ON THE DETERMINATION OF SETS BY SETS OF SUMS OF FIXED ORDER

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1. Introduction. The present investigation is based on two papers: "On the determination of numbers by their sums of a fixed order," by J. L. Selfridge and E. G. Straus **(4)**, and "On the determination of sets by the sets of sums of a certain order," by B. Gordon, A. S. Fraenkel, and E. G. Straus **(2)**.

First of all, we explain the terms implicit in the above titles. Throughout these considerations we use the term "set" to mean "a totality having possible multiplicities," so that two sets will be counted as equal if, and only if, they have the same elements with identical multiplicities. In the most general sense the term "numbers" of **(4)** can be replaced by "elements of any given torsion-free Abelian group."

We can now state the

Problem. For any given (s, n) in $Z \times Z$, with $2 \leqslant s \leqslant n$, we choose arbitrarily an n-set $X = \{x_1, \ldots . x_n\}$, then form the set $P(X) = \{\sigma_i\}$ of all sums of s distinct elements of X and ask whether or not there exists an n-set X' different from X giving rise to the same set of sums as does X. More formally, we can describe the problem as follows: Define a mapping P from the set $\{X\}$ of all n-sets to the set of all $\binom{n}{s}$-sets by the rule:

$$X \to P(X) = \{x_{i_1} + \ldots + x_{i_s}: 1 \leqslant i_1 < \ldots < i_s \leqslant n\}$$

and try to determine (for the given pair (s, n)) whether or not P is one-to-one.

We take as our point of departure the following

THEOREM 4 (of **4**). *For each given pair (s, n) in $Z \times Z$, with $2 \leqslant s \leqslant n$, let the function $f(n, k)$ be defined by:*

$$f(n, k) = \sum_{i=1}^{s} (-1)^{i-1} \cdot \binom{n}{s-i} \cdot i^{k-1},$$

for each k in Z^+. If $f(n, k) \neq 0$, for each k in $\{1, \ldots , n\}$, then the mapping P is one-to-one.

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The above form of $f(n, k)$ was first established in **(2)**. In §2 we give an elementary proof of this result. To describe the contents of §§3 and 4 we introduce some notation. For a given pair (s, n), with $2 \leqslant s \leqslant n$, we define an equivalence relation $\sim$ on the set $\{X\}$ of all n-sets by the rule:

$$X \sim Y \Leftrightarrow P(X) = P(Y).$$

Also, we define $F_s(n)$ as the greatest number of sets which can fall into one equivalence class. In §§3 and 4 we settle the estimates $F_4(12) \leqslant 2$ and $F_3(6) \leqslant 6$ of **(3)** to $F_4(12) = 1$ and $F_3(6) = 4$, respectively. The first of these two results shows that the condition $f(n, k) = 0$, though necessary, is not sufficient. Finally, after **(4; 2)** and the results of this paper, the only cases left open under the orders $s = 3, 4$ are: (3, 27), (3, 486), and (4, 8).

2. The general case. Before stating the main theorem of this section, we introduce some further notation. For given (s, n) in $Z \times Z$, with $2 \leqslant s \leqslant n$, let $N = \binom{n}{s}$, and for each k in Z^+, put

$$S_k = \sum_{i=1}^{n} x_i^{\,k} \quad \text{and} \quad \Sigma_k = \sum_{i=1}^{N} \sigma_i^{\,k}.$$

Also, recall that for an arbitrary set X, $|X|$ denotes its cardinality. In this language we have an alternative way of saying that the mapping $P\colon \{X\} \to \{Y\}$ is one-to-one: viz.,

$$|P^{-1}(Y)| \leqslant 1 \quad \text{for each } Y \text{ in } \{Y\}.$$

Here, it should be borne in mind that $P^{-1}(Y)$ is a collection of sets, where the term "collection" carries the usual meaning of set. Now, let us assume (as we shall within the statement of our theorem) that we are given a set $\{\sigma_i\}$ of sums, N in number. Then for arbitrary k in Z^+, we consider the symmetric function

$$s!\,\Sigma_k = \sum\nolimits_{D(s)} (x_{i_1} + \ldots + x_{i_s})^k,$$

where $D(s)$ means to extend the sum over all ordered s-tuples $(i_1, \ldots, i_s)$ of distinct elements of $\{1, \ldots, n\}$. From symmetric function theory we know that the right side of this equation can be expressed in terms of the power-sum symmetric functions $S_1, S_2, \ldots$. It is further not difficult to see that apart from its coefficient the typical term in the complete expansion of the above-mentioned right side has the appearance

$$S_1^{j_1}\, S_2^{j_2} \ldots S_k^{j_k},$$

$(j_1, j_2, \ldots, j_k)$ a solution of

$$1j_1 + 2j_2 + \ldots + kj_k = k,$$

the $j_i \geqslant 0$ and at most s of them >0. The import of the main theorem is that the term involving S_k, corresponding to $(0, 0, \ldots, 0, 1)$, has a coefficient given by the formula

$$s!f(n, k) = s!\sum_{i=1}^{s} (-1)^{i-1}\cdot\binom{n}{s-i}\cdot i^{k-1}.$$

In **(4)** the authors outline a proof of this fact; however, the proof depends on the theory of group characters of the symmetric group on s symbols. In this paper we give a new and simple derivation, avoiding the theory of group characters entirely. Furthermore, our proof subsumes a protion of **(3)**. For one of the tasks of that paper was to simplify the form of $f(n, k)$.

THEOREM 1. *Let n, s denote given integers, with $n \geqslant s \geqslant 2$. If $f(n, k) \neq 0$, for each k in $\{1, \ldots, n\}$, then $|P^{-1}(Y)| \leqslant 1$, for each given set*

$$Y = \{\sigma_1, \ldots, \sigma_N\}, \quad N = \binom{n}{s}.$$

Proof. We break the proof up into a sequence of lemmas, first introducing some further notation.

