PERIODIC RADICAL PRODUCTS OF TWO LOCALLY NILPOTENT SUBGROUPS†

by BURKHARD HÖFLING

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1. Introduction. A group G is the *product of its subgroups A and B* if G equals the set $AB = \{ab \mid a \in A, b \in B\}$. A subgroup H of G is called *prefactorized* if it is the product of a subgroup of A and a subgroup of B; thus H is prefactorized if and only if $H = (H \cap A)(H \cap B)$. A prefactorized subgroup H of G is *factorized* if it contains $A \cap B$. If H is any subgroup of $G = AB$, then the intersection X of all factorized subgroups of G containing H is itself factorized; see for example [**2**, Lemma 1.1.2]. This subgroup, which is evidently the smallest factorized subgroup of G which contains H, is called the *factorizer* of H in $G = AB$.

Let π be a set of primes. Often, the question arises whether a Sylow π-subgroup, that is, a maximal π-subgroup, of $G = AB$ is prefactorized. Wielandt has shown in [**17**] that every finite soluble product G of two subgroups possesses prefactorized Sylow π-subgroups; see also [**2**, Lemma 1.3.2]. These results have been generalized to U-groups by Černikov [**4**, Lemma 5]. Here a locally finite group G a U-group if, for every subgroup H of G and every set of primes π, the maximal π-subgroups of H are conjugate in H (note that by [**8**], such a group has a finite normal series with locally nilpotent factors). Moreover, Franciosi, de Giovanni and Sysak have proved in [**6**] that, if the periodic radical group G is the product of a locally nilpotent subgroup A and a hypercentral subgroup B, then there exist prefactorized Sylow π-subgroups in G.

On the other hand, Sysak [**15**] (see also [**16**]) has constructed a locally finite group $G = AB = AN = BN$, where N is a normal Sylow p-subgroup of G, and A and B are locally nilpotent p'-groups. Thus the unique Sylow p-subgroup N of G may have trivial intersection with A and B and hence is not prefactorized.

Another question which has attained considerable interest is whether the Hirsch-Plotkin radical, that is, the maximal locally nilpotent normal subgroup of a product $G = AB$ of two locally nilpotent subgroups is factorized. This could be answered positively under various hypotheses on G, including the case when G is finite or an U-group; for an overview of these and related results see for example [**1**] and [**2**]. Moreover, the Hirsch-Plotkin radical of a periodic radical group $G = AB$ is also factorized if one of the subgroups A and B is hypercentral [**6**].

On the other hand, Sysak's example cited above also shows that the Hirsch-Plotkin radical of a periodic radical product of two locally nilpotent subgroups need not be prefactorized. Recall that a group is *radical* if it has an ascending normal series with locally nilpotent factors.

Let the periodic radical group G be the product of two locally nilpotent subgroups A and B. In the present paper, we will show that the existence of prefactorized Sylow π-subgroups of $G = AB$ is indeed closely related to the question whether the Hirsch-Plotkin radical of G is factorized:

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THEOREM 4.3. *Let the periodic radical group G be the product of two locally nilpotent subgroups A and B. Then the following statements are equivalent:*

(a) *$\{A_pB_p \mid p \in P\}$ is a Sylow generating basis of G.*

(b) *$\langle A_{p'}, B_{p'}\rangle$ is a p'-group for every prime p.*

(c) *For every set of primes π and every normal subgroup N of G, the set $A_\pi B_\pi N/N$ is a Sylow π-subgroup of G/N.*

(d) *For every prime p and every normal subgroup N of G, the set A_pB_pN/N is a Sylow p-subgroup of G/N.*

(e) *For every normal subgroup N of G, the Hirsch-Plotkin radical $R(G/N)$ of G/N is factorized.*

(f) *Every term of the Hirsch-Plotkin series of G is factorized.*

(g) *The group G possesses an ascending series of prefactorized subgroups with locally nilpotent factors.*

Let P denote the set of all primes. A set of subgroups $\{G_p \mid p \in P\}$ of the group G is a *Sylow generating basis of G* if it satisfies the following conditions:

(SB1) $G_pG_q = G_qG_p$ for all primes p and q.

(SB2) $\langle G_p \mid p \in \pi\rangle$ is a maximal π-subgroup of G for every set π of primes.

Most statements of Theorem 4.3 above are actually a consequence of Theorem 3.4, which relates the existence of prefactorized Sylow π- and π'-subgroups of a product $G = AB$ of two subgroups A and B having normal Sylow π- and π'-subgroups to the question whether $O_\pi(G)$ and $O_{\pi'}(G)$ are factorized.

In Section 5, we show that one, and hence all, of the equivalent conditions of Theorem 4.3, are satisfied if the product $G = AB$ belongs to a class of locally finite groups having a sufficient Sylow structure, such as the classes of all FC- or CC-groups or the class of all periodic locally soluble groups satisfying the minimal condition on p-subgroups for all primes p. Thus in Theorem 5.3 and Theorem 5.7 we obtain:

THEOREM. *Suppose that the locally finite group G is the product of its locally nilpotent subgroups A and B. If G is a* U*-group, an FC-group, a CC-group, or G is locally soluble and satisfies min-p for every prime p, then the following statements hold:*

(a) *G is locally finite-soluble.*

(b) *The set $\{A_pB_p \mid p \in P\}$ is a Sylow generating basis of G.*

(c) *For every set π of primes, $O_\pi(G)$ is a factorized subgroup of $A_\pi B\pi$, hence is a prefactorized subgroup of G.*

(d) *For every set π of primes, $O_{\pi',\pi}(G)$ is a factorized subgroup of G.*

(e) *The Hirsch-Plotkin radical of G is factorized.*

Our notation is standard and follows **[2]**, **[5]** and **[14]**.

2. Preliminary results. Our first lemma shows in particular that a finite subset of a product G of two subgroups is contained in a countable prefactorized subgroup of G. It also ensures that certain intersection of prefactorized subgroups are prefactorized.

LEMMA 2.1. *Suppose that the group G is the product of two subgroups A and B. Moreover, let S be a set of prefactorized subgroups of G which is closed with respect to arbitrary intersections of its members. Then every subset X of G is contained in a prefactorized subgroup H of*

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cardinality not exceeding $\max(\aleph_0, |X|)$, *such that* $H \cap S$ *is a prefactorized subgroup of* H *for every* $S \in \mathcal{S}$.

Proof. Suppose without loss of generality that $G \in \mathcal{S}$. For every $x \in G$, let G_x denote the intersection of all $S \in \mathcal{S}$ such that $x \in S$, then by hypothesis $G_x \in \mathcal{S}$ for every $x \in G$, and in particular G_x is prefactorized. Now define functions $a_1 : G \to A$, $b_1 : G \to B$, $a_2 : G \to A$, and $b_2 : G \to B$ as follows: For each $x \in G$, choose elements a, $a' \in A \cap G_x$ and b, $b' \in B \cap G_x$ such that $x = ab = b'a'$ and put $a_1(x) = a$, $b_1(x) = b$, $a_2(x) = a'$ and $b_2(x) = b'$.

