag{4.1}$$

is well defined between the set of elliptic zeta functions and the space of weight 2 modular forms $M_2(SL_2(\mathbb{Z}))$. In addition, this map is surjective.

Proof Let $k \in \mathbb{Z}$ be the weight of $\mathcal{Z}$ and set

$$f(\tau) = \frac{\Phi_{\Lambda_\tau}}{\Psi_{\Lambda_\tau}}, \quad \tau \in \mathbb{H}.$$

Since Φ_{Λ_τ} and Ψ_{Λ_τ} are meromorphic in τ, so is $f(\tau)$. Now let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$. Since Φ_Λ and Ψ_Λ are homogeneous of weights $k-1$ and $k+1$ respectively, we have

$$\Phi_{\Lambda_{\gamma\tau}} = (c\tau+d)^{-k+1}\Phi_{(a\tau+b)\mathbb{Z}+(c\tau+d)\mathbb{Z}} = (c\tau+d)^{-k+1}\Phi_{\Lambda_\tau},$$
$$\Psi_{\Lambda_{\gamma\tau}} = (c\tau+d)^{-k-1}\Psi_{(a\tau+b)\mathbb{Z}+(c\tau+d)\mathbb{Z}} = (c\tau+d)^{-k-1}\Psi_{\Lambda_\tau}.$$

Therefore,

$$f(\gamma\tau) = (c\tau+d)^2 f(\tau).$$

Hence the map is well defined as Φ_Λ and Ψ_Λ are uniquely determined by $\mathcal{Z}$. We now prove that the map is onto. Let $f \in M_2(\mathrm{SL}_2(\mathbb{Z}))$ and set

$$\Phi_\Lambda = \frac{1}{\omega_1^2} f\Big(\frac{\omega_2}{\omega_1}\Big), \quad \Psi_\Lambda = 1,$$

for $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. The map Φ_Λ is well defined in the sense that it is independent of the choice of the basis (ω_1, ω_2). Indeed, if $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$, then

$$\begin{aligned}\frac{1}{(a\omega_1+b\omega_2)^2} f\Big(\frac{c\omega_1+d\omega_2}{a\omega_1+b\omega_2}\Big) &= \frac{1}{(a\omega_1+b\omega_2)^2} f\Big(\frac{d\,\frac{\omega_2}{\omega_1}+c}{b\,\frac{\omega_2}{\omega_1}+a}\Big) \\ &= \frac{(b\,\frac{\omega_2}{\omega_1}+a)^2}{(a\omega_1+b\omega_2)^2} f\Big(\frac{\omega_2}{\omega_1}\Big) = \frac{1}{\omega_1^2} f\Big(\frac{\omega_2}{\omega_1}\Big).\end{aligned}$$

Thus, we have an elliptic zeta function

$$\mathcal{Z}(\Lambda, z) = \frac{1}{\omega_1^2} f\Big(\frac{\omega_2}{\omega_1}\Big) z + \zeta(z)$$

of weight -1 that is sent to $f(\tau)$ by the map (4.1). ∎

5 Equivariant Functions

We introduced the notion of equivariant functions earlier as being meromorphic functions on $\mathbb{H}$ that commute with the action of the modular group. They were extensively studied in [3,4,7,8] in connection with modular forms, vector-valued modular forms, and other topics. In particular, each modular form of any weight (even with a character) gives rise to an equivariant function. Indeed, if f is a modular form of weight

https://doi.org/10.4153/CMB-2017-034-7 Published online by Cambridge University Press

k, then the function

$$h_f(\tau) = \tau + k\frac{f(\tau)}{f'(\tau)}$$

is equivariant with respect to $\mathrm{SL}_2(\mathbb{Z})$.

Not all the equivariant functions arise in this way from a modular form. In fact a necessary and sufficient condition for an equivariant function h to be equal to h_f for some modular form f is that the poles of $1/(h(\tau)-\tau)$ in $\mathbb{H}\cup\{\infty\}$ are all simple with rational residues [4]. Such functions are referred to as the rational equivariant functions.