Notation. For each given n-set $X = \{x_1, \ldots, x_n\}$ and $\alpha_1, \ldots, \alpha_u$ in Z^+, with $1 \leqslant u \leqslant n$, set

$$S_{\alpha_1, \ldots, \alpha_u} = \sum\nolimits_{D(u)} x_{i_1}^{\alpha_1} \ldots x_{i_u}^{\alpha_u}.$$

LEMMA 1. *The coefficient of $S_{\alpha_1+\ldots+\alpha_u}$ in the expression of $S_{\alpha_1,\ldots,\alpha_u}$ in terms of the power-sum symmetric functions is*:

$$(-1)^{u-1}\cdot(u-1)!.$$

Proof of Lemma 1. Recall the identity

$$S_{\alpha_1,\ldots,\alpha_t} = S_{\alpha_1,\ldots,\alpha_{t-1}}\cdot S_{\alpha_t} - S_{\alpha_1+\alpha_t,\alpha_2,\ldots,\alpha_{t-1}} - \ldots - S_{\alpha_1,\ldots,\alpha_{t-2},\alpha_{t-1}+\alpha_t}$$

and use induction on u.

LEMMA 2.

$$\binom{n}{s-q} = \sum_{v=q}^{s}\binom{n-v}{s-v}\binom{v-1}{q-1}.$$

Proof of Lemma 2. Use induction on n, starting with $n = s$.

Now, returning to the proof of our theorem, we allow the k's of $S_{k_1,\ldots,k_s}$ to be 0, and thus derive:

$$S_{k_1,\ldots,k_{s-1},0} = (n-s+1)S_{k_1,\ldots,k_{s-1}}.$$

For a given t in $\{0, 1, \ldots, s-1\}$, suppose that exactly t of the k's in $\{k_1, \ldots, k_s\}$ are zero and, without loss of generality, put

$$\{k_1, \ldots, k_s\} = \{l_1, \ldots, l_{s-t}, 0, \ldots, 0\},$$

so that all l's are positive integers and the remaining t elements of the right-hand set are all 0. Then we successively apply the above procedure to derive

$$S_{k_1,\ldots,k_s} = (n-s+1)\ldots(n-s+t)S_{l_1,\ldots,l_{s-t}}.$$

We now appeal to Lemma 1 to write:

$$(*)\qquad S_{k_1,\ldots,k_s} = (n-s+1)\ldots(n-s+t)(-1)^{s-t-1}(s-t-1)!\,S_k + \text{terms involving } S_1,\ldots,S_{k-1},$$

where $k = k_1 + \ldots + k_s$. And now, let us suppose explicitly that we are given $Y = \{\sigma_1, \ldots, \sigma_N\}$, where $N = \binom{n}{s}$, and further that we are viewing the elements of $X = \{x_1, \ldots, x_n\}$ as unknown. Hence, choose any k in Z^+ and write:

$$\begin{aligned} s!\,\Sigma_k &= \sum_{D(s)} (x_{i_1} + \ldots + x_{i_s})^k \\ &= \sum_{D(s)} \sum \frac{k!}{k_1!\ldots k_s!} x_{i_1}^{k_1}\ldots x_{i_s}^{k_s} \\ &= \sum \frac{k!}{k_1!\ldots k_s!} \sum_{D(s)} x_{i_1}^{k_1}\ldots x_{i_s}^{k_s} \\ &= \sum \frac{k!}{k!\ldots k_s!} S_{k_1,\ldots,k_s}, \end{aligned}$$

where for the undescribed sigma sums one extends the sum over all non-negative integral solutions of

$$\sum_{i=1}^{s} k_i = k.$$

Now, substitute from $(*)$ to write

$$s!\,\Sigma_k = S_k\left\{\sum_{t=0}^{s-1} \frac{(n-s+t)!}{(n-s)!}(-1)^{s-t-1}(s-t-1)!\sum \frac{k!}{k_1!\ldots k_s!}\right\} + \ldots,$$

where for the second sum one extends the sum over all non-negative integral solutions of

$$\sum_{i=1}^{s} k_i = k,$$

for which exactly t of the k's are 0. To evaluate the inner sum choose just t of the elements $k_1, \ldots, k_s$ to be 0, set $r = s - t$, denote the new non-zero k's by the letter k, as well, and write:

$$\begin{aligned} \sum \frac{k!}{k_1!\ldots k_r!} &= \sum \frac{k!}{k_1!\ldots k_r!} - \binom{r}{1}\sum \frac{k!}{k_1!\ldots k_{r-1}!} \\ &\qquad + \binom{r}{2}\sum \frac{k!}{k_1!\ldots k_{r-2}!} - \ldots \\ &= \sum_{m=0}^{r-1} (-1)^m \binom{r}{m}(r-m)^k \\ &= \sum_{l=1}^{r} (-1)^{r-l}\binom{r}{l} l^k, \end{aligned}$$

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setting $l = r - m$. We remark that the sum on the left in the above sequence of equations extends over all positive integral solutions of $\sum k_i = k$. Thus, we have derived:

$$\sum \frac{k!}{k_1! \ldots k_{s-t}!} = \sum_{l=1}^{s-t} (-1)^{s-t-l} \binom{s-t}{l} l^k,$$

and since there are precisely $\binom{s}{t}$ ways of choosing just t of the elements to be 0 (order immaterial), we have:

$$s!\, \Sigma_k = S_k \left\{ \sum_{t=0}^{s-1} \frac{(n-s+t)!}{(n-s)!} (-1)^{s-t-1} (s-t-1)! \binom{s}{t} \times \sum_{l=1}^{s-t} (-1)^{s-t-l} \binom{s-t}{l} l^k \right\} + \ldots .$$

