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Asymptotic Behavior of Optimal Circle Packings in a Square

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Abstract. A lower bound on the number of points that can be placed in a square of side σ such that no two points are within unit distance from each other is proven. The result is constructive, and the series of packings obtained contains many conjecturally optimal packings.

1 Introduction

The problem of finding the maximum radius of n non-overlapping congruent *circles* in a unit square, or equivalently, the problem of maximizing the minimum distance between any two of *n points* in a unit square (this distance is denoted by d_n, and a corresponding packing is said to be *optimal*) has received a lot of attention; see [1], [3], [6] for recent results and references to earlier work. In the results to be presented here still another equivalent formulation will be used: Determine $N_p(\sigma)$, the maximum number of points with mutual distance at least 1 that can be placed into a square of side σ.

In [1] it is pointed out that it is not difficult to show that d_n behaves asymptotically according to

$$d_n \sim \sqrt{\frac{2}{\sqrt{3}n}}$$

as n tends to infinity. In the same place, good bounds for the error term are further asked for. Already in the early 1960s, Oler [7] proved a theorem that implies the following corollary: If X is a compact convex subset of the Euclidean plane, then the number of points with mutual distance at least 1 is at most

$$\frac{2}{\sqrt{3}}A(X) + \frac{1}{2}P(X) + 1,$$

where $A(X)$ is the area and $P(X)$ is the perimeter of the convex subset (see [2] for an elementary proof). If X is a square ($A(X) = \sigma^2$, $P(X) = 4\sigma$), then this result leads directly to the upper bound

$$\frac{\sqrt{3}}{2}N_p(\sigma) \leq \sigma^2 + \sqrt{3}\sigma + \frac{\sqrt{3}}{2} \tag{1}$$

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(where actually, in discussing the asymptotic behavior, the value of the constant term is insignificant).

In this paper, a lower bound on $N_p(\sigma)$ is obtained. The bound is constructive: packings that attain the lower bound are used in the proof. Some of the packings are very good. These packings, which are the cornerstones of the proof, are possibly optimal for many values of n (there is possibly even an infinite sequence of optimal packings of this type). This is in contrast to the series discussed in [3], which all contain only a finite number of optimal packings for very small values of n.

Before proceeding, we would like to point out that the dual problem of *covering* a square by circles has been studied earlier from a similar point-of-view. Kershner [4] and Verblunsky [8] obtained the following upper and lower bounds on $N_c(\sigma)$, the least number of circles of unit radius which can cover a square of side σ.

Theorem 1 *There is a* $c \geq \frac{1}{2}$ *such that, for all sufficiently large* σ, $\sigma^2 + c\sigma < \frac{3\sqrt{3}}{2} N_c(\sigma) < \sigma^2 + 8\sigma + 16$.

2 The Proof

We shall now give a construction of packings, which will lead to a desired lower bound on $N_p(\sigma)$. In the following presentation, however, we will explicitly consider packings in a unit square. The packings have the pattern depicted in Figure 1, where the side of the solid square is of unit length.

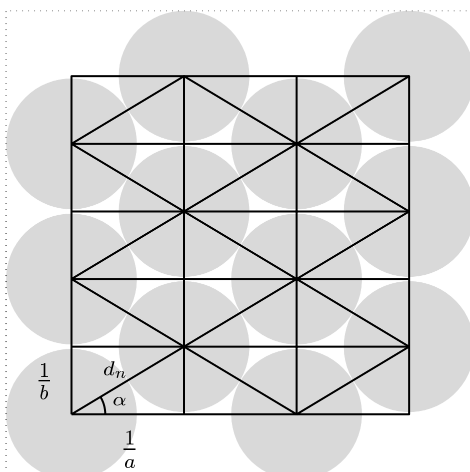

Figure 1

We have $a + 1$ columns and $b + 1$ rows of points (or circles) in Figure 1. Without loss of generality, we require that $b \geq a$. Furthermore, a circle in a column touches one or two circles in adjacent columns, but not necessarily (in practice, not) other circles in the same

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column. The total number of points is

$$n = \left\lfloor \frac{(a+1)(b+1)+1}{2} \right\rfloor, \tag{2}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. For circles not to overlap, we must have that

$$\frac{d_n}{2} \le \frac{1}{b}.$$

Straight-forward calculations give that $\alpha \ge \frac{\pi}{6}$ and, as $\tan\alpha = \frac{a}{b}$, that

$$b \le \sqrt{3}a.$$

Finally,

$$d_n = \sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = \frac{\sqrt{a^2+b^2}}{ab}. \tag{3}$$

We now consider the following values of a and b: $a_i = i$, $b_i = \lfloor \sqrt{3}i \rfloor$, $b'_i = \lfloor \sqrt{3}i \rfloor - 1$, and $b''_i = \lfloor \sqrt{3}i \rfloor - 2$ for $i \ge 3$. Using (2), these values of a and b lead to packings for values n_i, n'_i, and n''_i, respectively. It is a matter of direct calculations to show that $n_{i-1} < n''_i$, so we have that $\cdots < n_{i-2} < n''_{i-1} < n'_{i-1} < n_{i-1} < n''_i < n'_i < n_i < \cdots$. For other values of n, we construct packings by taking the smallest value in this series which is greater than or equal to n and by removing points from that packing.