Important applications were obtained regarding the critical points of modular forms and their q-expansion in [6]. As an example, recall the Eisenstein series $G_2(\tau)$ defined by

$$G_2(\tau) = \frac{1}{2}\sum_{n\neq 0}\frac{1}{n^2} + \frac{1}{2}\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^2}$$

and the normalized weight two Eisenstein series

$$E_2(\tau) = \frac{6}{\pi^2}G_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n, \quad q = e^{2\pi i\tau},$$

where $\sigma_1(n)$ is the sum of positive divisors of n. One can easily deduce from the definition of the Weierstrass ζ-function [5] that

$$\eta(1) = G_2(\tau), \quad \eta(\tau) = \tau G_2(\tau) - 2\pi i,$$

and since we have

$$E_2(\tau) = \frac{1}{2\pi i}\frac{\Delta'(\tau)}{\Delta(\tau)},$$

where Δ is the weight 12 cusp form (the discriminant)

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24}, \quad q = e^{2\pi i\tau},$$

we deduce the following proposition.

Proposition 5.1 *The equivariant function $h(\tau) = \frac{\eta(\tau)}{\eta(1)}$ from Proposition 2.7 is rational with*

$$h(\tau) = \tau + 12\,\frac{\Delta}{\Delta'}.$$

Let us denote by Eq the set of all equivariant functions with respect to $\mathrm{SL}_2(\mathbb{Z})$. Although $h(\tau)=\tau$ is trivially equivariant, it will be excluded from Eq.

Recall that if $f \in M_2(\mathrm{SL}_2(\mathbb{Z}))$ and $\gamma = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbb{C})$ we denote $\frac{af(\tau)+b}{cf(\tau)+d}$ by $\gamma f(\tau)$. Now recall from Section 3 the matrix

$$M_{(1,\tau)} = \begin{bmatrix} \tau & \eta(\tau) \\ 1 & \eta(1)\end{bmatrix},$$

which is invertible thanks to the Legendre relation.

Theorem 5.2 *The map from $M_2(SL_2(\mathbb{Z}))$ to* Eq, *$f \mapsto M_{(1,\tau)}f$ is a bijection. The inverse map is given by $h \mapsto M^{-1}_{(1,\tau)}h$.*

https://doi.org/10.4153/CMB-2017-034-7 Published online by Cambridge University Press

Proof Let $f \in M_2(\mathrm{SL}_2(\mathbb{Z}))$ and set

$$h(\tau) = M_{(1,\tau)}f(\tau) = \frac{\tau f(\tau) + \eta(\tau)}{f(\tau) + \eta(1)}.$$

For $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$h(\gamma\tau) = \frac{\gamma\tau f(\gamma\tau) + \eta_{\Lambda_{\gamma\tau}}(\gamma\tau)}{f(\gamma\tau) + \eta_{\Lambda_\tau}(1)}.$$

Since

$$\begin{aligned} \eta_{\Lambda_{\gamma\tau}}(\gamma\tau) &= (c\tau + d)(a\eta(\tau) + b\eta(1)), \\ \eta_{\Lambda_\tau}(1) &= (c\tau + d)(c\eta(\tau) + d\eta(1)), \\ f(\gamma\tau) &= (c\tau + d)^2 f(\tau), \end{aligned}$$

we have

$$h(\gamma\tau) = \frac{(a\tau + b)f(\tau) + a\eta(\tau) + b\eta(1)}{(c\tau + d)f(\tau) + c\eta(\tau) + d\eta(1)} = \gamma h(\tau).$$

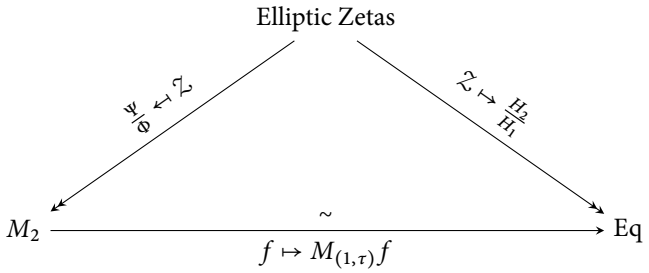
Similarly, one can prove that if $h \in Eq$, then $M^{-1}_{(1,\tau)}h \in M_2(\mathrm{SL}_2(\mathbb{Z}))$. ∎