Now, we are interested in the bracketed expression. Therefore, set $s - t = v$ and reverse the order of summation, getting (with P_0 used as an abbreviation):

$$\begin{aligned} P_0 &= \sum_{v=1}^{s} \frac{(n-v)!}{(n-s)!} (-1)^{v-1} (v-1)! \frac{s!}{v!(s-v)!} \sum_{l=1}^{s-t} (-1)^{v-l} \frac{v!}{l!(v-l)!} l^k \\ &= s! \sum_{l=1}^{s} (-1)^{l-1} l^{k-1} \sum_{v=1}^{s} \binom{n-v}{s-v} \binom{v-1}{l-1} \\ &= s! \sum_{l=1}^{s} (-1)^{l-1} l^{k-1} \binom{n}{s-l}. \end{aligned}$$

However, P_0 is just $s!f(n, k)$. Finally, to complete the proof of our theorem, we observe that: if $f(n, k) \neq 0$, for each k in $\{1, \ldots, n\}$, we can recursively determine the S_k in terms of $\Sigma_1, \ldots, \Sigma_n$. But as soon as we know $S_1, \ldots, S_n$, we equivalently know $a_1, \ldots, a_n$, the elementary symmetric functions in the x's. In a word, under our present hypotheses, $|P^{-1}(Y)| \leqslant 1$.

3. The case (4, 12). In **(2)** the authors established the following general theorem.

THEOREM. *For a fixed $s > 2$ and n sufficiently large, we have $|P^{-1}(Y)| \leqslant 1$, always.*

However, the present state of the art dictates that for small n each case $2 \leqslant s \leqslant n$ be investigated separately. In this regard much has already been done: e.g., sets Y having more than one original under P have actually been constructed for the case $s = 2$, $n = 8$. Also, in cases not as thoroughly investigated, an upper bound has been found for $|P^{-1}(Y)|$, as Y ranges over all possibilities. In every case where $|P^{-1}(Y)| > 1$, for some Y, we have:

(1) $$f(n, k) = 0 \quad \text{for some } k \text{ in } \{1, \ldots, n\}.$$

It is our purpose to present in this section a special case: viz., $s = 4$, $n = 12$, for which the condition (1) holds and yet $|P^{-1}(Y)| \leqslant 1$, always. Heretofore such an example has not been found. Accordingly, let $Y = \{\sigma_1, \ldots, \sigma_N\}$ be given, where $N = \binom{12}{4} = 495$. For the proof we shall also use a weak part of another result established by Gordon, Fraenkel, and Straus.

THEOREM. *If $|P^{-1}(Y)| > 1$, for some set Y, then $|P^{-1}(Y)| > 1$, for some $Y \subset Z$, the ring of integers.*

We begin by choosing arbitrarily k in Z^+ and considering the problem of expressing

$$4!\,\Sigma_k = \sum\nolimits_{D(4)} (x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4})^k$$

in terms of $S_1, \ldots, S_k$. By effecting a simple translation we may assume, without loss of generality, that $S_1 = 0$. The first 12 and the 14th equations are given below. In **(2)** it is established that $|P^{-1}(Y)| \leqslant 2$, for the case $s = 4$, $n = 12$.

From symmetric function theory we shall also need the following lemma, which we state without proof. (For the proof, see p. 6 of **(3)**.)

LEMMA (*). *If $m > n$ and $S_1, S_2, \ldots$ are the power-sum symmetric functions in $x_1, \ldots, x_n$, then*

$$\frac{1}{m} S_m = \sum_{\substack{1p_1 + 2p_2 + \ldots = m;\\ p_m = 0}} (-1)^{\Sigma_p} \frac{S_1^{p_1} S_2^{p_2} \ldots}{1^{p_1} 2^{p_2} \ldots p_1!\, p_2! \ldots},$$

where

(1) $\Sigma_1 = 0,$

(2) $\Sigma_2 = 120 S_2,$

(3) $\Sigma_3 = 48 S_3,$

(4) $\Sigma_4 = -48 S_4 + 84 S_2^2,$

(5) $\Sigma_5 = -120 S_5 + 140 S_2 S_3,$

(6) $\Sigma_6 = 0.S_6 - 120 S_2 S_4 + 40 S_3^2 + 90 S_2^3,$

(7) $\Sigma_7 = 648 S_7 - 714 S_2 S_5 - 350 S_3 S_4 + 420 S_2^2 S_3,$

(8) $\Sigma_8 = 1632 S_8 - 896 S_2 S_6 - 1120 S_3 S_5 - 280 S_4^2 + 0.S_2^2 S_4 + 560 S_2 S_3^2 + 105 S_2^4,$

(9) $\Sigma_9 = -3480 S_9 + 4824 S_2 S_7 + 1176 S_3 S_6 + 1764 S_4 S_5 + 0.S_3^3 - 3024 S_2^2 S_5 - 2520 S_2 S_3 S_4 + 1260 S_2^3 S_3,$

(10) $\Sigma_{10} = -59{,}520 S_{10} + 42{,}840 S_2 S_8 + 29{,}280 S_3 S_7 + 23{,}520 S_4 S_6 + 12{,}600 S_5^2 - 15{,}120 S_2^2 S_6 - 8{,}400 S_3^2 S_4 - 9450 S_2 S_4^2 - 25{,}000 S_2 S_3 S_5 + 3150 S_2^3 S_4 + 6300 S_2^2 S_3^2,$

(11) $\Sigma_{11} = -407{,}352 S_{11} + 222{,}530 S_2 S_9 + 196{,}350 S_3 S_8 + 155{,}100 S_4 S_7 + 150{,}612 S_5 S_6 - 55{,}440 S_2^2 S_7 - 55{,}440 S_3^2 S_5 - 46{,}200 S_3 S_4^2 - 120{,}120 S_2 S_3 S_6 - 97{,}020 S_2 S_4 S_5 + 6930 S_2^3 S_5 + 15{,}400 S_2 S_3^3 + 34{,}650 S_2^2 S_3 S_4,$

