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MODULES OVER HEREDITARY NOETHERIAN PRIME RINGS, II

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1. Introduction. Let R be a hereditary noetherian prime ring ((hnp)-ring) with enough invertible ideals. Torsion modules over bounded (hnp)-rings were studied by the author in [**10**; **11**]. All the results proved in [**10**; **11**] also hold for torsion R-modules having no completely faithful submodules. In Section 2, indecomposable injective torsion R-modules which are not completely faithful are studied, and they are shown to have finite periodicities (Theorem (2.8) and Corollary (2.9)). These results are used to determine the structure of quasi-injective and quasi-projective modules over bounded (hnp)-rings (Theorems (2.13), (2.14) and (2.15)). It was proved by Eisenbud and Robson [**3**] that if R has only finitely many maximal idempotent ideals, then R is an intersection of Dedekind prime rings. In Section 3, it is shown that any (hnp)-ring with enough invertible ideals is an intersection of Dedekind prime rings. The notations and terminology are essentially the same as in [**10**; **11**] except that '(hnp)-ring, not right primitive' has been replaced by 'bounded (hnp)-ring' in view of Lenagan [**8**].

2. Periodicity theorem. Throughout R is an (hnp)-ring with enough invertible ideals and Q is its classical quotient ring. Eisenbud and Robson [**2**; **3**] called a module M to be *completely faithful* if every submodule of each of its factor modules is faithful. The following is an immediate consequence of [**3**, Theorem (3.1)].

LEMMA (2.1). *If U is any uniform torsion R-module, then either U is completely faithful or every finitely generated submodule of U is unfaithful.*

The proof of the following theorem is essentially the same as that of [**10**, Theorem 4].

THEOREM (2.2). *Let E be an indecomposable injective torsion R-module, such that E is not completely faithful. There exists an infinite properly ascending chain of submodules*

(1) $(0) = x_0R < x_1R < x_2R < \ldots < x_MR < \ldots < E$

such that $x_{i+1}R/x_iR$ is a simple R-module, the members of the chain are the only submodules of E and $E = \bigcup_m x_mR$. Further, there exists a non-negative integer n such that $x_{i+1}R/x_iR \cong x_{j+1}R/x_jR$ if and only if $i \equiv j (\text{mod } n)$.

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This n is called the periodicity of E and the series (1) given above is called the *composition series* of E. If $n > 0$, E is said to be of *finite periodicity*. One of the main results of this section is that any indecomposable injective, torsion R-module which is not completely faithful, is of finite periodicity (Theorem (2.8)).

Eisenbud and Robson [**3**, p. 91] introduced the concept of cycles of maximal ideal. Let P and P' be two non-zero idempotent maximal ideals of R. Using the fact that $P \cap P'$ contains an invertible ideal X, [**3**, Proposition (1.6)] yields that $0_l(P) = 0_l(P')$ if and only if $P = P'$, where $0_l(P) = \{x \in Q | xP \subset P\}$. This fact gives that any two cycles of prime ideals in R are disjoint or equal. Let A be a maximal invertible ideal of R. A is an intersection of a cycle of prime ideals $P_1, P_2, \ldots, P_n$ [**3**]. We say that each P_i *belongs* to A. By [**3**, Corollary (4.7)] every non-zero prime ideal P of R belongs to a cycle, and hence to a maximal invertible ideal. Let M be a non-faithful simple R-module and $P = \text{ann}_R(M)$. If P belongs to a maximal invertible ideal A, we say M belongs to A.

In the notation of Theorem (2.2), if $P_i = \text{ann}(x_iR/x_{i-1}R)$ for every $i > 0$, then the sequence $(P_1, P_2, P_3, \ldots)$ is called the *prime sequence* of E. The prime ideals $P_1, P_2, P_3, \ldots$ are said to be *associated* with E. If E is of finite periodicity n, then the above prime sequence is periodic and its first n members $P_1, P_2, \ldots, P_n$ are all distinct.

Henceforth E will be an indecomposable injective torsion R-module such that E is not completely faithful and

$$(0) = x_0R < x_1R < x_2R < \ldots < x_mR < \ldots < E$$

is its composition series. Further $(P_1, P_2, \ldots, P_n \ldots)$ is the prime sequence of E.

LEMMA (2.3). *Let X be any uniserial module over a right artinian ring S and let*

$$X = X_0 > X_1 > X_2 > \ldots > X_t = (0)$$

be its unique composition series. If for any i with $0 \leqq i \leqq t - 1$, $P_i = \text{ann}(X_i/X_{i+1})$, then $X_iP_i = X_{i+1}$.