Usually the definition of a meromorphic modular forms also involves the behavior at the cusps. More precisely, if f is a modular form for $\mathrm{SL}_2(\mathbb{Z})$, then $f(\tau+1) = f(\tau)$ for all $\tau \in \mathbb{H}$ and thus has a Fourier expansion that is a Laurent series in $q = \exp(2\pi i\tau)$. We say that f is meromorphic at the cusp ∞ if this Laurent series has only finitely many negative powers of q. In the meantime, if h is equivariant for $\mathrm{SL}_2(\mathbb{Z})$ then $h(\tau + 1) = h(\tau) + 1$. Hence, $h(\tau) - \tau$ is also periodic of period one and thus has a Fourier expansion in q. The proper behavior of h at the cusp at infinity is that $h(\tau) - \tau$ is meromorphic in q; see [4]. If a weight two modular form f and an equivariant function h correspond to each other by Theorem 5.2, then $h(\tau) = \frac{\tau f(\tau)+\eta(\tau)}{f(\tau)+\eta(1)}$ and thus, using the Legendre relation (2.4), we have

$$h(\tau) - \tau = \frac{2\pi i}{f(\tau) + \eta(1)}.$$

Since $\eta(1) = G_2(\tau)$ is holomorphic in q, we see that the behavior at infinity for both f and h is preserved under the correspondence of Theorem 5.2.

Taking into account the results of the above sections, we have established a correspondence between the set of elliptic zeta functions, the space of modular forms of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$ and the set of equivariant functions for $\mathrm{SL}_2(\mathbb{Z})$ summarized as follows:



https://doi.org/10.4153/CMB-2017-034-7 Published online by Cambridge University Press

where H_1 and H_2 are the quasi-periods of the elliptic zeta function $\mathcal{Z}$, Φ and Ψ are such that $\mathcal{Z}(\Lambda, z) = \Phi\ z + \Psi\zeta(z) + E$ with E elliptic and $M_{(1,\tau)}$ as above. Of course, this diagram is commutative, and each map is surjective.

6 The Case of Modular Subgroups

So far the constructions in the previous sections involve the full modular group $\mathrm{SL}_2(\mathbb{Z})$. In the meantime, the notion of modular forms or equivariant functions can be restricted to any finite index subgroup. Thus, we need to define the notion of elliptic zeta functions for any such subgroup.

Fix a modular subgroup Γ of finite index in $\mathrm{SL}_2(\mathbb{Z})$. Set

$$\mathbb{M} = \left\{ (\omega_1, \omega_2) \in \mathbb{C}^2 : \Im(\omega_2/\omega_1) > 0 \right\}.$$

The group Γ acts on $\mathbb{M}$ in the usual way: $\gamma(\omega_1, \omega_2) = (\omega_1, \omega_2)\gamma^t$. Denote by Ω_Γ the quotient $\Gamma\backslash\mathbb{M}$ and the class of (ω_1, ω_2) by $[\omega_1, \omega_2]$. Also, $\mathbb{C}^*$ acts on $\mathbb{M}$ in the usual way and this action extends to Ω_Γ as

$$\alpha[\omega_1, \omega_2] = [\alpha\omega_1, \alpha\omega_2].$$

If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then $[\omega_1, \omega_2]$ is identified with the lattice $\Lambda_{(\omega_1,\omega_2)} = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, but for an arbitrary finite index subgroup Γ, the situation is different. Following the ideas in [2], Ω_Λ is identified with the set of pairs of lattices (Λ, Λ') with Λ' being a finite index sub-lattice of Λ fixed by Γ and Λ' is the smallest such lattice (and thus defined as the intersection of all such sub-lattices that are Γ-invariant). If such pair (Λ, Λ') is given, and as $\mathrm{SL}_2(\mathbb{Z})$ acts by automorphisms of Λ by a change of basis, Γ would be defined by

$$\Gamma = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma\Lambda' \subseteq \Lambda'\}.$$

For example, if $\Gamma = \Gamma(N)$ is the principal congruence subgroup of level $N \geq 1$, then $\Lambda' = N\omega_1\mathbb{Z} + N\omega_2\mathbb{Z}$, which is a sub-lattice of $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ of index N^2 . If $\Gamma = \Gamma_0(N)$, then $\Lambda' = \omega_1\mathbb{Z} + N\omega_2\mathbb{Z}$ of index N in $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. However, we will not need this identification in what follows.