Theorem 2 $\sigma^2 + \frac{1-\sqrt{3}}{2}\sigma \le \frac{\sqrt{3}}{2}N_p(\sigma) \le \sigma^2 + \sqrt{3}\sigma + \frac{\sqrt{3}}{2}$.

Proof The upper bound is from (1) and we prove here that the lower bound holds for all packings with $n_{i-1} < n \le n_i$ for $i \ge 3$. We consider the three cases $n'_i < n \le n_i$, $n''_i < n \le n'_i$, and $n_{i-1} < n \le n''_i$, and finally take the worst of the three bounds obtained. All packings in one of these intervals have the same minimum distance.

Case 1: *($n'_i < n \le n_i$)* Now $a = a_i$ and $b = b_i = \lfloor \sqrt{3}a \rfloor$. We first use (3) to get

$$d_n = \sqrt{\frac{1}{a^2} + \frac{1}{b^2}} \ge \sqrt{\frac{1}{a^2} + \frac{1}{3a^2}} = \frac{2}{\sqrt{3}a}.$$

When this packing is enlarged to get a packing with minimum distance 1, we get that the side of the square is $\sigma \le \frac{\sqrt{3}}{2}a$, so $a \ge \frac{2}{\sqrt{3}}\sigma$. The number of points is

$$\begin{aligned} n \ge n'_i + 1 &= \left\lfloor \frac{(a_i+1)(b_i-1+1)+1}{2} \right\rfloor + 1 \ge \left\lfloor \frac{(a+1)(\sqrt{3}a-1)+1}{2} \right\rfloor + 1 \\ &= \left\lfloor \frac{\sqrt{3}a^2 + (\sqrt{3}-1)a}{2} \right\rfloor + 1 \ge \frac{\sqrt{3}}{2}a^2 + \frac{\sqrt{3}-1}{2}a. \end{aligned}$$

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By combining this inequality and $a \geq \frac{2}{\sqrt{3}}\sigma$, we get that

$$\frac{\sqrt{3}}{2}n \geq \sigma^2 + \frac{\sqrt{3}-1}{2}\sigma. \tag{4}$$

Case 2: *($n_i'' < n \leq n_i'$)* Now $a = a_i$ and $b = b_i' = \lfloor\sqrt{3}a\rfloor - 1$. Again, we derive a lower bound on d_n. This is somewhat trickier than in Case 1. Using (3) we get that

$$\begin{aligned} d_n^2 &\geq \frac{4a^2 - 2\sqrt{3}a + 1}{3a^4 - 2\sqrt{3}a^3 + a^2}, \text{ so} \\ \sigma^2 &\leq \frac{3a^4 - 2\sqrt{3}a^3 + a^2}{4a^2 - 2\sqrt{3}a + 1} = \frac{3}{4}a^2 - \frac{\sqrt{3}}{8}a - \frac{1}{8} + \frac{1 - \sqrt{3}a}{8(4a^2 - 2\sqrt{3}a + 1)} \\ &\leq \frac{3}{4}a^2 - \frac{\sqrt{3}}{8}a - \frac{1}{8}, \end{aligned}$$

when $a \geq \frac{1}{\sqrt{3}}$. Furthermore,

$$\begin{aligned} n \geq n_i'' + 1 &= \left\lfloor \frac{(a_i+1)(b_i-2+1)+1}{2} \right\rfloor + 1 \\ &\geq \left\lfloor \frac{(a+1)(\sqrt{3}a-2)+1}{2} \right\rfloor + 1 = \left\lfloor \frac{\sqrt{3}a^2 + (\sqrt{3}-2)a - 1}{2} \right\rfloor + 1 \\ &\geq \frac{\sqrt{3}}{2}a^2 + \frac{\sqrt{3}-2}{2}a - \frac{1}{2} = \frac{2}{\sqrt{3}}\left(\frac{3}{4}a^2 + \frac{3-2\sqrt{3}}{4}a - \frac{\sqrt{3}}{4}\right) \\ &\qquad \text{(here we do not explicitly have } a \text{ as a function of } \sigma) \\ &= \frac{2}{\sqrt{3}}\left(\frac{3}{4}a^2 - \frac{\sqrt{3}}{8}a - \frac{1}{8} + \frac{6-3\sqrt{3}}{8}a + \frac{1-2\sqrt{3}}{8}\right) \\ &\geq \frac{2}{\sqrt{3}}\left(\sigma^2 + \frac{2\sqrt{3}-3}{4}\sigma + \frac{1-2\sqrt{3}}{8}\right). \end{aligned}$$