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$$(12)\quad \Sigma_{12} = -2{,}203{,}488S_{12} + 964{,}128S_2 S_{10} + 998{,}800S_3 S_9 + 827{,}640S_4 S_8 + 744{,}480S_5 S_7 + 373{,}296S_6^2 - 69{,}300S_4^3 - 178{,}200S_2^2S_8 - 258{,}720S_3^2S_6 - 182{,}952S_2 S_5^2 - 459{,}360S_2 S_3 S_7 -415{,}800S_2 S_4 S_6 - 443{,}520S_3 S_4 S_5 + 15{,}400S_3^4 + 13{,}860S_2^3S_6 + 83{,}160S_2^2S_3 S_5 + 138{,}600S_2 S_3^2S_4 + 51{,}975S_2^2S_4^2,$$

$$(14)\quad \Sigma_{14} = -48{,}517{,}440S_{14} + 14{,}260{,}792S_2 S_{12} + 18{,}521{,}776S_3 S_{11} + 17{,}649{,}632S_4 S_{10} + 15{,}095{,}080S_5 S_9 + 14{,}030{,}016S_6 S_8 + 7{,}008{,}144S_7^2 - 1{,}513{,}512S_2^2S_{10} - 3{,}723{,}720S_3^2S_8 - 3{,}783{,}780S_4^2S_6 - 3{,}531{,}528S_4 S_5^2 - 2{,}270{,}268S_2 S_6^2 - 5{,}005{,}000S_2 S_3 S_9 - 5{,}675{,}670S_2 S_4 S_8 - 5{,}045{,}040S_2 S_5 S_7 - 7{,}687{,}680S_3 S_4 S_7 - 6{,}726{,}720S_3 S_5 S_6 + 45{,}045S_2^3S_8 + 560{,}560S_3^3S_5 + 525{,}525S_2 S_4^3 + 378{,}378S_2^2S_5^2 + 1{,}051{,}050S_3^2S_4^2 + 360{,}360S_2^2S_3 S_7 + 630{,}630S_2^2S_4 S_6 + 840{,}840S_2 S_3^2S_6 + 2{,}522{,}520S_2 S_3 S_4 S_5.$$

The problem then is to decide whether (i) $|P^{-1}(Y)| = 2$, for some Y or (ii) $|P^{-1}(Y)| \leqslant 1$, always. To this end, we first express S_{14} in terms of $S_2, S_3, \ldots, S_{12}$ with the help of

$$\frac{S_{14}}{14} = \frac{1}{24} S_2 S_{12} + \frac{1}{33} S_3 S_{11} + \frac{1}{40} S_4 S_{10} + \frac{1}{45} S_5 S_9 + \frac{1}{48} S_6 S_8 - \frac{1}{80} S_2^2 S_{10} - \frac{1}{54} S_2 S_3 S_9 - \frac{1}{64} S_2 S_4 S_8 - \frac{1}{144} S_3^2 S_8 - \frac{14}{144} S_2 S_6^2 - \frac{1}{90} S_3 S_5 S_6 - \frac{1}{192} S_4^2 S_6 + \frac{1}{384} S_2^3 S_8 + \frac{1}{192} S_2^2 S_4 S_6 + \frac{1}{216} S_2 S_3^2 S_6 - \frac{1}{2304} S_2^4 S_6 + \ldots .$$

(This follows from Lemma (*).) Then we substitute for $S_2, S_3, \ldots, S_{12}$ and collect coefficients of S_6^2 and S_6 to obtain

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$$(14') \qquad \Sigma_{14} = \left\{\frac{2\cdot 3^2\cdot 7\cdot 11\cdot 901}{5\cdot 17\cdot (1093)}\cdot \Sigma_2\right\} S_6^{\,2}$$

$$+\left\{-\frac{7\cdot 2159}{2^2\cdot 3\cdot 17}\Sigma_8 + \frac{7(54{,}779{,}291{,}131)}{2^4\cdot 3^3\cdot 5^2\cdot 17\cdot 29\cdot 16{,}973}\Sigma_3\Sigma_5\right.$$

$$-\frac{7\cdot 36{,}941}{2^4\cdot 3^3\cdot 17\cdot 31}\Sigma_4^{\,2} + \frac{7(3{,}143{,}656{,}870{,}861)}{2^{12}\cdot 3^4\cdot 17\cdot 31\cdot (1093)}\Sigma_2^{\,2}\Sigma_4$$

$$-\frac{7(24{,}630{,}735{,}707{,}477{,}213)}{2^8\cdot 3^5\cdot 5^2\cdot 17\cdot 29\cdot (1093)\cdot 16{,}973}\Sigma_2\,\Sigma_3^{\,2}$$

$$\left.+\frac{7(45{,}882{,}318{,}990{,}331)}{2^{14}\cdot 3^4\cdot 5^4\cdot 17\cdot 31\cdot (1093)}\Sigma_2^{\,4}\right\} S_6 + \ldots .$$

Let us now state the main result.

THEOREM 2. *The equation* (14′) *cannot have two distinct positive real roots, say* S_6 *and* S_6'.

Proof. For otherwise, we can assume, without loss of generality, that $0 \leqslant S_6 < S_6'$.

We can further assume that $Y = \{\sigma_1, \ldots, \sigma_N\}$ does not consist entirely of 0's, since in the contrary case it is clear that $P^{-1}(Y) = \{\theta\}$, where θ is the 12-set consisting entirely of 0's. So, $\Sigma_2 > 0$ and the coefficient of S_6^2 in (14′) is positive. Hence, the coefficient of S_6 must be negative. However, we shall now show that the coefficient of S_6 is always positive. We, first of all, establish the following simple lemma.

LEMMA 3. (i) $\Sigma_8 \leqslant \Sigma_4^2 \leqslant \Sigma_2^2\Sigma_4 \leqslant \Sigma_2^4$,

(ii) $\Sigma_2\,\Sigma_3^2 \leqslant \Sigma_2^2\Sigma_4$,

(iii) $|\Sigma_3\,\Sigma_5| \leqslant \Sigma_2^2\Sigma_4$.