Let $X_0 = X$ and $A_0 = B_0 = \emptyset$. For every integer $i \geq 0$, define sets A_{i+1}, B_{i+1} and X_{i+1} containing A_i, B_i and X_i respectively, as follows: if i is even, let $A_{i+1} = \langle A_i, a_1(x) \mid x \in X_i \rangle$, $B_{i+1} = \langle B_i, b_1(x) \mid x \in X_i \rangle$ and $X_{i+1} = A_{i+1}B_{i+1}$. If i is odd, put $A_{i+1} = \langle A_i, a_2(x) \mid x \in X_i \rangle$, $B_{i+1} = \langle B_i, b_2(x) \mid x \in X_i \rangle$ and $X_{i+1} = B_{i+1}A_{i+1}$. Then in both cases, X_i is contained in X_{i+1}, and the cardinalities of A_{i+1}, B_{i+1} and X_{i+1} do not exceed $\max(\aleph_0, |X|)$.

Now let $A_\infty = \bigcup_{i \in N} A_i$ and $B_\infty = \bigcup_{i \in N} B_i$, then it is easy to verify that $A_\infty B_\infty \subseteq B_\infty A_\infty \subseteq A_\infty B_\infty$ so that $H = A_\infty B_\infty$ is a prefactorized subgroup of G whose cardinality does not exceed $\max(\aleph_0, |X|)$.

Let $S \in \mathcal{S}$, then it remains to show that $H \cap S$ is prefactorized. Choose $x \in H \cap S$, then $x \in X_i \subseteq X_{i+1}$ for some integer i so that i may be assumed odd. Then $X_i = A_iB_i$ and $x \in X_i$ is the product of $a = a_1(x) \in A_\infty$ and $b = b_1(x) \in B_\infty$. By construction, a and b belong to G_x and $G_x \leq S$ because $x \in S$. This shows that $x = ab \in (A \cap H \cap S)(B \cap H \cap S)$ and $H \cap S$ is prefactorized.

In order to show that the Sylow π-subgroups of a periodic locally nilpotent product of two subgroups are always prefactorized, we need the following:

LEMMA 2.2. *Suppose that the group* $G = M \times N$ *is the product of two subgroups* A *and* B. *If* $A = (A \cap M)(A \cap N)$ *and* $B = (B \cap M)(B \cap N)$, *then* $M = (M \cap A)(M \cap B)$.

Proof. Clearly, $G = AB = (A \cap M)N(B \cap M)$. Therefore

$$\begin{aligned} M &= M \cap (A \cap M)N(B \cap M)(A \cap M) = (M \cap N(B \cap M)) \\ &= (A \cap M)(M \cap N)(B \cap M) = (A \cap M)(B \cap M). \end{aligned}$$

by Dedekind's modular law.

In particular, if the sets of primes involved in M and N are disjoint, this leads to:

COROLLARY 2.3. *Let* G *be a group and suppose that* G *is the direct product of a normal Sylow* π*-subgroup* G_π *and a normal Sylow* π'*-subgroup* $G_{\pi'}$. *If* $G = AB$ *for two subgroups* A *and* B, *then* $G_\pi = A_\pi B_\pi$ *and* $G_{\pi'} = A_{\pi'}B_{\pi'}$, *where* A_π, $A_{\pi'}$, B_π *and* $B_{\pi'}$ *are normal Sylow* π*- and Sylow* π'*-subgroups of* A *and* B, *respectively. In particular, if* G *is a periodic locally nilpotent group, then the set* $\{A_pB_p \mid p \in P\}$ *is the unique Sylow generating basis of* G.

The next lemma is a weak version of the Schur-Zassenhaus theorem which, however, holds for arbitrary locally finite groups.

LEMMA 2.4. *Suppose that* G *is a locally finite group and* π *a set of primes such that* G/N *is a* π*-group for some subgroup* $N \leq Z(G)$. *Then* G *has a unique Sylow* π*-subgroup* G_π *and a unique Sylow* π'*-subgroup* $G_{\pi'}$ *and* $G = G_\pi G_{\pi'}$.

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Proof. Let $N_{\pi'}$ be the unique Sylow π'-subgroup of N, then also $G/N_{\pi'}$ is a π-group. Therefore assume without loss of generality that N is a π'-group. Let S be the set of all π-elements of G. If $g, h \in S$, then $F = \langle g, h\rangle$ is a finite group. Hence by the theorem of Schur and Zassenhaus, $F = F_\pi(F \cap N)$, where F_π is a Hall π-subgroup of F. Since $F \cap N \leq Z(F)$, the subgroup F_π is the unique Hall π-subgroup of F and so $gh \in F_\pi$ is a π-element. Hence $gh \in S$ and S is a π-subgroup of G.

Let $x \in G$, then there exists a π-number $n \in \mathbb{N}$ such that $x^n \in N$. Since N is a π'-group, there exists $y \in N$ such that $y^n = x^n$. Therefore $xy^{-1} \in S$ and $x = (xy^{-1})y$ is contained in SN, as required.

The following proposition states some criteria for a periodic product of two subgroups to have prefactorized Sylow subgroups.

PROPOSITION 2.5. *Suppose that the periodic group G is the product of two subgroups A and B and that $A = A_\pi A_{\pi'}$ and $B = B_\pi B_{\pi'}$, where A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ are π- and π'-subgroups of A and B, respectively.*

(a) (N. S. Černikov [**4**, Lemma 2]). *If $\langle A_\pi, B_\pi\rangle$ is a π-group and $\langle A_{\pi'}, B_{\pi'}\rangle$ is a π'-group, then $A_\pi B_\pi N/N = B_\pi A_\pi N/N$ is a Sylow π-subgroup of G/N for every normal subgroup N of G (and $A_{\pi'}B_{\pi'}N/N$ is a Sylow π'-subgroup of G/N).*

(b) *If $\mathcal{N}$ is a set of normal subgroups of G such that $\bigcap_{N\in\mathcal{N}} N = 1$ and for every $N \in \mathcal{N}$, the subgroups $\langle A_\pi, B_\pi\rangle N/N$ and $\langle A_{\pi'}, B_{\pi'}\rangle N/N$ are a π- and a π'-subgroup of G/N, respectively, then $A_\pi B_\pi$ and $A_{\pi'}B_{\pi'}$ are a Sylow π- and π'-subgroup of G.*

(c) *If G is locally finite, $N \leq G$ is contained in the centre of G, and $\langle A_\pi, B_\pi\rangle N/N$ and $\langle A_{\pi'}, B_{\pi'}\rangle N/N$ are a π- and a π'-subgroup of G/N, respectively, then $A_\pi B_\pi$ and $A_{\pi'}B_{\pi'}$ are a Sylow π- and a π'-subgroup of G.*