A Γ-elliptic zeta function with respect to Γ is a map

$$\mathcal{Z}\colon \Omega_\Gamma \times \mathbb{C} \longrightarrow \mathbb{C} \cup \{\infty\}$$

satisfying the following.

(a) For each $[\omega_1, \omega_2] \in \Omega_\Gamma$, the map

$$\mathcal{Z}([\omega_1, \omega_2], \cdot)\colon \mathbb{C} \longrightarrow \mathbb{C} \cup \{\infty\}$$

is quasi-periodic with respect to $\Lambda_{(\omega_1,\omega_2)}$; that is, for all $z \in \mathbb{C}$ and all $\omega \in \Lambda_{(\omega_1,\omega_2)}$ we have

$$\mathcal{Z}\big([\omega_1, \omega_2], z + \omega\big) = \mathcal{Z}\big([\omega_1, \omega_2], z\big) + H_{[\omega_1,\omega_2]}(\omega).$$

(b) The map $\mathcal{Z}$ is homogeneous; that is, there exists an integer k, referred to as the weight of $\mathcal{Z}$, such that for all $\alpha \in \mathbb{C}^*$, $[\omega_1, \omega_2] \in \Omega_\Gamma$ and $z \in \mathbb{C}$ we have

$$\mathcal{Z}\big(\alpha[\omega_1, \omega_2], \alpha z\big) = \alpha^k \mathcal{Z}\big([\omega_1, \omega_2], z\big).$$

https://doi.org/10.4153/CMB-2017-034-7 Published online by Cambridge University Press

(c) The maps

$$\tau \longmapsto H_{[1,\tau]}(\tau) \quad \text{and} \quad \tau \longmapsto H_{[1,\tau]}(1)$$

are meromorphic in $\mathbb{H}$.

From this definition, it is clear that the quasi-period map $H_{[\omega_1,\omega_2]}$ is $\mathbb{Z}$-linear on the lattice $\Lambda_{(\omega_1,\omega_2)}$, and thus it is completely determined by its values on ω_1 and ω_2. It is also homogeneous of weight k:

$$H_{[\alpha\omega_1,\alpha\omega_2]}(\alpha\omega) = \alpha^k H_{[\omega_1,\omega_2]}(\omega), \quad \omega \in \Lambda_{(\omega_1,\omega_2)}, \quad \alpha \in \mathbb{C}^*.$$

Using the same arguments as in Section 3, one can easily establish the following proposition.

Proposition 6.1 *Let $\mathcal{Z}\colon \Omega_\Gamma \times \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be a Γ-elliptic zeta function. There exists unique maps $\Phi_{[\omega_1,\omega_2]}$ of weight $k-1$ and $\Psi_{[\omega_1,\omega_2]}$ of weight $k+1$ such that for all $[\omega_1,\omega_2] \in \Omega_\Gamma$ and $z \in \mathbb{C}$ we have*

$$\mathcal{Z}([\omega_1,\omega_2],z) = \Phi_{[\omega_1,\omega_2]} z + \Psi_{[\omega_1,\omega_2]}\zeta(z) + E_{[\omega_1,\omega_2]}(z),$$

where $E_{[\omega_1,\omega_2]}(z)$ is an $\Lambda_{(\omega_1,\omega_2)}$-elliptic function.

Notice that $\Phi_{[\omega_1,\omega_2]}$ and $\Psi_{[\omega_1,\omega_2]}$ can be shown to be independent of the choice of the representative of the class $[\omega_1,\omega_2]$ in the same way as for Proposition 3.2 using transformations from Γ instead of $\mathrm{SL}_2(\mathbb{Z})$.