Proof of Lemma 3.

$$\Sigma_2^{\,2} = \Sigma_4 + 2\sum_{1\leqslant i<j\leqslant N}\sigma_i^{\,2}\sigma_j^{\,2} \geqslant \Sigma_4$$

and

$$\Sigma_4^{\,2} = \Sigma_8 + 2\sum_{1\leqslant i<j\leqslant N}\sigma_i^{\,4}\sigma_j^{\,4} \geqslant \Sigma_8.$$

This proves (i). By Schwarz's inequality we have

$$\Sigma_3^{\,2} = \left|\sum_{j=1}^{N}\sigma_j^{\,3}\right|^2 \leqslant \sum_{j=1}^{N}\sigma_j^{\,2}\cdot\sum_{j=1}^{N}\sigma_j^{\,4} = \Sigma_2\,\Sigma_4,$$

and (ii) follows. Finally, the same argument shows that

$$|\Sigma_5| \leqslant \Sigma_2^{1/2}\Sigma_8^{1/2} \leqslant \Sigma_2^{1/2}\Sigma_4.$$

But

$$|\Sigma_3| \leqslant \Sigma_2^{1/2}\Sigma_4^{1/2} \leqslant \Sigma_2^{3/2},$$

whence

$$|\Sigma_3\ \Sigma_5| \leqslant \Sigma_2^2\Sigma_4.$$

Now, after all cancellations have been effected and equation (14′) has been multiplied by the LCD of the resulting fractions, the coefficient of S_6 becomes:

$$\begin{aligned}
&-7{,}466{,}483{,}185{,}927{,}856{,}640{,}000\cdot\Sigma_8\\
&+427{,}642{,}380{,}608{,}641{,}459{,}200\cdot\Sigma_3\ \Sigma_5\\
&-114{,}474{,}275{,}132{,}088{,}960{,}000\cdot\Sigma_4^2\\
&+11{,}605{,}210{,}155{,}034{,}416{,}277{,}500\cdot\Sigma_2^2\Sigma_4\\
&-1{,}221{,}684{,}491{,}090{,}869{,}764{,}800\cdot\Sigma_2\ \Sigma_3^2\\
&+67{,}752{,}172{,}219{,}391{,}261{,}481\cdot\Sigma_2^4.
\end{aligned}$$

And now, in view of our lemma, it is easy to check that the "$\Sigma_2^2\Sigma_4$" term dominates the possibly negative terms in the above sum, whence it is always positive.

But now going back to the equations following Lemma (*), we see that the first five Σ's uniquely determine the first five S's, and once a value of S_6 is known, this and the Σ's uniquely determine S_7, S_8, S_9, S_{10}, S_{11}, and S_{12}. Since the Σ's uniquely determine S_6, as well, they uniquely determine all of ths S's and hence the x's.

4. The case (3, 6).

THEOREM 3. $2 \leqslant F_3(6) \leqslant 4$.

Proof. Without loss of generality, let us assume that all sets under consideration are sets of integers. Then to see that $2 \leqslant F_3(6)$ it suffices to choose any $X = \{x_1, \ldots, x_6\} \subset Z$ not symmetric with respect to the origin, so that $X \neq -X$, and write

$$x_{i_1} + x_{i_2} + x_{i_3} = (-x_{j_1}) + (-x_{j_2}) + (-x_{j_3}),$$

where $1 \leqslant i_1 < i_2 < i_3 \leqslant 6$, $1 \leqslant j_1 < j_2 < j_3 \leqslant 6$, and $\{i_1, i_2, i_3\} \cap \{j_1, j_2, j_3\} = \emptyset$. Since the j indices trace all possibilities as do the i indices, we have $X \sim -X$, whence $2 \leqslant F_3(6)$.

The arguments for demonstrating that $F_3(6) \leqslant 4$ are similar to those used in §3. They are sketched below, omitting the details. The normalization $S_1 = 0$ makes the set of sums symmetric with respect to the origin. Hence, $\Sigma_k = 0$ for odd k. S_2 and S_4 are determined uniquely from the equations for Σ_2 and Σ_4. The equation for Σ_6 is used to express S_6 in terms of S_3^2. The equation

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for Σ_8 involves S_3^2, S_3, S_5, S_8, and the equation for Σ_{10} involves S_3^2, $S_3 S_5$, S_5^2, S_7, S_8, S_{10}. Now, S_7, S_8, and S_{10} can be expressed by terms of lower-order S's, using Lemma (*). This eliminates S_8 in the equation for Σ_8, and permits writing it in the form $S_3 S_5 = \alpha S_3^2 + \beta$, where α and β are rational functions of the Σ's and $\alpha > 0$. It also eliminates S_7, S_8, S_{10} from the equation for Σ_{10}. The resulting equation for Σ_{10} is now multiplied by S_3^2, and $S_3 S_5$ is replaced by $\alpha S_3^2 + \beta$. This gives a quartic equation in S_3, and it turns out that the coefficient of S_3^4 does not vanish identically. Hence, $F_3(6) \leqslant 4$.

We now sharpen our last result by the following theorem.

THEOREM 4. $F_3(6) = 4$.

Proof. To see this, it suffices to exhibit four distinct sets giving rise to the same set of sums. But we wish to do more. We wish to characterize the four-member equivalence classes. And since the one-member classes are solely determined by the sets symmetric to the origin, all classes will be determined up to the appearance of the elements of the sets in two-member classes.