Proof. (a) Since the hypotheses are inherited by every factor group G/N of G, it clearly suffices to consider the case when $N = 1$. Now assume that the π-group $\langle A_\pi, B_\pi\rangle$ is contained in a π-group P of G and let $g \in P$. Since $G = AB = A_\pi A_{\pi'} B_{\pi'} B_\pi$, the element g can be written as $g = a_\pi a_{\pi'} b_{\pi'} b_\pi$, where $a_\pi \in A_\pi$, $a_{\pi'} \in A_{\pi'}$, $b_\pi \in B_\pi$ and $b_{\pi'} \in B_{\pi'}$. Therefore $a_{\pi'} b_{\pi'} = a_\pi^{-1} g b_\pi^{-1}$ is contained in $P \cap \langle A_{\pi'}, B_{\pi'}\rangle = 1$. Hence $g = a_\pi b_\pi$ is contained in the set $A_\pi B_\pi$ and so $\langle A_\pi, B_\pi\rangle = A_\pi B_\pi$ is a Sylow π-subgroup of G. A similar argument shows that $A_{\pi'}B_{\pi'}$ is a Sylow π'-subgroup of G.

(b) Let $S = \langle A_\pi, B_\pi\rangle$, then by hypothesis, $S/S \cap N \cong SN/N$ is a π-group for every $N \in \mathcal{N}$. Since G is periodic, this shows that S is a π-group. Similarly, $\langle A_{\pi'}, B_{\pi'}\rangle$ is a π'-group. Now the result follows from (a).

(c) Let $H = \langle A_\pi, B_\pi\rangle N$, then H has a normal Sylow π-subgroup H_π by Lemma 2.4. Since $A_\pi H_\pi$ and $B_\pi H_\pi$ are π-subgroups of H, it follows that $\langle A_\pi, B_\pi\rangle \leq H_\pi$ is a π-group. Similarly, $\langle A_{\pi'}, B_{\pi'}\rangle$ is a π'-group. The desired result follows from (a).

Note that in Proposition 2.5 (a), we do *not* claim that the Sylow subgroups $A_\pi B_\pi$ and $A_{\pi'}B_{\pi'}$ of $G = AB$ permute. See Theorem 3.4 for an additional hypothesis which ensures that $G = (A_\pi B_\pi)(A_{\pi'}B_{\pi'})$.

LEMMA 2.6. *Let π be a set of primes and suppose that the group G is the product of two subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, where A_π, $A_{\pi'}$ B_π and $B_{\pi'}$ are π- and π'-subgroups of A and B, respectively. Further, assume that $A_\pi B_\pi$ is a π-subgroup of G. If N is a normal*

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π-subgroup of G contained in $A_\pi B_\pi$, then the factorizer of N is the direct product of its maximal π-subgroup $A_\pi N \cap B_\pi N$ and its maximal π'-subgroup $A_{\pi'} \cap B_{\pi'}$.

Proof. Let $X = AN \cap BN$ denote the factorizer of N and put $P = A_\pi N \cap B_\pi N$. Then P is a subgroup of $A_\pi B_\pi$, hence is a factorized subgroup of $A_\pi B_\pi$. Therefore $A_{\pi'} \cap B_{\pi'}$ centralizes $P = (P \cap A_\pi)(P \cap B_\pi)$ and so $Y = (A_{\pi'} \cap B_{\pi'}) \times P = (A_{\pi'} \cap B_{\pi'})(P \cap A)(P \cap B)$ is a prefactorized subgroup of G. Since $A \cap B = (A_{\pi'} \cap B_{\pi'})(A_\pi \cap B_\pi)$, the factorized subgroup Y contains $A \cap B$. Moreover, $N \leq Y \leq X$ and so $Y = X$ by the definition of X.

Černikov's criterion can also be used to show that under relatively weak hypotheses, prefactorized Sylow subgroups reduce into prefactorized subgroups.

PROPOSITION 2.7. *Suppose that the periodic group G is the product of its subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, where A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ are Sylow π- and Sylow π'-subgroups of A and B, respectively. Further, assume that $\langle A_\pi, B_\pi\rangle$ and $\langle A_{\pi'}, B_{\pi'}\rangle$ are a π- and a π'-subgroup of G. If S is a prefactorized subgroup of G, then $(S \cap A_\pi)(S \cap B_\pi) = S \cap A_\pi B_\pi$ and $(S \cap A_{\pi'})(S \cap B_{\pi'}) = S \cap A_{\pi'} B_{\pi'}$ are a Sylow π- and a Sylow π'-subgroup of S. Hence the Sylow subgroups $A_\pi B_\pi$ and $A_{\pi'} B_{\pi'}$ reduce into S.*

Proof. Clearly, $S \cap A = (S \cap A_\pi)(S \cap A_{\pi'})$ and $S \cap B = (S \cap B_\pi)(S \cap B_{\pi'})$. Since $\langle S \cap A_\pi, S \cap B_\pi\rangle \leq A_\pi B_\pi$ is a π-group and $\langle S \cap A_{\pi'}, S \cap B_{\pi'}\rangle$ is a π'-group, the subgroup $S = (S \cap A)\ (S \cap B)$ satisfies the hypotheses of Proposition 2.5 (a). Therefore $(S \cap A_\pi)(S \cap B_\pi)$ is a Sylow π-subgroup of S. Since this Sylow subgroup is contained in the π-subgroup $S \cap A_\pi B_\pi$ of G, it follows that $(S \cap A_\pi)(S \cap B_\pi) = S \cap A_\pi B_\pi$. The corresponding result about $A_{\pi'} B_{\pi'}$ follows by exchanging π and π'.

Observe that the following corollary holds in particular if $A_\pi B_\pi N/N$ is a Sylow π-subgroup of G/N for every normal subgroup N of G.

COROLLARY 2.8. *Let π be a set of primes and suppose that the group G is the product of two subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, where A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ are π- and π'-subgroups of A and B, respectively. If the set $A_\pi B_\pi$ is a π-subgroup of G which contains $O_\pi(G)$, then the factorizer $X = AO_\pi(G) \cap BO_\pi(G)$ of $O_\pi(G)$ is the direct product of its maximal π-subgroup $A_\pi O_\pi(G) \cap B_\pi O_\pi(G)$ and its maximal π'-subgroup $A_{\pi'} \cap B_{\pi'}$. Moreover, if $O_{\pi',\pi}(G)$ is contained in $A_\pi B_\pi O_{\pi'}(G)$ then the factorizer of $O_{\pi',\pi}(G)$ is an extension of a π'-group by a π-group.*

Proof. Put $N = O_\pi(G)$, then the first statement follows directly from Lemma 2.6. Now let X denote the factorizer of $O_{\pi',\pi}(G)$; then by [**2**, Lemma 1.1.2], $X/O_{\pi'}(G)$ is the factorizer of $O_{\pi',\pi}(G)/O_{\pi'}(G) = O_\pi(G/O_{\pi'}(G))$ in $G/O_{\pi'}(G)$ and hence is an extension of a π'-group by a π-group, as required.