Let $M_2(\Gamma)$ denote the space of meromorphic weight two modular forms with respect to Γ and $\mathrm{Eq}(\Gamma)$ be the set of Γ-equivariant functions, that is the set of meromorphic functions on $\mathbb{H}$ that commute with the action of Γ. It is clear that the matrix $M_{(1,\tau)}$ of the previous section provides a bijection between $M_2(\Gamma)$ and $\mathrm{Eq}(\Gamma)$. Using the fact that by definition of Ω_Γ, when $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$, we have

$$[\omega_1,\omega_2] = [a\omega_1 + b\omega_2, c\omega_1 + d\omega_2],$$

we deduce, in the same way as in the previous sections, the following theorem.

Theorem 6.2 *If Γ is a finite index subgroup of $SL_2(\mathbb{Z})$, then:*

(i) *The map*

$$\mathcal{Z} \longmapsto \frac{H_{[1,\tau]}(\tau)}{H_{[1,\tau]}(1)} \tag{6.1}$$

is a well-defined map from the set of Γ-elliptic zeta functions to $\mathrm{Eq}(\Gamma)$.

(ii) *The map*

$$\mathcal{Z} \longmapsto \frac{\Phi_{[1,\tau]}}{\Psi_{[1,\tau]}} \tag{6.2}$$

is well defined between the set of Γ-elliptic zeta functions and $M_2(\Gamma)$. It is also onto as for each $f \in M_2(\Gamma)$

$$\mathcal{Z}([\omega_1,\omega_2],z) = \frac{1}{{\omega_1}^2} f\Big(\frac{\omega_2}{\omega_1}\Big)\, z + \zeta(z)$$

is a Γ-elliptic zeta function of weight -1 that maps to f by (6.2).

https://doi.org/10.4153/CMB-2017-034-7 Published online by Cambridge University Press

Remark 6.3 Using the bijection between $M_2(\Gamma)$ and $Eq(\Gamma)$ and the surjective map (6.2), one can also see that the map (6.1) is surjective. Thus we have shown that each Γ-equivariant function arises from a Γ-elliptic zeta function. Notice that the trivial equivariant function τ is also the quotient of the quasi-periods of the trivial Γ-elliptic zeta function $\mathcal{Z}(z) = z$.

Remark 6.4 It is worth explaining the behavior at the cusps as has been discussed at the end of Section 5 for the cusp at infinity. In the case of a modular subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, there are more than one cusp that are not in the same Γ-orbit. Meanwhile, the analytic behavior of a meromorphic modular form at a rational cusp is well defined (see [9, Chapter 1] for instance), and that of an equivariant function has been established in [4, §3]. It is not difficult to show that the two behaviors at a rational cusp are well preserved under the correspondence between a weight 2 modular form for Γ and a Γ-equivariant function when Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

7 Examples

In this section, we study an important class of elliptic functions given by integrals of the powers $\wp^n$ of the Weierstrass $\wp$-function. These integrals were treated in [8].

Let $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, $\Im(\omega_2/\omega_1) > 0$, be a lattice in $\mathbb{C}$. The Eisenstein series g_2 and g_3 are defined by

$$g_2(\Lambda) = 60 \sum_{\omega\in\Lambda-\{0\}} \frac{1}{\omega^4}, \quad g_3(\Lambda) = 140 \sum_{\omega\in\Lambda-\{0\}} \frac{1}{\omega^4}.$$

When $\Lambda = \mathbb{Z}+\tau\mathbb{Z}$, $\tau \in \mathbb{H}$, g_2 and g_3, as functions of τ are modular forms of weight four and six respectively. For a non-negative integer n, the power $\wp^n(z)$ can be written as a linear combination of 1, $\wp$ and successive derivatives of $\wp$:

$$\wp^n(\Lambda, z) = \Phi_n(\Lambda) - \Psi_n(\Lambda)\wp(\Lambda, z) + \sum_{k=1}^{n-1} \alpha_k \wp^{(2k)},$$

where the coefficients α_k are polynomials in g_2 and g_3 with rational coefficients; see [10, p. 108]. In particular, $\Phi_0 = 1$, $\Psi_0 = 0$, $\Phi_1 = 0$ and $\Psi_1 = -1$.