So, let there be given any two equivalent 6-sets $X = \{x_1, \ldots, x_6\}$ and $Y = \{y_1, \ldots, y_6\}$, with elements ordered according to size: i.e.,

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_6 \quad \text{and} \quad y_1 \leqslant y_2 \leqslant \ldots \leqslant y_6.$$

This ordering of X and Y induces an ordering on $P(X)$ and $P(Y)$, so that (with x_{ijk} used as abbreviation for $x_i + x_j + x_k$) we have:

$$x_{123} \leqslant x_{124} \leqslant x_{i_1 i_2 i_3} \quad \text{and} \quad x_{456} \geqslant x_{356} \geqslant x_{i_1 i_2 i_3}$$

for all $1 \leqslant i_1 < i_2 < i_3 \leqslant 6$. We make a similar statement about the y-sums and therefore must have

$$x_{123} = y_{123}, \quad x_{124} = y_{124}, \quad x_{356} = y_{356}, \quad x_{456} = y_{456}.$$

Now, the next smallest x-sum after x_{124} must be either x_{125} or x_{134}. We make a similar statement about the y-sums and thus assert that exactly one of the following three alternatives must prevail:

$$\begin{array}{lcl lcl} x_{123} = y_{123} & & x_{123} = y_{123} & & x_{123} = y_{123} \\ x_{124} = y_{124} & \text{or} & x_{124} = y_{124} & \text{or} & x_{124} = y_{124} \\ x_{125} = y_{125} & & x_{134} = y_{134} & & x_{125} = y_{134} \end{array}$$

Since we are only interested in comparing the sets X and Y up to their being either identical or one being the set of negatives of the other, we can assume, without loss of generality, that the second of these alternatives holds. For putting $z_i = -x_{7-i}$, for each i in $\{1, \ldots, 6\}$, we have $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_6$ if and only if $z_1 \leqslant z_2 \leqslant \ldots \leqslant z_6$, whence

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$$z_{123} = -x_{654} = x_{123}, \quad z_{124} = -x_{653} = x_{124},$$

$$z_{125} = -x_{652} = x_{134}, \quad z_{134} = -x_{643} = x_{125}.$$

$$(*) \qquad x_{256} = y_{256}, \quad x_{356} = y_{356}, \quad x_{456} = y_{456},$$

there are (as a first approximation to solution) 14! ways of pairing the remaining sums. However, the following lemma greatly simplifies the matter.

LEMMA 4. *If alternative* 2 *holds and therefore the complementary couplings* $(*)$; *and if a single pair of* x- *and* y-*sums which are indexed the same way, not including the six mentioned pairs, are equal, then we must have* $Y = X$ *or* $Y = -X$.

Proof. Because of symmetry it suffices to choose the additional equality from the following seven possibilities:

$$x_{125} = y_{125}, \quad x_{126} = y_{126}, \quad x_{135} = y_{135}, \quad x_{136} = y_{136},$$

$$x_{145} = y_{145}, \quad x_{146} = y_{146}, \quad x_{156} = y_{156}.$$

For each $i \in \{1, \ldots, 6\}$, put $\Delta_i = x_i - y_i$. Then

$$\Delta_3 = \Delta_4 = \Delta, \text{ say};$$

$$x_{12} = y_{12} - \Delta;$$

and

$$x_{56} = y_{56} - \Delta.$$

The diagram

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 2\Delta & -\Delta & -\Delta & -\Delta & -\Delta_5 & -\Delta_6 \end{array}$$

is to be interpreted as follows. Add the (signed) Δ's to the x's under which they stand to obtain

$$y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5 \quad y_6.$$

The hypothesis and any one of the above seven equations imply that either $\Delta = 0$ (in which case $Y = X$) or one of the two patterns

$$(*_1) \qquad \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 2\Delta & -\Delta & -\Delta & -\Delta & -\Delta & 2\Delta \end{array}$$

$$(*_2) \qquad \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 2\Delta & -\Delta & -\Delta & -\Delta & 2\Delta & -\Delta \end{array}$$

must prevail. Therefore, suppose $\Delta \neq 0$ and $(*_1)$. Then equal sums correspond to the indices 123, 124, 125, 134, 135, 145, 456, 356, 346, 256, 246, 236 while unequal sums correspond to 126, 136, 146, 156, 345, 245, 235, 234. Hence

$$x_{126} = y_{234}, \quad x_{136} = y_{235}, \quad x_{146} = y_{245}, \quad x_{156} = y_{345}.$$

From the first of these equations

$$\Delta = -(x_1 - x_3 - x_4 + x_6)/3,$$

while from the second we derive

$$\Delta = -(x_1 - x_2 - x_5 + x_6)/3.$$

Therefore

$$x_3 + x_4 = x_2 + x_5.$$

But then

$$\begin{aligned} 3\Delta &= -x_1 + x_3 + x_4 - x_6 \\ &= -x_1 - x_2 - x_3 - x_4 - x_5 - x_6 + x_2 + x_5 + 2(x_3 + x_4) \\ &= 3(x_3 + x_4) = 3(x_2 + x_5), \end{aligned}$$

whence

$$\Delta = x_3 + x_4 = x_2 + x_5.$$

So,

$$x_3 - y_3 = x_3 + x_4 \to y_3 = -x_4,$$

$$x_4 - y_4 = x_3 + x_4 \to y_4 = -x_3,$$

$$x_2 - y_2 = x_2 + x_5 \to y_2 = -x_5,$$

and

$$x_5 - y_5 = x_2 + x_5 \to y_5 = -x_2.$$

Further,

$$y_1 = x_1 + 2\Delta = x_1 + (x_2 + x_5) + (x_3 + x_4) = -x_6,$$

and

$$y_6 = x_6 + 2\Delta = x_6 + (x_2 + x_5) + (x_3 + x_4) = -x_1.$$

Thus $Y = -X$. If $\Delta \neq 0$ and $(*_2)$ holds, the same argument leads to

$$y_1 = -x_5, \quad y_2 = -x_6, \quad y_3 = -x_4,$$

$$y_4 = -x_3, \quad y_5 = -x_1, \quad y_6 = -x_2.$$

This proves the lemma.