3. Sylow π-subgroups of π-separable locally finite groups. Recall that a group G belongs to the class $\acute{P}(S_{\pi'}S_\pi)$ if it is locally finite-soluble and has an ascending normal series whose factors are either π-group or π'-groups.

PROPOSITION 3.1. *Let π be a set of primes and suppose that the group $G \in \acute{P}(S_{\pi'}S_\pi)$ is the product of two subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, where A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ are*

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π- and π'-subgroups of A and B, respectively. If $\langle A_\pi, B_\pi\rangle$ and $\langle A_{\pi'}, B_{\pi'}\rangle$ are a π-group and a π'-group, respectively, then $O_{\pi',\pi}(G)$ is factorized. Moreover, $O_\pi(G)$ is a factorized subgroup of the Sylow π-subgroup $A_\pi B_\pi$ of G. Hence $O_\pi(G)$ is a prefactorized subgroup of $G = AB$.

Proof. First, we show that $O_{\pi',\pi}(G)$ is factorized. By Proposition 2.5, $A_\pi B_\pi N/N$ and $A_{\pi'}B_{\pi'}N/N$ are a maximal π-subgroup and a maximal π'-subgroup of G/N for every normal subgroup N of G. Thus it suffices to consider the case when $O_{\pi'}(G) = 1$. Now by Corollary 2.8, $A_{\pi'} \cap B_{\pi'}$ centralizes $O_\pi(G)$ and so by [**5**, 4.1.1], $A_{\pi'} \cap B_{\pi'} = 1$. Therefore the factorizer X of $O_\pi(G)$ is a π-group. Exchanging the roles of π and π', it follows by the same arguments that $Y/O_\pi(G)$ is a π'-group, where Y is the factorizer of $O_{\pi,\pi'}(G)$. Since the factorized subgroup Y contains $O_\pi(G)$, we have $X \leq Y$, and so the π-group X must be contained in $O_\pi(G)$. Hence $O_\pi(G)$ is factorized.

To prove the second statement, observe that, exchanging the roles of π and π', it follows from the first part, applied to $G/O_\pi(G)$, and [**2**, Lemma 1.1.2] that $O_{\pi,\pi'}(G)$ is factorized. Therefore $O_\pi(G) = A_\pi B_\pi \cap O_{\pi,\pi'}(G)$ is a factorized subgroup of $A_\pi B_\pi$, hence is a prefactorized subgroup of G.

The following lemma extends [**6**, Lemma 2.2].

LEMMA 3.2. *Let π be a set of primes and suppose that the group $G \in \acute{P}(S_{\pi'}S_\pi)$ is the product of a π-group A and a π'-group B. If Γ is a finite group of automorphisms of G and X is a finite subset of G, then there exists a finite factorized Γ-invariant subgroup of G containing X.*

Proof. Observe first that $A \cap B = 1$, so that every prefactorized subgroup of G is factorized. By hypothesis, there exists an ordinal β such that G possesses an ascending series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_\beta = G$$

whose factors are π- or π'-groups, and clearly the G_i may be assumed characteristic in G. Since AN/N and BN/N are maximal π- and π'-subgroups of G/N for every normal subgroup N of G, we show by induction on α that every G_α is factorized: if α is a limit ordinal, we have

$$G_\alpha = \bigcup_{\beta<\alpha} G_\beta = \bigcup_{\beta<\alpha}(A \cap G_\beta)(B \cap G_\beta)$$

which is clearly contained in $(A \cap G_\alpha)(B \cap G_\alpha)$. Therefore suppose that $\alpha - 1$ exists. If $G_\alpha/G_{\alpha-1}$ is a π-group, then $G_\alpha \leq AG_{\alpha-1}$. Therefore $G_\alpha = G_\alpha \cap AG_{\alpha-1} = (G_\alpha \cap A)G_{\alpha-1}$, and since $G_{\alpha-1}$ is factorized by hypothesis, G_α is contained in $(A \cap G_\alpha)(B \cap G_\alpha)$. Thus G_α is factorized. Otherwise, the π'-group $G_\alpha/G_{\alpha-1}$ is contained in $G_{\alpha-1}B$, and a similar argument shows that G_α is factorized also in the last case.

Now let α be the least ordinal such that G possesses a finite Γ-invariant subgroup K containing X such that KG_α is factorized, and assume that $\alpha > 0$. By the modular law, we have

$$A \cap KG_\alpha = A \cap K(B \cap G_\alpha)(A \cap G_\alpha) = (A \cap K(B \cap G_\alpha))(A \cap G_\alpha).$$

Let $A_0 = A \cap KB$ and $B_0 = B \cap KA$, the $A \cap KG_\alpha$ is contained in $A_0(A \cap G_\alpha)$ and similarly, $B \cap KG_\alpha \leq B_0(B \cap G_\alpha)$. Now the sets A_0 and B_0 are contained in the factorizer Y of K by [**2**, Lemma 1.1.3]. Since KG_α is factorized, it contains Y, and so we have $A_0 \leq A \cap KG_\alpha$ and $B_0 \leq B \cap KG_\alpha$. This shows that $A \cap KG_\alpha = A_0(A \cap G_\alpha)$ and $B \cap KG_\alpha = B_0(B \cap G_\alpha)$.

Moreover, K is obviously contained in the set A_0B_0. Since KG_α is Γ-invariant and $\langle A_0, B_0\rangle$ is finite, there exists a Γ-invariant finite subgroup F of G such that $\langle A_0, B_0\rangle \leq F \leq KG_\alpha$. Thus, applying the modular law twice, we obtain

$$F = F \cap KG_\alpha = F \cap A_0G_\alpha B_0 = A_0(F \cap G_\alpha B_0) = A_0(F \cap G_\alpha)B_0.$$

Assume that α is a limit ordinal, then $F \cap G_\alpha = F \cap G_\beta$ for some $\beta < \alpha$ and so $FG_\beta = A_0G_\beta B_0 = A_0(A \cap G_\beta)(B \cap G_\beta)B_0$. Therefore FG_β is factorized, contradicting the choice of α.

Therefore $\alpha - 1$ exists, and we may assume without loss of generality that $\alpha = 1$ and that G_1 is a π'-group. Then $F \cap G_1$ is a subgroup of B and $F = A_0(F \cap B)B_0$ is factorized. This final contradiction proves the lemma.

Our next lemma is a slight extension of [**6**, Lemma 2.3].

LEMMA 3.3. *Let π be a set of primes and suppose that $G \in \acute{P}(S_{\pi'}S_\pi)$ is a countable locally finite group. If N is a normal subgroup of G such that G/N is a π-group and N is the product of a π-group A_0 and a π'-group B, then there exists a π-subgroup A of G such that $G = AB$.*

Proof. Since G is countable, G is the union of an ascending chain of finite subgroups $G_1 \leq G_2 \leq \ldots$ of type ω. We define an ascending chain $\{K_i \mid i \in N\}$ of finite subgroups of N as follows: put $K_1 = 1$. If $i > 1$, then by Lemma 3.2, there exists a finite G_i-invariant subgroup K_i of $N = A_0B$ which contains $G_i \cap N$ and K_{i-1} and satisfies $K_i = (A_0 \cap K_i)(B \cap K_i)$.