For each lattice Λ and $z \in \mathbb{C}$, a primitive $\int \wp^n(u)du$ of $\wp^n$ has the form

$$\Phi_n(\Lambda)\, z + \Psi_n(\Lambda)\zeta(\Lambda, z) + E_n(\Lambda, z),$$

where for each Λ, $E_n(\Lambda, z)$ is a Λ-elliptic function. We define

$$\mathcal{Z}_n(\Lambda, z) := \Phi_n(\Lambda)z + \Psi_n(\Lambda)\zeta(\Lambda, z).$$

It is clear that for each Λ, $\mathcal{Z}_n(\Lambda, z)$ is quasi-periodic with the quasi-period map given by

$$H_n(\omega) = \Phi_n(\Lambda)\, \omega + \Psi_n(\Lambda)\eta(\omega),$$

where η is the quasi-period map for the Weierstrass ζ-function. If there is no confusion, we will write Φ_n for $\Phi_n(\Lambda)$ and Ψ_n for $\Psi_n(\Lambda)$. According to [10, p. 109] (see also [8, §9]), Φ_n, Ψ_n, and thus H_n satisfy the same three-term recurrence relation

$$u_{n+1} = \frac{2n-1}{4(2n+1)}\, g_2 u_{n-1} + \frac{n-1}{2(2n+1)}\, g_3 u_{n-2},$$

https://doi.org/10.4153/CMB-2017-034-7 Published online by Cambridge University Press

with the following initial conditions

$$\Phi_{-1} = \Psi_{-1} = H_{-1} = 0$$
$$\Phi_0 = 1, \Psi_0 = 0, H_0(\omega) = \omega$$
$$\Phi_1 = 0, \Psi_1 = -1, H_1(\omega) = -\eta(\omega).$$

One can easily see that when $\Lambda = \Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$, $\tau \in \mathbb{H}$, H_n is polynomial in $g_2, g_3, \eta(1)$ and $\eta(\tau)$, and thus $H_n(1)$ and $H_n(\tau)$ are meromorphic functions of τ. It follows that the map $\mathcal{Z}_n(\Lambda, z)$ satisfies the axioms of a Weierstrass elliptic zeta function of weight $-2n+1$.

Let us put $\Phi_n(\tau) = \Phi_n(\Lambda_\tau)$, $\Psi_n(\tau) = \Psi_n(\Lambda_\tau)$, $\eta_1 = \eta(1)$ and $\eta_2 = \eta(\tau)$. Then $\Phi_2(\tau) = \frac{1}{12}\, g_2(\tau)$, $\Phi_3(\tau) = \frac{1}{10}\, g_3(\tau)$, $\Psi_2(\tau) = 0$ and $\Psi_3(\tau) = \frac{-3}{20}\, g_2(\tau)$. More generally, one can show by induction that

Proposition 7.1 *For each positive integer n, Φ_n and Ψ_n are weighted homogeneous polynomials in g_2 and g_3 with rational coefficients and of degrees $2n$ and $2(n-1)$, respectively, and these degrees are also their weights as holomorphic modular forms.*

For small weights, it is clear that Φ_n and Ψ_n are simple monomials. In light of Sections 4 and 5, for each elliptic zeta function $\mathcal{Z}_n$, there correspond, on one hand, a weight two modular form

$$f_n(\tau) = \frac{\Phi_n(\tau)}{\Psi_n(\tau)},$$

which is a rational function of g_2 and g_3 with rational coefficients, and on the other hand, an equivariant function

$$h_n(\tau) = \frac{H_n(\tau)}{H_n(1)} = \frac{\Phi_n(\tau)\,\tau + \Psi_n(\tau)\eta_2}{\Phi_n(\tau) + \Psi_n(\tau)\eta_1}.$$