Let us call either of the cases $Y = X$ and $Y = -X$ trivial. Then any other case will be called non-trivial. With the help of our lemma we argue that the first four equations of any pairings of the x- and y-sums which would lead to a non-trivial case must be:

$$x_{123} = y_{123}, \quad x_{124} = y_{124}, \quad x_{134} = y_{134}, \quad x_{125} = y_{234}.$$

Now, there are three possibilities for the fifth equation:

$$\text{(I)}\ x_{234} = y_{125}, \quad \text{(II)}\ x_{126} = y_{125}, \quad \text{or} \quad \text{(III)}\ x_{135} = y_{125}.$$

Case I. Here we have

$$x_{125} = y_{234} \qquad x_{346} = y_{156}$$

and

$$x_{234} = y_{125} \qquad x_{156} = y_{346}$$

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Subtracting the second of the right pair of equations from the first of the left pair we have

$$x_2 - y_2 = x_6 - y_6.$$

Hence, the pattern $(*_2)$ must prevail. So, once more either $\Delta = 0$ (in which case $Y = X$) or $\Delta \neq 0$ and $Y = -X$.

Case II. Here the equations

$$x_{125} = y_{234} \quad \text{and} \quad x_{126} = y_{125}$$

imply

$$y_5 = x_6 - \Delta.$$

So,

$$y_1 = x_1 + 2\Delta, \quad y_2 = x_2 - \Delta, \quad y_3 = x_3 - \Delta,$$

$$y_4 = x_4 - \Delta, \quad y_5 = x_6 - \Delta,$$

whence $-y_6 = -x_5 - 2\Delta$, i.e., $y_6 = x_5 + 2\Delta$.

If $\Delta = 0$, the problem is settled. If $\Delta \neq 0$, then those sums not paired are summarized by

$$\begin{aligned} y_{126} &= x_{125} + 3\Delta, & y_{234} &= x_{234} - 3\Delta, \\ y_{136} &= x_{135} + 3\Delta, & y_{235} &= x_{236} - 3\Delta, \\ y_{146} &= x_{145} + 3\Delta, & y_{245} &= x_{246} - 3\Delta, \\ y_{156} &= x_{156} + 3\Delta, & y_{345} &= x_{346} - 3\Delta. \end{aligned}$$

Without loss of generality, we can assume that $\Delta > 0$, whence

$$x_{125} < y_{125} \leqslant y_{136} \leqslant y_{146} \leqslant y_{156}$$

and therefore we must have $x_{125} = y_{234}$, the smallest y-sum in the complementary set. But then

$$x_{125} = x_{234} - 3\Delta, \qquad x_{135} = x_{236} - 3\Delta,$$

whence

$$\Delta = x_2 + x_6 = x_3 + x_4$$

and $Y = -X$, as in the proof of the lemma for the pattern $(*_2)$.

Case III. Here the reader will find it instructive to construct trees (as we have partially done at the end of this exposition) to show all possible orderings of the x- and y-sums induced by the orderings of the elements. Straightforward analysis then reveals that (i) $Y = X$ or (ii) $Y = -X$ or (iii) one of the following five descriptions must fit X and Y, where A, B, α, β, and d are parameters used for this purpose (i.e., of describing X and Y):

(1) $X = \{A, A + 3B, A + 4B, A + 5B, A + 6B, A + 8B\}$,
$Y = \{A + 2B, A + 2B, A + 3B, A + 4B, A + 6B, A + 9B\}$,
$A = x_1, \quad B = -\Delta, \quad \text{and } \Delta_5 = 0$;

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(2) $X = \{A, A + 6B, A + 9B, A + 12B, A + 15B, A + 24B\}$,
$Y = \{A + 4B, A + 4B, A + 7B, A + 10B, A + 16B, A + 25B\}$,
$A = x_1, \quad B = \Delta_5, \quad \text{and } \Delta = -2\Delta_5$;

(3) $X = \{A, A + 8B, A + 9B, A + 10B, A + 13B, A + 16B\}$,
$Y = \{A + 4B, A + 6B, A + 7B, A + 8B, A + 12B, A + 19B\}$,
$A = x_1, \quad B = -\Delta_5, \quad \text{and } \Delta = 2\Delta_5$;

(4) $X = \{A, A + 4B, A + 5B, A + 6B, A + 8B, A + 11B\}$,
$Y = \{A + 2B, A + 3B, A + 4B, A + 5B, A + 8B, A + 12B\}$,
$A = x_1, \quad B = -\Delta, \quad \text{and } \Delta_5 = 0$;

(5) $X = \{\alpha, \frac{1}{2}(\alpha + \beta), \frac{1}{2}(\alpha + \beta) + d, \frac{1}{2}(\alpha + \beta) + 2d, \beta, \frac{1}{2}(3\beta - \alpha) - d\}$,
$Y = \{\alpha + 2d, \frac{1}{2}(\alpha + \beta) - d, \frac{1}{2}(\alpha + \beta), \frac{1}{2}(\alpha + \beta) + d, \beta, \frac{1}{2}(3\beta - \alpha)\}$,
$\alpha = x_1, \quad \beta = x_5, \quad d = -\Delta, \quad \text{and } \Delta_5 = 0.$

Making the convention that the y-sums shall always be on the left in the tree analysis, we are able to list the first sequence of 10 equations giving rise to each of these families as follows:

(1)	(2)	(3)
$y_{123} = x_{123}$	$y_{123} = x_{123}$	$y_{123} = x_{123}$
$y_{124} = x_{124}$	$y_{124} = x_{124}$	$y_{124} = x_{124}$
$y_{134} = x_{134}$	$y_{134} = x_{134}$	$y_{134} = x_{134}$
$y_{234} = x_{125}$	$y_{234} = x_{125}$	$y_{234} = x_{125}$
$y_{125} = x_{135}$	$y_{125} = x_{135}$	$y_{125} = x_{135}$
$y_{135} = x_{126}$	$y_{135} = x_{145}$	$y_{135} = x_{145}$
$y_{145} = x_{136}$	$y_{145} = x_{126}$	$y_{145} = x_{126}$
$y_{235} = x_{145}$	$y_{126} = x_{136}$	$y_{235} = x_{136}$
$y_{126} = x_{146}$	$y_{136} = x_{146}$	$y_{245} = x_{146}$
$y_{136} = x_{156}$	$y_{146} = x_{156}$	$y_{126} = x_{156}$