Suppose now that $G_{i-1}K_{i-1} = A_{i-1}(B \cap G_{i-1})$ for a π-subgroup A_{i-1} of G_{i-1}. Since K_i is G_i-invariant, G_iK_i is a finite subgroup of G, hence is π-separable. Therefore A_{i-1} is contained in a Hall π-subgroup A_i of G_iK_i. Since $G_i \cap N \leq K_i$, the factor group $G_iK_i/K_i \cong G_i/G_i \cap K_i$ is a π-group and $B \cap K_i$ is a Hall π'-subgroup of G_iK_i so that $G_iK_i = A_i(B \cap K_i)$. Thus $A = \cup_{i \in N} A_i$ is the required π-subgroup of G.

We can now formulate the relation between the existence of prefactorized Sylow subgroups and of certain prefactorized characteristic subgroups of a group G which is the product of two subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$.

THEOREM 3.4. *Let π be a set of primes and suppose that the group $G \in \acute{P}(S_{\pi'}S_\pi)$ is the product of two subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, where A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ are π- and π'-subgroups of A and B, respectively. Then the following statements are equivalent:*

(a) *$\langle A_\pi, B_\pi\rangle$ is a π-group and $\langle A_{\pi'}, B_{\pi'}\rangle$ is a π'-group.*

(b) *For every normal subgroup N of G, $A_\pi B_\pi N/N$ is a Sylow π-subgroup of G/N and $A_{\pi'}B_{\pi'}N/N$ is a Sylow π'-subgroup of G; moreover, $G = (A_\pi B_\pi)(A_{\pi'}B_{\pi'})$.*

(c) *$O_{\pi',\pi}(G/N)$ and $O_{\pi,\pi'}(G/N)$ are factorized for every normal subgroup N of G.*

(d) *$O_\pi(G/N)$ and $O_{\pi'}(G/N)$ are prefactorized for every normal subgroup N of G.*

(e) *The group G possesses an ascending normal series of prefactorized subgroups whose factors are either π- or π'-groups.*

Proof. The implication (a) $\Rightarrow$ (c) has been proved in Proposition 3.1.

(c) $\Rightarrow$ (d). Since $O_{\pi',\pi}(G/N) \cap O_{\pi,\pi'}(G/N) = O_\pi(G/N) \times O_{\pi'}(G/N)$ is factorized by [**2**, Lemma 1.1.2], this follows from Corollary 2.3.

Since the implications (d)$\Rightarrow$(e) and (b)$\Rightarrow$(a) are trivial, it remains to show that (e)$\Rightarrow$(b).

In view of Proposition 2.5, it clearly suffices to consider the case when $N = 1$. Let $\{N_\alpha\}_{\alpha\leq\beta}$ be an ascending chain of prefactorized normal subgroups such that $N_{\alpha+1}/N_\alpha$ is a π-group or a π'-group for every $\alpha < \beta$. By transfinite induction on β, the sets $(A_\pi \cap N_\alpha)$ $(B_\pi \cap N_\alpha)$, $(A_{\pi'} \cap N_\alpha)(B_{\pi'} \cap N_\alpha)$ are Sylow π- and π'-subgroups of N_α such that

$$N_\alpha = (A_\pi \cap N_\alpha)(B_\pi \cap N_\alpha)(A_{\pi'} \cap N_\alpha)(B_{\pi'} \cap N_\alpha)$$

for every $\alpha < \beta$. Thus if β is a limit ordinal, then

$$A_\pi B_\pi = \Big(\bigcup_{\alpha<\beta} A_\pi \cap N_\alpha\Big)\cdot\Big(\bigcup_{\alpha<\beta} B_\pi \cap N_\alpha\Big) \subseteq \bigcup_{\alpha<\beta}(A_\pi \cap N_\alpha)(B_\pi \cap N_\alpha)$$

and so $A_\pi B_\pi$ is a π-group. By similar arguments, $A_{\pi'}B_{\pi'}$ is a π'-group and $G = (A_\pi B_\pi)$ $(A_{\pi'}B_{\pi'})$.

Therefore assume that β possesses a predecessor $\beta - 1$ and set $N = N_{\beta-1}$. Exchanging π and π' if necessary, we may also assume that G/N is a π-group, so that $\langle A_{\pi'}, B_{\pi'}\rangle = (A_{\pi'} \cap N_{\beta-1})(B_{\pi'} \cap N_{\beta-1}) = A_{\pi'}B_{\pi'}$ is a Sylow π'-group of G. Now suppose that $\langle A_\pi, B_\pi\rangle$ is a π-group. Then it follows from Proposition 2.5 that $A_\pi B_\pi$ is a Sylow π-subgroup of G and that $A_\pi B_\pi N/N$ is a Sylow π-subgroup of G/N. Hence

$$G = A_\pi B_\pi N = (A_\pi B_\pi)(A_{\pi'} \cap N_{\beta-1})(B_{\pi'} \cap N_{\beta-1}) = (A_\pi B_\pi)(A_{\pi'}B_{\pi'}),$$

as required.

Thus it remains to show that $\langle A_\pi, B_\pi\rangle$ is a π-group. Let A_0 and B_0 be arbitrary finite subsets of A_π and B_π, respectively, then it clearly suffices to show that $\langle A_0, B_0\rangle$ is a π-group. By Lemma 2.1, there exists a countable prefactorized subgroup H of G containing A_0 and B_0 such that $H \cap N_\alpha$ is prefactorized in H for every $\alpha \leq \beta$. Therefore we may assume without loss of generality that $G = H$ and so G is countable. Then by Lemma 3.3, there exists a maximal π-subgroup P of G such that $G = PA_{\pi'}B_{\pi'}$. By Lemma 3.2, $\langle A_0, B_0\rangle$ is contained in a finite subgroup F satisfying $F = (F \cap P)(F \cup A_{\pi'}B_{\pi'})$. Let Q be a Hall π-subgroup of F containing A_0, then $B_0 \leq Q^g$ for some $g \in F$, since F is π-separable. Since $F = Q(F \cap A_{\pi'}B_{\pi'})$, we may clearly assume that $g \in F \cap A_{\pi'}B_{\pi'}$. Write $g = ab^{-1}$ with $a \in A_{\pi'}$ and $b \in B_{\pi'}$, then $A_0 = A_0^a$ is contained in Q^a and also $B_0 = B_0^b$ is contained in $Q^{gb} = Q^a$. Therefore $\langle A_0, B_0\rangle$ is a π-group, as required.