Also, using the Legendre relation, f_n and h_n are related by

$$h_n(\tau) = \tau + \frac{2\pi i}{f_n(\tau) + \eta_1}.$$

The following table gives Φ_n, Ψ_n, f_n, and h_n for $1 \le n \le 6$:

n	Φ_n	Ψ_n	f_n	h_n
1	0	-1	0	$\frac{\eta_2}{\eta_1} = \tau + \frac{2\pi i}{\eta_1}$
2	$\frac{1}{12} g_2$	0	$-$	τ
3	$\frac{1}{10}\, g_3$	$\frac{-3}{20}\, g_2$	$\frac{-2}{3}\,\frac{g_3}{g_2}$	$\tau + \frac{6\pi i g_2}{-2g_3+3g_2\eta_1}$
4	$\frac{5}{336}\, g_2^2$	$\frac{-2}{14}\, g_3$	$\frac{-5}{48}\,\frac{g_2^2}{g_3}$	$\tau + \frac{96\pi i g_3}{-5g_2^2+48g_3\eta_1}$
5	$\frac{1}{30}\, g_2 g_3$	$\frac{-7}{240}\, g_2^2$	$\frac{-8}{7}\,\frac{g_3}{g_2}$	$\tau + \frac{14\pi i g_2}{-8g_3+7g_2\eta_1}$
6	$\frac{15}{4928}\, g_2^3 + \frac{1}{55}\, g_3^2$	$\frac{-87}{1540}\, g_2 g_3$	$\frac{-25}{464}\,\frac{g_2^2}{g_3} - \frac{28}{87}\,\frac{g_3}{g_2}$	$\tau + \frac{2784\pi i g_2 g_3}{-75g_2^3-448g_3^2+1392g_2g_3\eta_1}$

https://doi.org/10.4153/CMB-2017-034-7 Published online by Cambridge University Press

References

[1] M. M. Brady, *Meromorphic solutions of a system of functional equations involving the modular group.* Proc. Amer. Math. Soc. **30**(1971), 271–277. http://dx.doi.org/10.1090/S0002-9939-1971-0280712-5

[2] J. H. Conway, *Understanding groups like* $\Gamma_0(N)$. In: Groups, difference sets, and the Monster (Columbus, OH, 1993), Ohio State Univ. Math. Res. Inst. Publ., 4, de Gruyter, Berlin, 1996, pp. 327–343.

[3] A. Elbasraoui and A. Sebbar, *Equivariant forms: structure and geometry.* Canad. Math. Bull. **56**(2013), no. 3, 520–533. http://dx.doi.org/10.4153/CMB-2011-195-2

[4] ———, *Rational equivariant forms.* Int. J. Number Theory **8**(2012), no. 4, 963–981. http://dx.doi.org/10.1142/S1793042112500571

[5] S. Lang, *Elliptic functions.* Second ed., Graduate Texts in Mathematics, 112, Springer-Verlag, New York, 1987. http://dx.doi.org/10.1007/978-1-4612-4752-4

[6] H. Saber and A. Sebbar, *On the critical points of modular forms.* J. Number Theory **132**(2012), no. 8, 1780–1787. http://dx.doi.org/10.1016/j.jnt.2012.03.004

[7] ———, *Equivariant functions and vector-valued modular forms.* Int. J. Number Theory **10**(2014), no. 4, 949–954. http://dx.doi.org/10.1142/S1793042114500092

[8] A. Sebbar and A. Sebbar, *Equivariant functions and integrals of elliptic functions.* Geom. Dedicata **160**(2012), 373–414. http://dx.doi.org/10.1007/s10711-011-9688-7

[9] G. Shimura, *Introduction to the arithmetic theory of automorphic functions.* Kan Memorial Lectures, 1, Publications of the Mathematical Society of Japan, 11, Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, NJ, 1971.

[10] J. Tannery and J. Molk, *Eléments de la théorie des fonctions elliptiques.* Gauthier-Villars, Paris, 1990.

Department of Mathematics and Statistics, University of Ottawa, Ottawa ON K1N 6N5
e-mail: asebbar@uottawa.ca ialsh010@uottawa.ca