Hence

$$\frac{\sqrt{3}}{2}n \geq \sigma^2 + \frac{2\sqrt{3}-3}{4}\sigma + \frac{1-2\sqrt{3}}{8}. \tag{5}$$

Case 3: *($n_{i-1} < n \leq n_i''$)* This case is similar to Case 2. Now $a = a_i$ and $b = b_i'' =$

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$\lfloor\sqrt{3}a\rfloor - 2$. Using (3) we get that

$$\begin{aligned}
d_n^2 &\geq \frac{4a^2 - 4\sqrt{3}a + 4}{3a^4 - 4\sqrt{3}a^3 + 4a^2}, \text{ so} \\
\sigma^2 &\leq \frac{3a^4 - 4\sqrt{3}a^3 + 4a^2}{4a^2 - 4\sqrt{3}a + 4} = \frac{3}{4}a^2 - \frac{\sqrt{3}}{4}a - \frac{1}{2} + \frac{2 - \sqrt{3}a}{4a^2 - 4\sqrt{3}a + 4} \\
&\leq \frac{3}{4}a^2 - \frac{\sqrt{3}}{4}a - \frac{1}{2}
\end{aligned}$$

when $a \geq \frac{2}{\sqrt{3}}$. Now

$$\begin{aligned}
n \geq n_{i-1} + 1 &= \left\lfloor \frac{(a_{i-1}+1)(b_{i-1}+1)+1}{2} \right\rfloor + 1 \\
&\geq \left\lfloor \frac{a(\sqrt{3}(a-1)-1+1)+1}{2} \right\rfloor + 1 = \left\lfloor \frac{\sqrt{3}a^2 - \sqrt{3}a + 1}{2} \right\rfloor + 1 \\
&\geq \frac{\sqrt{3}}{2}a^2 - \frac{\sqrt{3}}{2}a + \frac{1}{2} = \frac{2}{\sqrt{3}}\left(\frac{3}{4}a^2 - \frac{3}{4}a + \frac{\sqrt{3}}{4}\right) \\
&= \frac{2}{\sqrt{3}}\left(\frac{3}{4}a^2 - \frac{\sqrt{3}}{4}a - \frac{1}{2} + \frac{\sqrt{3}-3}{4}a + \frac{2+\sqrt{3}}{4}\right) \\
&\geq \frac{2}{\sqrt{3}}\left(\sigma^2 + \frac{1-\sqrt{3}}{2}\sigma + \frac{2+\sqrt{3}}{4}\right) \\
&\geq \frac{2}{\sqrt{3}}\left(\sigma^2 + \frac{1-\sqrt{3}}{2}\sigma\right).
\end{aligned}$$

Hence

$$\frac{\sqrt{3}}{2}n \geq \sigma^2 + \frac{1-\sqrt{3}}{2}\sigma. \tag{6}$$

Comparing the bounds (4), (5), and (6) for packings with at least $n \geq n_2 + 1 = 7$ points reveals that (6) from Case 3 is worst. An analysis of this bound for small values of σ gives that the theorem holds for all positive values of σ. ■

3 A Series of Conjecturally Optimal Packings

Many of the packings used in the proof in the previous section are very good and possibly optimal. We shall now discuss a subseries of these packings; we conjecture that all packings in this subseries are optimal.

To get a possibly good packing, we want the fraction

$$\frac{a}{b}$$

to be very close to $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$ in the construction discussed earlier (then the interior of the packing is close to the hexagonal lattice packing in the plane). From elementary Diophantine approximation theory it is known that a "good approximation" to an irrational number is necessarily a partial fraction in the continued fraction expansion of this number; see [5, Ch. 7]. Furthermore, the continued fraction expansion of a quadratic irrational number is periodic [5, Theorem 7.19]. We now have that

$$\frac{\sqrt{3}}{3} = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cdots}}}}}}.$$

This expansion gives a sequence of partial fractions

$$\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{3}{5}, \frac{4}{7}, \frac{11}{19}, \frac{15}{26}, \frac{41}{71}, \dots,$$

where every second value, starting with 1, is greater than $\frac{\sqrt{3}}{3}$ and gives a valid, conjecturally optimal packing (when a is given the value of the numerator, and b is given the value of the denominator). The first four packings in this subseries have 2, 12, 120, and 1512 points (the packing with 12 points is shown in Figure 1).

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