NUMERICAL METHODS IN CONVECTION THEORY

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SUMMARY

Two and three-dimensional computations have enlarged our understanding of nonlinear convection, particularly in Boussinesq fluids. However, we cannot adequately predict the relationship between convective heat transport and the superadiabatic temperature gradient. Nor is there any indication of a preferred length scale, other than the depth of the convecting layer, in a compressible fluid.

1. INTRODUCTION

The standard procedure for calculating the structure of stellar convection zones is to use mixing length theory, calibrated to fit the sun and neighbouring stars on the main sequence. Mixing length theory is based on plausible physical assumptions and seems to provide qualitatively acceptable results but, as Dr. Gough has emphasized, it lacks any firm theoretical basis. The principal need in astrophysical convection is for a soundly based theory that can confirm, or replace, the procedure now adopted. In particular, we would like to establish the functional relationship between the convective heat transport and the superadiabatic temperature gradient, and to determine the preferred scale of convective motion. Spiegel's (1971b, 1972) excellent review of astrophysical convection contains a thorough discussion of the basic fluid dynamical problem and includes a full list of references, which has been brought up to date by Gough (1976). So I shall limit myself to describing recent progress towards understanding nonlinear convection by solving model problems numerically on a computer.

Direct observation of solar convection reveals cellular patterns. Hot gas rises, cold gas sinks and the lifetime of an individual cell is of the same order as the time taken for fluid to turn over in it. The photospheric granulation has a horizontal scale similar to the local density or pressure scale height, and comparable with the thickness of the strongly superadiabatic layer at the top of the convective zone. Supergranules have diameters about 15 times larger; their relationship to features with strong magnetic fields implies that they correspond to more deep-seated convection. There are also suggestions of motion on a scale comparable to the depth of the convection zone, while speckle photometry indicates that there may be large scale convective cells in the outer layers of red giants.

Observations provide few constraints on the relationship between heat flux and temperature gradient. Nor can the parameters appropriate for astrophysical convection be modelled in laboratory experiments. Hence we must attempt to solve the governing equations which, since they are nonlinear, have to be tackled on a computer. The full problem is still too difficult. So it is necessary to make various simplifying geometrical and fluid dynamical approximations. We hope that a better description of astrophysical convection will eventually emerge from the results of a sequence of idealized numerical experiments.

2. THE IDEALIZED PROBLEM

Let us consider convection in a horizontal layer, heated uniformly from below and confined between the planes z = 0, d, where the z-axis points vertically upwards. In the absence of motion the superadiabatic temperature gradient

$$\beta = -\left[\frac{dT}{dz} + \beta_{ad}\right] ,$$

where T is the temperature and the adiabatic gradient

$$\beta_{ad} = \frac{g\alpha T}{C_p}$$

Here $\underline{g}$ is the gravitational acceleration, C_p the specific heat at constant pressure and α the coefficient of thermal expansion (for a perfect gas $\alpha = 1/T$). In the Boussinesq approximation we assume that the layer depth d is much smaller than the temperature scale height $C_p/g\alpha$ and that the Mach number $U/c_s \ll 1$ (where U is a typical velocity and c_s is the velocity of sound): then the velocity $\underline{u}$ and density ρ satisfy the equations

$$\nabla.\underline{u} = 0 \qquad , \qquad \rho = \rho_0\left[1 - \alpha(T - T_0)\right] ,$$

where ρ_0, T_0 are constant, and the configuration is described by two dimensionless parameters, the Rayleigh number

$$R = \frac{g\alpha\beta d^4}{\kappa\nu}$$

and the Prandtl number $\sigma = \nu/\kappa$, where κ, ν are the thermal and viscous diffusivities. The heat flux can be expressed in terms of the dimensionless number

$$N = \frac{F - \kappa\beta_{ad}}{\kappa\beta} ,$$

where the total heat flux is $C_p\rho F$. For an infinite layer N is a function of R and σ only.