4. Sylow generating bases of periodic radical products of two locally nilpotent subgroups. The results obtained so far are of special interest when G is a periodic radical group which is the product of two locally nilpotent subgroups A and B. We will show that if $A_\pi B_\pi$ is a Sylow π-group of G for every set of primes π, then the set $\{A_pB_p \mid p \in P\}$ even forms a Sylow generating basis of G. First, we study the factorizer of the Hirsch-Plotkin radical of G.

LEMMA 4.1. *Suppose that the periodic group G is the product of two locally nilpotent subgroups A and B. If p is a prime such that the set A_pB_p is a p-group containing $O_p(G)$, then the factorizer $X = AR \cap BR$ of the Hirsch-Plotkin radical R of G is an extension of a p'-group by a p-group.*

Proof. Let $R_p = O_p(G)$ and $R_{p'}$ be the Sylow p- and p'-subgroups of R, respectively, and denote with X the factorizer of R. Then $R/R_{p'}$ is contained in $A_pB_pR_{p'}/R_{p'}$. Now by Lemma 2.6, the factorizer $X/R_{p'}$ of $R/R_{p'}$ is an extension of a p'-group by a p-group. Therefore also X is an extension of a p'-group by a p-group.

In particular, this result can be applied to locally finite groups which are the product of two locally nilpotent subgroups.

COROLLARY 4.2. *Suppose that the locally finite group G is the product of two locally nilpotent subgroups A and B. If the set A_pB_p is a Sylow p-subgroup of G for every prime p, then the factorizer of the Hirsch-Plotkin radical of G is locally nilpotent.*

Proof. Since $A_pB_pO_p(G)$ is a p-group, G satisfies the hypothesis of Lemma 4.1 for every prime p. Thus if X denotes the factorizer of the Hirsch-Plotkin radical of G and Y is a finite subgroup of X, then $Y/O_{p'}(Y)$ is a p-group, hence nilpotent, for every $p \in P$. It follows that Y is nilpotent, and so X is locally nilpotent.

In the following theorem, we collect the main properties of a periodic radical group which is the product of two locally nilpotent subgroups. Note that, despite the similarities with Theorem 3.4, the results for radical groups are slightly stronger, mainly because it suffices to consider respectively p'- and p-groups in (b) and (d) below.

THEOREM 4.3. *Let the periodic radical group G be the product of two locally nilpotent subgroups A and B. Then the following statements are equivalent:*

(a) *$\{A_pB_p \mid p \in P\}$ is a Sylow generating basis of G.*

(b) *$\langle A_{p'}, B_{p'}\rangle$ is a p'-group for every prime p.*

(c) *For every set of primes π and every normal subgroup N of G, the set $A_\pi B_\pi N/N$ is a Sylow π-subgroup of G/N.*

(d) *For every prime p and every normal subgroup N of G, the set A_pB_pN/N is a Sylow p-subgroup of G/N.*

(e) *For every normal subgroup N of G, the Hirsch-Plotkin radical $R(G/N)$ of G/N is factorized.*

(f) *Every term of the Hirsch-Plotkin series of G is factorized.*

(g) *The group G possesses an ascending normal series of prefactorized subgroups with locally nilpotent factors.*

Proof. (a) ⇒ (b) follows directly from the definition of a Sylow generating basis.

(b) ⇒ (c). Clearly, A and B are the direct product of their Sylow π- and π'–subgroups and $\langle A_\pi, B_\pi\rangle$ is obviously contained in the π-group.

$$\bigcap_{q \in P\setminus\pi} \langle A_{q'}, B_{q'}\rangle.$$

Therefore $\langle A_\pi, B_\pi\rangle$ and $\langle A_{\pi'}, B_{\pi'}\rangle$ are a π- and a π'-subgroup, and so (c) follows from Proposition 2.5 (a).

The implication (c) ⇒ (d) is trivial.

(d) ⇒ (e). Let $R_0 = N$ and for every ordinal α, define $R_{\alpha+1}/R_\alpha = R(G/R_\alpha)$; moreover, put $R_\alpha = \bigcup_{\gamma<\alpha} R_\gamma$ if α is a limit ordinal. Then $G = R_\beta$ for some ordinal β since G is radical.

For every ordinal α, let X_α denote the factorizer of R_α. Then for every α, the factor group $X_{\alpha+1}/R_\alpha$ is locally nilpotent by Corollary 4.2. Therefore by [**13**, II, p. 10], the subgroup X_α/R_α is a serial subgroup of $X_{\alpha+1}/R_\alpha$ for every α, hence of G/R_α for every $\alpha \leq \beta$. Now suppose that $\alpha \leq \beta$ has a predecessor $\alpha - 1$. Since G is locally finite, it follows from [**9**, Lemma 3] that the serial locally nilpotent subgroup $X_\alpha/R_{\alpha-1}$ is contained in $R_\alpha/R_{\alpha-1}$, and thus we have $X_\alpha = R_\alpha$. So R_α is factorized for every α that is not a limit ordinal. For limit ordinals α, we have

$$R_\alpha = \bigcup_{\gamma<\alpha} R_\gamma = \bigcup_{\gamma<\alpha} (R_\gamma \cap A)(R_\gamma \cap B) \subseteq (R_\alpha \cap A)(R_\alpha \cap B).$$

This shows that $R_\alpha = (R_\alpha \cap A)(R_\alpha \cap B)$, and so R_α is factorized.

The implications (e) ⇒ (f) and (f) ⇒ (g) are trivial.

Suppose now that (g) holds. For every set σ of primes, let A_σ and B_σ denote the (unique) Sylow σ-subgroup of A and B, respectively. Let π be any set of primes, then by Corollary 2.3, an ascending series of prefactorized subgroups with locally nilpotent factors can be refined to an ascending normal series of prefactorized subgroups whose factors are π- or π'-groups. Therefore it follows from Theorem 3.4 that $A_\pi B_\pi$ and $A_{\pi'}B_{\pi'}$ are a Sylow π- and a Sylow π'-subgroup of G such that $G = (A_\pi B_\pi)(A_{\pi'}B_{\pi'})$. Moreover, if p and q are two primes, then the same argument, applied to the Sylow $\{p, q\}$-group $H = A_pA_qB_pB_q$ of G shows that $H = (A_pB_p)(A_qB_q)$, whence $\{A_pB_p \mid p \in P\}$ is a Sylow generating basis of G. This proves (a).

We mention a number of particularly useful consequences of the preceding theorem. The first shows that in many cases, the question whether the Hirsch-Plotkin radical of a periodic radical product of two locally nilpotent subgroups is prefactorized is equivalent to the question whether it is factorized.

COROLLARY 4.4. *Suppose that the periodic radical group G is the product of its locally nilpotent subgroups A and B. If the Hirsch-Plotkin radical $R(G/N)$ of G/N is prefactorized for every normal subgroup N of G, then $R(G/N)$ is factorized for all $N \trianglelefteq G$.*

Theorem 4.3 also gives a useful criterion for the Hirsch-Plotkin radical of a product G of two locally nilpotent subgroups (and, indeed, of every factor group of G) to be factorized.