(4)	(5)
$y_{123} = x_{123}$	$y_{123} = x_{123}$
$y_{124} = x_{124}$	$y_{124} = x_{124}$
$y_{134} = x_{134}$	$y_{134} = x_{134}$
$y_{234} = x_{125}$	$y_{234} = x_{125}$
$y_{125} = x_{135}$	$y_{125} = x_{135}$
$y_{135} = x_{145}$	$y_{135} = x_{145}$
$y_{145} = x_{126}$	$y_{145} = x_{234}$
$y_{235} = x_{234}$	$y_{126} = x_{235}$
$y_{126} = x_{235}$	$y_{136} = x_{245}$
$y_{136} = x_{245}$	$y_{146} = x_{345}$

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Now, we shall see that cases (1), (2), and (4) are but special cases of case (5). We get (1) from (5) simply by setting $\alpha = A$, $d = B$, and $\beta = A + 6B$. To obtain (4) from (5) set $\alpha = A$, $d = B$, and $\beta = A + 8B$. To see that (2) is a special case of (5), we argue as follows: For each i in $\{1, \ldots, 6\}$,

$$-3y_{7-i} = \left(-y_{7-i} + \sum_{j=1}^{6} y_j\right) - 2y_{7-i},$$

whence it follows that under (2)

$$-Y = \{A + 18B, A + 18B, A + 15B, A + 12B, A + 6B, A - 3B\}$$

and

$$X = \{A + 24B, A + 15B, A + 12B, A + 9B, A + 6B, A\},$$

where we have reversed the ordering of the elements to show off the fact that (2) is a special case of (5), up to either one or perhaps both of the sets being replaced by the corresponding sets of negatives. Simply take $\alpha = A + 24B$, $\beta = A + 6B$, and $d = -B$.

Owing to the condition $S_1 = 0$, we can also describe (5) as a two-parameter family of classes. To this end, set

$$a = \tfrac{1}{2}(\alpha + \beta), \quad b = \tfrac{1}{2}(\alpha + \beta) + d, \quad \text{and } c = \beta.$$

Then

$$2b - a = \tfrac{1}{2}(\alpha + \beta) + 2d, \quad 2a - c = \alpha, \quad 2c - b = \tfrac{1}{2}(3\beta - \alpha) - d,$$
$$2a - b = \tfrac{1}{2}(\alpha + \beta) - d, \quad 2c - a = \tfrac{1}{2}(3\beta - \alpha), \quad 2b - c = \alpha + 2d.$$

Now, it easily follows (because of $S_1 = 0$) that $a + b + c = 0$, whence

$$\begin{aligned} X &= \{a, b, -a - b, 2b - a, -2a - 3b, 3a + b\}, \\ Y &= \{a, b, -a - b, 2a - b, -2b - 3a, 3b + a\}, \end{aligned} \tag{5}$$

where now the sets are not ordered according to size.

The condition $S_1 = 0$ implies

$$A = -(28/3)B \tag{$*'$}$$

for family (3). Thus, we have proved the following theorem.

THEOREM 5. *There are two distinct families of four-member classes of sets. One family is described by two parameters a and b as by* (5) *above. The other family is a one-parameter family described by*

$$\begin{aligned} X &= \{A, A + 8B, A + 9B, A + 10B, A + 13B, A + 16B\}, \\ Y &= \{A + 4B, A + 6B, A + 7B, A + 8B, A + 12B, A + 19B\}, \end{aligned} \tag{3}$$

where $B \neq 0$ and A is a rational multiple of B according to ($*'$).

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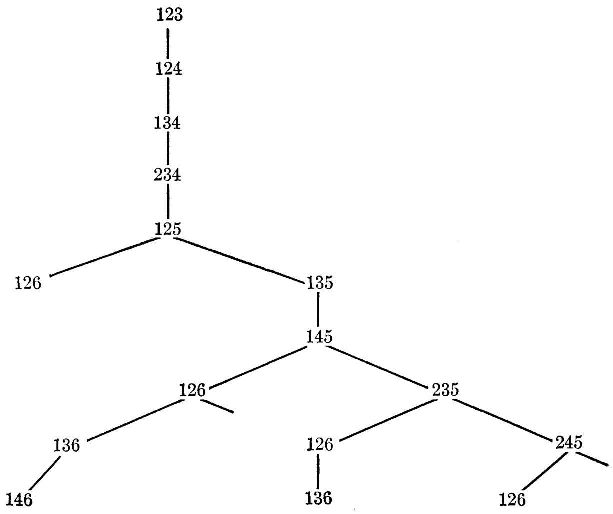

FIGURE 1. The Y tree.

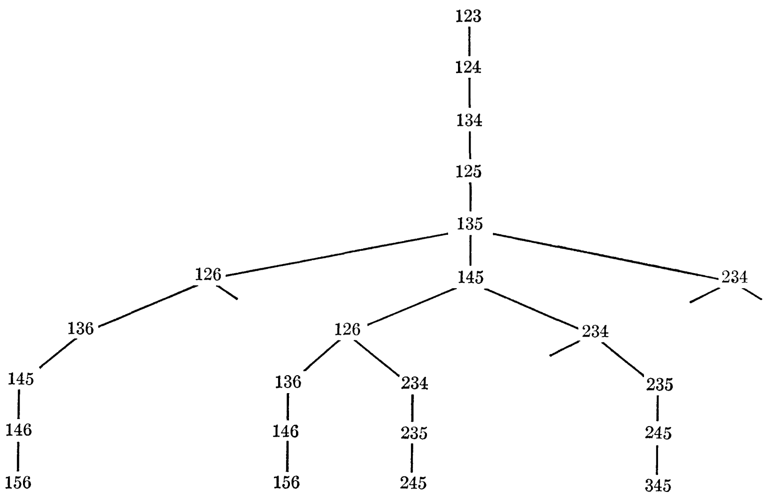

FIGURE 2. The X tree.

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