In most laboratory experiments the convecting fluid is confined between rigid boundaries at which $\underline{u}$ vanishes, and the resulting flow is dominated by viscous boundary layers. These boundary conditions are inappropriate for stars and it is usual to assume that the tangential stress and normal velocity vanish at the surfaces z = 0, d, which are held at fixed temperatures $T_0 + \beta d$, T_0. These "free" boundary conditions are dynamically fairly passive and mathematically convenient. Nevertheless, any technique or theory should be capable of describing experimental results correctly before it is applied to astrophysical convection.

A bewildering array of power laws has been put forward for the function $N(R, \sigma)$. For $R \gg 1$ an asymptotic upper bound with $N \sim R^{\frac{1}{2}}$ has been established (Howard 1963, Busse 1969). At high Prandtl numbers $N \sim R^{1/3}$ for free boundaries but the radiative conductivity is high in stars and the Prandtl number is therefore small. For $\sigma \ll 1$ we might expect that the energy flux should not depend explicitly

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on the viscosity ν , so that $N = N(S)$, where $S = \sigma R$. In the sun, $\sigma \approx 10^{-9}$ but S is typically of order 10^{12}. Arguments can be found for suggesting asymptotic power laws of the form $N \sim S^r$ with $r = 2$, ½, 1/3, 1/4, 1/5 (Spiegel 1971 a,b; Gough and Weiss 1976; Jones et al. 1976; Gough et al. 1975) but it is not obvious which, if any, of these exponents is correct.

3. BOUSSINESQ CONVECTION

In the Boussinesq approximation the pressure can be eliminated by taking the curl of the equation of motion. The time-dependent equations then become

$$\frac{\partial T}{\partial t} = -\underline{u}\cdot(\nabla T + \beta_{ad}\,\underline{e}_z) + \kappa\nabla^2 T$$

and

$$\frac{\partial \underline{\omega}}{\partial t} = \nabla\wedge(\underline{u}\wedge\underline{\omega}) - \alpha\nabla T\wedge\underline{g} + \nu\nabla^2\underline{\omega}$$

together with $\nabla.\underline{u} = 0$, where the vorticity $\underline{\omega} = \nabla\wedge\underline{u}$ and $\underline{e}_z$ is a unit vector in the z-direction. In two dimensions, with motion confined to the xz plane and independent of the y co-ordinate, the vorticity has only a y-component and the velocity can be expressed in terms of a stream function ψ such that

$$\underline{u} = \nabla\psi\wedge\underline{e}_y \quad , \qquad \omega = -\nabla^2\psi \ ,$$

where $\underline{e}_y$ is a unit vector in the y-direction.
The vorticity equation then reduces to

$$\frac{\partial\omega}{\partial t} = -\underline{u}.\nabla\omega - g\alpha\frac{\partial T}{\partial x} + \nu\nabla^2\omega \ .$$

3.1 Rigid boundaries

Convection sets in at the critical Rayleigh number R_c and two-dimensional solutions for $R \lesssim 1000\ R_c$ have been available for some time (e.g. Fromm 1965, Schneck and Veronis 1967, Plows 1968). Busse (1967) first showed that two-dimensional solutions at infinite Prandtl number may be unstable to three-dimensional perturbations and the development of rolls into three-dimensional and time-dependent regimes has been studied experimentally (e.g. Busse and Whitehead 1974) for fluids with high Prandtl numbers. The stability of two-dimensional rolls was systematically investigated by Clever and Busse (1974): for Prandtl numbers of order unity, the rolls develop a wavelike oscillatory instability when $R \gtrsim 3.5\ R_c$. The most thoroughly investigated case is convection in air ($\sigma = 0.7$) for which Veltishchev and Želnin (1975) and Lipps (1976) have computed three-dimensional solutions with $R \lesssim 15\ R_c$. At low Rayleigh numbers Lipps' numerical experiments show the development of rolls whose preferred width differs from that which maximizes the heat transport. As R is increased, the oscillatory instability appears and solutions become time-dependent. For $R \approx 15\ R_c$, motion is three-dimensional and aperiodic. However the change from two to three dimensional convection does not greatly affect the time-averaged Nusselt number. These numerical results are all supported by experiments (Willis and Deardorff 1967, 1970; Krishnamurti 1970a,b, 1973; Brown 1973). Unfortunately, apart from the experiments by Rossby (1969), few results are available for low Prandtl number convection.