COROLLARY 4.5. *Suppose that the periodic radical group G is the product of its locally nilpotent subgroups A and B. If every factor group of G possesses a prefactorized locally nilpotent normal subgroup, then the Hirsch-Plotkin radical of G is factorized.*

In view of Theorem 3.4 (b) and Theorem 4.3, we also obtain the following result.

COROLLARY 4.6. *Suppose that the periodic radical group G is the product of its locally nilpotent subgroups A and B. For every set of primes, let A_π and B_π denote the (unique) Sylow π-subgroups of A and B, respectively, and suppose that $\langle A_\pi, B_\pi \rangle$ is a π-group for every set of primes π. Then $\{A_pB_p \mid p \in P\}$ is a Sylow generating basis of G, and in particular $G = (A_\pi B_\pi)(A_{\pi'}B_{\pi'})$ for every set π of primes.*

The next theorem is a direct consequence of Proposition 2.7 and Theorem 4.3.

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THEOREM 4.7. *Let the periodic radical group G be the product of its locally nilpotent subgroups A and B, and suppose that the set $\{A_pB_p \mid p \in P\}$ is a Sylow generating basis of G. If S is a prefactorized subgroup of G, then $\{A_pB_p \mid p \in P\}$ reduces into S.*

Proof. By Proposition 2.7, $(S \cap A_\pi)(S \cap B_\pi) = S \cap A_\pi B_\pi$ is a Sylow π-subgroup of S for every set of primes π. Therefore S satisfies Theorem 4.3 (b) and so $\{A_pB_p \cap S \mid p \in P\}$ is a Sylow generating basis of S.

As a consequence, a periodic radical product G of two locally nilpotent subgroups has at most one Sylow generating basis consisting of prefactorized Sylow subgroups of G.

COROLLARY 4.8. *Let the periodic radical group G be the product of its locally nilpotent subgroups A and B, and suppose that the set $\{A_pB_p \mid p \in P\}$ is a Sylow generating basis of G. Then $\{A_pB_p \mid p \in P\}$ is the unique Sylow generating basis of G which consists of prefactorized subgroups of G.*

Proof. Let $\{G_p \mid p \in P\}$ be a Sylow generating basis of G which consists of prefactorized subgroups of G and let $q \in P$. Then $\{A_pB_p \mid p \in P\}$ reduces into G_q, and so $A_qB_q \cap G_q = G_q$. This shows that $G_q = A_qB_q$.

5. Applications. The following proposition restates the results of Proposition 2.5 for Sylow generating bases of periodic radical products of two locally nilpotent subgroups.

PROPOSITION 5.1. *Suppose that the periodic radical group G is the product of two locally nilpotent subgroups A and B.*

(a) *If $\mathcal{N}$ is a set of normal subgroups of G such that $\bigcap_{N \in \mathcal{N}} N = 1$ and for every $N \in \mathcal{N}$, the set $\{A_pB_pN/N \mid p \in P\}$ is a Sylow generating basis of G/N, then $\{A_pB_p \mid p \in P\}$ is a Sylow generating basis of G.*

(b) *If $N \leq Z(G)$ and $\{A_pB_pN/N \mid p \in P\}$ is a Sylow generating basis of G/N, then $\{A_pB_p \mid p \in P\}$ is a Sylow generating basis of G.*

Proof. In view of the equivalence of statements (a) and (b) of Theorem 3.4, both statements follow from Proposition 2.5.

Recall that a group G is an FC-group (a CC-group) if $G/C_G(x^G)$ is finite (a Černikov group) for every $x \in G$.

LEMMA 5.2. *Let G be a CC-group. Then G has a local system of normal subgroups which are central-by-Černikov. Moreover G has a local system of central-by-finite subgroups.*

Proof. Let X be a finite subset of G and put $N = X^G$. Now $Z(N) = \bigcap_{x \in X} C_N(x^G)$, and since N is a CC-group and X is finite, $N/Z(N)$ is Černikov and N is central-by-Černikov. In particular, every finitely generated subgroup of G is central-by-finite.

THEOREM 5.3. *Suppose that the locally finite group G is the product of its locally nilpotent subgroups A and B. If G is a U-group, an FC-group, or a CC-group, then the following statements hold:*

(a) *G is locally finite-soluble.*

(b) *The set $\{A_pB_p \mid p \in P\}$ is a Sylow generating basis of G.*

(c) *For every set π of primes, $O_\pi(G)$ is a factorized subgroup of $A_\pi B_\pi$, hence is a prefactorized subgroup of G.*

(d) *For every π of primes, $O_{\pi',\pi}(G)$ is a factorized subgroup of G.*

(e) *The Hirsch-Plotkin radical of G is factorized.*

Proof. (a) It clearly suffices to consider the case when G is a CC-group. Let $x \in G$, then $G/C_G(x^G)$ is a Černikov group, hence abelian-by-finite. Since every finite image of G is soluble by the theorem of Kegel and Wielandt [**11**, **18**], the factor group $G/C_G(x^G)$ is soluble. Since $\bigcap_{x\in G} C_G(x^G) = Z(G)$, the group $G/Z(G)$ has a descending series of type $\leq \omega$ whose factors are abelian. Consequently G has such a descending normal series of type $\leq \omega + 1$. Now let X be a finite subset of G, then its normal closure $N = X^G$ is central-by-Černikov by Lemma 5.2. Therefore $\langle X\rangle Z(N)/Z(N)$ is finite, hence soluble, and so also $\langle X\rangle$ is soluble.

(b) If G is a U-group, then $\{A_pB_p \mid p \in P\}$ is a Sylow generating basis of G by [**4**, Lemma 5]. If G is a CC-group, then $G/C_G(x^G)$ is a U-group for every $x \in G$, and since $Z(G) = \bigcap_{x\in G} C_G(x^G)$, $\{A_pB_pZ(G)/Z(G) \mid p \in P\}$ is a Sylow generating basis of $G/Z(G)$ by Proposition 5.1 (a). Therefore by Proposition 5.1 (b), $\{A_pB_p \mid p \in P\}$ is a Sylow generating basis of G.

The statements (c), (d) and (e) now follow from Theorem 4.3 and Proposition 3.1.

It may also be of interest that Theorem 5.3 (e) remains true for general CC-groups. To prove this, we need the following results, which might already be known.

LEMMA 5.4. *Every CC-group which is residually (locally nilpotent) is locally nilpotent.*

Proof. Let G be a CC-group and $\mathcal{N}$ a set of subgroups of G such that $\bigcap_{N\in\mathcal{N}} N = 1$ and G/N is locally nilpotent for every $N \in \mathcal{N}$. Then every finitely generated subgroup U of G is central-by-finite by Lemma 5.2. Moreover, for every $N \in \mathcal{N}$, the finitely generated group $U/U \cap N$ is nilpotent of class at most $|U : Z(U)|$. Therefore U is nilpotent and G is locally nilpotent.