3.2 Free boundaries

Convection in two-dimensional rolls has been studied in numerical experiments with $R \leqslant 1000\ R_c$ (Fromm 1965, Veronis 1966, Moore and Weiss 1973). For high Prandtl number ($\sigma >> R^{2/3}$) there are steady solutions with $N \propto R^{1/3}$, which are apparently stable (Straus 1972). For $\sigma << 1$, the Nusselt number depends only on R and for R>>1 $N \sim R^{0.36}$ (Moore and Weiss 1973). In these laminar solutions the vorticity ω is nearly constant on the streamlines. The nonlinear term in the vorticity equation remains small even when the Reynolds number is large: the rolls behave like flywheels and are slowly accelerated until, after they have turned over many times, the buoyancy torque is eventually balanced by friction. However, the oscillatory instability sets in near the critical Rayleigh number for $\sigma << 1$ (Busse 1972) and the rolls should break down into three-dimensional cells.

In the two-dimensional solutions, rising and falling plumes are exactly symmetrical but this symmetry is no longer present in, say, a hexagonal cell where fluid can rise in a central column and sink around the perimeter of the cell. It was conjectured that this geometrical change might affect the physics so that N could depend on S for $\sigma << 1$. So we investigated axisymmetric convection in a cylindrical cell (Jones et al. 1976). This idealized model is mathematically two-dimensional but geometrically three-dimensional, though the cells cannot be packed together to fill a plane. Referred to cylindrical polar co-ordinates (r,φ,z) the velocity is given by a Stokes stream function $\psi(r,z)$ such that

$$\underline{u} = \nabla\left(\frac{\psi}{r}\right) \wedge \underline{e}_\varphi ,$$

where $\underline{e}_\varphi$ is a unit vector in the φ-direction, and the vorticity

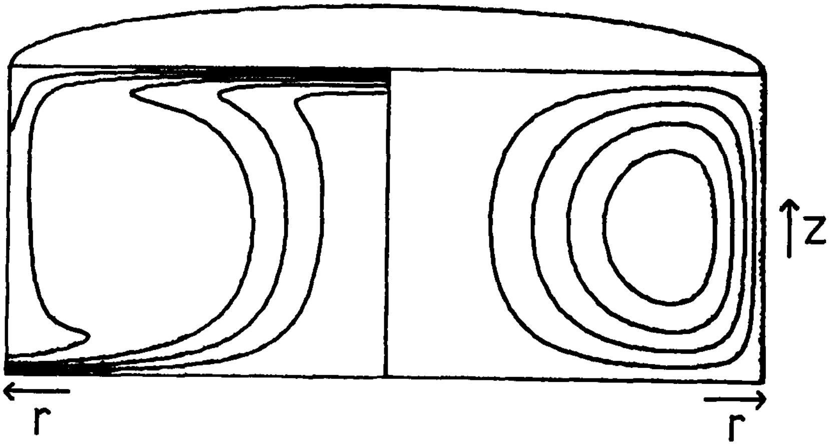
$$\underline{\omega} = r\Omega\, \underline{e}_\varphi ,$$

where

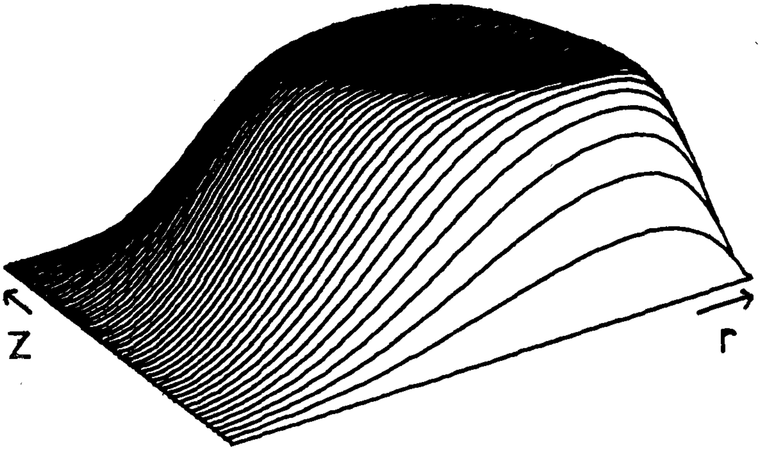
$$\frac{\partial\Omega}{\partial t} = -\underline{u}.\nabla\Omega - \frac{g\alpha}{r}\frac{\partial T}{\partial r} + \nu\nabla.\left[\frac{1}{r^2}\nabla(r^2\Omega)\right] .$$

We found, however, that the convective flux was similar to that for two-dimensional rolls. For high Prandtl numbers $N \propto R^{1/3}$ again, while for $\sigma << 1$ $N \sim R^{0.4}$ approximately. So the Nusselt number is still independent of σ and approaches closer to the upper bound. The form of the solutions is displayed in Fig. 1, which shows streamlines, isotherms and profiles of the modified vorticity Ω for a steady solution with $\sigma = 0.01$, $R = 100\ R_c$. Vortex tubes are stretched as they move away from the axis with the fluid and Ω is nearly constant along streamlines. So flywheel solutions exist in a cylinder and would also, presumably, appear in hexagons.

Are these solutions stable? Jones and Moore (1977) have recently shown that the axisymmetric flow is unstable to non-axisymmetric perturbations. Thus a cylindrical cell can fragment into sectors, like unstable vortex rings in laboratory experiments (Widnall and Sullivan 1973, Widnall, 1975). As the Reynolds number increases, three-dimensional convection cells should therefore become unstable and split up. Such a phenomenon is observed in the sun: large granules explode and break up into smaller



(a)



(b)

Figure 1. Axisymmetric convection in a cylindrical cell. Results for $R = 100\ R_c$, $p = 0.01$. (a) Isotherms and streamlines: equally spaced contours of T(left) and ψ. (b) Profiles of the modified vorticity Ω, which is nearly constant along the streamlines.

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cells (Musman 1972). In a tesselated convection pattern there may also be collective instabilities which allow vortex rings to reconnect, so that cells are swallowed up and disappear.

3.3 The modal approximation

An alternative approach to three-dimensional convection has been to adopt a truncated modal expansion. For example, the vertical velocity w can be expanded in eigenfunctions of the two-dimensional Laplacian operator:

$$w(x, y, z) = \sum_i W_i(z) f_i(x, y) \quad , \qquad \nabla^2 f_i(x, y) = -a_i^2 f_i \, .$$

The single mode expansion, normalized so that $\overline{f} = 0$, $\overline{f^2} = 1$, $\overline{f^3} = 2C$ (where the bars denote horizontal averages) has been studied in great detail (Gough et al. 1975; Toomre et al. 1977) and numerical solutions have been obtained for Rayleigh numbers up to 10^{25}. In this approximation the plan form of a convection cell is prescribed by the linear eigenfunction, and enters the equations through the parameter C. For two-dimensional rolls C = 0 and the equations reduce to the mean field approximation; for cylinders C = 0.18 and for hexagons C = 0.41.

With rigid boundaries the results for a single mode agree quite well with experiments; with free boundaries $N \sim R^{1/3}$ when $\sigma \gg 1$ but $N \sim (S \ln S)^{1/5}$ for $R^{-1} \ll \sigma \ll 1$. The imposed plan form generates a large nonlinear term in the vorticity equation. If the flow is constrained only to be laminar and steady then it can adjust its plan form to make $\nabla \wedge (\underline{u} \wedge \underline{\omega})$ very small and the effective dissipation can therefore be reduced.