LEMMA 5.5. *Let G be a CC-group. If $R_x/C_G(x^G)$ denotes the Hirsch-Plotkin radical of $G/C_G(x^G)$ and R equals the intersection of all R_x, then R is the Hirsch-Plotkin radical of G. Moreover, G/R is a periodic FC-group.*

Proof. Clearly, the Hirsch-Plotkin radical of G is contained in every R_x and hence in R. Since $R/C_R(x^G)$ is locally nilpotent for every $x \in G$, it follows from Lemma 5.4 that $R/Z(G)$ is locally nilpotent. Therefore also R is locally nilpotent and so the normal subgroup R of G equals the Hirsch-Plotkin radical of G. Now let X be a finite subset of G, then X^G is central-by-Černikov by Lemma 5.2. Therefore the Hirsch-Plotkin radical $R \cap X^G$ of X^G has finite index in X^G and so X^GR/R is finite. Thus every finite subgroup of G/R is contained in a finite normal subgroup of G/R, and so G/R is a periodic FC-group.

From this, we deduce the following result about the Hirsch-Plotkin radical of a CC-group which is the product of two locally nilpotent subgroups.

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THEOREM 5.6. *Let the CC-group G be the product of its locally nilpotent subgroups A and B. Then the Hirsch-Plotkin radical of G is factorized.*

Proof. For every $x \in G$, let $R_x/C_G(x^G)$ denote the Hirsch-Plotkin radical of $G/C_G(x^G)$, then by Lemma 5.5, the intersection $R = \bigcap_{x\in G} R_x$ equals the Hirsch-Plotkin radical of G. By Theorem 5.3 (e), the subgroups R_x of G are factorized for every $x \in G$, and so R is factorized by [**2**, Lemma 1.1.2].

A result like Theorem 5.3 can also be proved for periodic locally soluble groups satisfying the minimal condition on p-subgroups for every prime p. However, different arguments are required because such groups need not be radical; see for example [**3**, *Folgerungen* 4.5 and 5.4].

THEOREM 5.7. *Let G be a periodic locally soluble group satisfying min-p for every prime p. Suppose that G is the product of its locally nilpotent subgroups A and B. The following properties hold.*

(a) *G is countable and has a descending series of length $\leq \omega$ whose factors are abelian.*

(b) *The set $\{A_pB_p \mid p \in P\}$ is a Sylow generating basis of G.*

(c) *For every set π of primes, $O_{\pi',\pi}(G)$ is a factorized subgroup of G.*

(d) *For every set π of primes, $O_\pi(G)$ is a factorized subgroup of $A_\pi B_\pi$, hence is a prefactorized subgroup of G.*

(e) *The Hirsch-Plotkin radical of G is factorized.*

(f) *If U is a prefactorized subgroup of G, then the Sylow generating basis $\{A_pB_p \mid p \in P\}$ of G reduces into U.*

Proof. (a) Since the p-components of A and B are locally soluble and satisfy the minimal condition on subgroups, the p-components of A and B are Černikov groups (see for example [**12**, Theorem 1.E.6]), hence are countable. Therefore also A and B are countable, and so G is countable.

Moreover, since G is locally soluble, for every prime p, the factor group $G/O_{p'}(G)$ is a Černikov group by [**12**, Theorem 3.17]. Hence these factor groups are soluble by the theorem of Kegel and Wielandt. Since $\bigcap_{p\in P} O_{p'}(G) = 1$, it follows that G has a descending series of length $\leq \omega$ whose factors are abelian.

(b) Since $G/O_{p'}(G)$ is a soluble Černikov group and thus a U-group, it follows from Theorem 5.3 that for every prime p, $\{A_qB_qO_{p'}(G)/O_{p'}(G) \mid q \in P\}$ is a Sylow generating basis of $G/O_{p'}(G)$. Therefore

$$G/O_{p'}(G) = (A_pB_pO_{p'}(G)/O_{p'}(G)) \cdot (A_{p'}B_{p'}/O_{p'}(G))$$

and so $G_{p'} = A_{p'}B_{p'}$ is a Sylow p'-subgroup of $G = (A_pB_p)(A_{p'}B_{p'})$ for every prime p. Let π be a set of primes, then

$$G_\pi = \langle A_\pi, B_\pi \rangle \leq \bigcap_{q\in\pi'} G_{q'}$$

which shows that G_π is a π-group. Since this holds for every set of primes π, it follows from Proposition 2.5 (a) that $G_\pi = A_\pi B_\pi$ is a Sylow π-subgroup of G. Now let p and q be distinct primes, then

$$G_pG_q = G_p\Big(\bigcap_{r\in P\setminus\{q\}} G_{r'}\Big) = G_p\Big(G_{p'} \cap \bigcap_{r\in P\setminus\{p,q\}} G_{r'}\Big),$$

and by Dedekind's modular law, we obtain

$$G_pG_q - G_pG_{p'} \cap \Big(\bigcap_{r \in P \setminus \{p,q\}} G_{r'} \Big) = \bigcap_{r \in P \setminus \{p,q\}} G_{r'},$$

whence G_pG_q is a subgroup of G. This shows that $\{A_pB_p \mid p \in P\}$ is a Sylow generating basis of G.

(c) Let π be a set of primes, and for every prime p, set $P_p/O_{p'}(G) = O_{\pi',\pi}(G/O_{p'}(G))$. Then $O_{\pi',\pi}(G) = \bigcap_{p \in P} P_p$ since G is periodic. By Theorem 5.3, the subgroups $P_p/O_{p'}(G)$ are factorized for every $p \in P$, and so every P_p is factorized. Therefore by [**2**, Lemma 1.1.2], also their intersection $O_{\pi',\pi}(G)$ is factorized.

(d) By (c), $O_{\pi,\pi'}(G)$ is factorized. Therefore the subgroup $O_\pi(G) = O_{\pi,\pi'}(G) \cap A_\pi B_\pi$ is factorized in $A_\pi B_\pi$, hence is a prefactorized subgroup of G.

(e) Let $R(G)$ denote the Hirsch-Plotkin radical of G. Clearly, $R(G) = \bigcap_{p \in P} O_{p',p}(G)$ and so $R(G)$ is the intersection of factorized subgroups, hence is factorized by [**1**, Lemma 1.1.2].

(f) Since U likewise satisfies min-p for every prime p, the set

$$\{(U \cap A_p)(U \cap B_p) \mid p \in P\}$$

is a Sylow generating basis of U by (b). Since obviously $(U \cap A_\pi)(U \cap B_\pi) \leq U \cap A_\pi B_\pi$, it follows that $(U \cap A_\pi)(U \cap B_\pi) = U \cap A_\pi B_\pi$ for every set π of primes, as required.

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SCHOOL OF MATHEMATICAL SCIENCES
AUSTRALIAN NATIONAL UNIVERSITY
CANBERRA ACT 0200
AUSTRALIA