The modal expansion and the flywheel solutions are two extremes. We might expect that instabilities would limit the lifetimes of three dimensional convection cells, so that they are comparable with the turnover time and laminar flywheel solutions cannot be attained. Then N should depend on S, though it is not clear what power law would hold. This problem will not be resolved until the results of fully three-dimensional computations have become available. Meanwhile, the power law derived from mixing length theory ($N \sim S^{\frac{1}{2}}$ for $S \gg 1$) remains as good as any other.

4. COMPRESSIBLE CONVECTION

The Boussinesq approximation is manifestly inadequate for stellar atmospheres that extend over many scale heights (the density increases by a factor of 10^6 in the solar convection zone). It is commonly supposed that the dimensions of convection cells should be of the same order as the local density or pressure scale height. This assumption fits the photospheric granulation and some physical arguments can be adduced to support it (Schwarzschild 1961; Weiss 1976). In mixing length theory (which is essentially Boussinesq) the mixing length is generally set equal to some multiple of the local pressure scale height. It would be comforting to have some theoretical justification for this choice of length scale.

Linear theory gives no help: in a polytropic atmosphere convection sets in with a horizontal scale that is comparable with the layer depth, even for the complete atmosphere where the scale height shrinks to zero at the upper boundary (Spiegel 1965; Gough et al. 1976; Graham and Moore 1977). At supercritical Rayleigh numbers modes with smaller horizontal scales have higher growth rates, at least when dissipation is ignored (see Spiegel 1972). Böhm (1967) discussed the growth rates of linear modes in a model of the solar convection zone, neglecting turbulent viscosity, and found that the growth rate increased monotonically with the horizontal wavenumber. Vandakurov (1975a,b) has included the effects of an eddy viscosity and found a maximum growth rate for cells with a horizontal scale intermediate between those of granules and supergranules. In these gravest modes there is no reversal of the velocity, though the amplitude is strongly peaked near the surface. A preliminary study of the marginal stability problem (Böhm 1975) indicates that there may be internal nodes but their interpretation is obscure. (The reversal in the temperature perturbation reported by Vickers (1971) is apparently due to numerical error.) Smaller length scales seem to be produced not by the density variation but by the strongly superadiabatic gradient, coupled with ionization, near the top of the convective zone.

In nonlinear studies sound waves can be filtered out by using the anelastic approximation (Gough 1969) which is valid provided the Mach number remains small. This has been applied, using the modal approximations to study (inefficient) convection in A-type stars (Latour et al. 1976; Toomre et al. 1976). However, no careful study of the transition from Boussinesq to compressible convection has yet been carried out.

Dr. Graham will describe his numerical experiments on fully compressible nonlinear convection in two and three dimensions. For steady convection in two-dimensional rolls the eye of an eddy is no longer at the centre of the cell but is displaced downwards and towards the sinking plume (Graham 1975). This asymmetry is observed in solar granules, which show a broad column of hot gas, rising at their centres, surrounded by narrower, more rapidly sinking ring of cold material (Kirk and Livingston 1968; Deubner 1976). Graham finds no evidence for small scale motion: convective cells extend across the entire layer, even when the density varies by a factor of 30. In studying compressible convection it is most straightforward to assume that the dynamic viscosity $\rho\nu$ is uniform. Then the viscous term dominates the equation of motion near the upper boundary, where the density is small (Gough et al. 1976). If the aim is to represent turbulent dissipation by an eddy viscosity, then the kinematic viscosity ν can be obtained from a model of the convection zone. For the sun, ν is roughly constant (Cocke 1967, Böhm 1975). However, Graham finds that the cell size is not altered by setting ν constant across the convecting layer.

So far, the only suggestion of small scale motion has come from some nonlinear calculations by Deupree (1976), whose resolution is too coarse for the results to be credible. Unless further computations on compressible convection reveal some new pattern of behaviour, we shall have to suppose that the observed scales of convection in granules and supergranules are caused by boundary layers near the surface of the sun, rather than by the changing density scale height. If so, the mixing length cannot be locally determined and mixing length theory is, at best, reliable only near the surfaces of main sequence stars.

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