Proof. Since $X_iJ(S) = X_{i+1}$ and $P_i \supset J(S)$, the result follows.

LEMMA (2.4). *In E, $x_{i+1}P_{i+1} = x_iR$.*

Proof. Let $A = \text{ann}(x_{i+1}R)$. As $A \neq (0)$ by Lemma (2.1), R/A is generalized uniserial [**1**, Corollary (3.2)] and $P_{i+1} \supset A$, the result follows from Lemma (2.3).

LEMMA (2.5). *Let xR be a uniserial torsion, unfaithful R-module and $A = \text{ann}(xR)$. The ring $S = R/A$ is generalized uniserial and has homogeneous socle. Further, if $e_1S, e_2S, \ldots, e_nS$ is a Kupisch series of S satisfying $d(e_{i+1}S) = 1 + d(e_iS)$ for $i < n$, then $xR \cong e_nS$.*

Proof. Given in the proof of [**10**, Theorem 4].

LEMMA (2.6). *If E is of finite periodicity n, and $P_1, P_2, \ldots, P_n$ are first n members of the prime sequence of E, then the ideal $P_nP_{n-1}\ldots P_1$ is not eventually idempotent and the ideal $\bigcap_{i=1}^n P_i$ is a maximal invertible ideal.*

Proof. For any $k > 0$,

$$x_{kn}(P_nP_{n-1}\ldots P_1)^k = (0)$$

and by Lemma (2.4)

$$x_{kn}(P_nP_{n-1}\ldots P_1)^{k-1} = x_nR \neq (0).$$

Hence $B = P_nP_{n-1}\ldots P_1$ is not eventually idempotent. By [**3**, Theorem (4.2)], $B = XC$ for some invertible ideals X and an eventually idempotent ideal C. Clearly $X \neq R$. Hence there exists a maximal invertible ideal A containing X. We claim that $A \subset P_i$ for every i. Now the maximal ideals containing A^t for any $t > 0$ are among $P_1, P_2, \ldots, P_n$. Let $A \not\subset P_i$ for some i. For that i, R/A^t admits no simple module isomorphic as an R-module to a simple summand of R/P_i. Now, the number of summands in the expression of R/A^t as a direct sum of indecomposable right ideals is independent of t. Hence as $A^t \neq A^{t+1}$ for all t, for a large enough t, the generalized uniserial ring R/A^t admits a uniserial module M of length greater than n. Since the number of non-isomorphic simple R/A^t-module does not exceed n, the composition series of M, has at least two distinct isomorphic composition factors. Since the socle of M_R is a simple R/P_j-module for some j, the injective hull $E(M)$ is equivalent to E (see the definition in [**10**]). Thus the periodicity of $E(M)$ and hence of M is also n. Thus R/A^t admits n non-isomorphic simple modules. As R/A^t does not admit any simple module isomorphic to a simple summand of R/P_i, and the prime ideals of R containing A^t are among $P_1, P_2, \ldots, P_n$, we get that R/A^t has less than n non-isomorphic simple modules. This is a contradiction. Hence $A \subset \bigcap_{i=1}^n P_i$ and R/A admits n non-isomorphic simple modules. Then [**3**, Proposition (2.5) and Corollary (4.7)] yield that $P_1, P_2, \ldots, P_n$ constitute the set of all members of a cycle of prime ideals and $A = \bigcap_{i=1}^n P_i$. This completes the proof.

THEOREM (2.7). *Let M be a non-faithful simple module over an* (hnp)-*ring R with enough invertible ideals. If the cycle of prime ideals to which P =* ann (M) *belongs, is of length n, then the injective hull $E(M)$ of M is of periodicity n, and the members of the cycle to which P belongs constitute the totality of distinct prime ideals associated with $E(M)$.*

Proof. Now $E = E(M)$ is not completely faithful. We show that E is of finite periodicity. On the contrary let E be of zero periodicity. Then P is an idempotent maximal ideal. Let $(P = P_1, P_2, \ldots, P_n)$ be the cycle to which P belongs [**3**, Corollary (4.7)]. Then $X = \bigcap_i P_i$ is a maximal invertible ideal. On similar lines as in Lemma (2.6) for some large enough k, R/X^k admits a uniserial module N of length $> n$, and N has repeated composition factors. Hence $E(N)$ is of finite periodicity. Further the prime ideals associated with $E(N)$ are among P_i's $(1 \leqq i \leqq n)$ and as seen in proof of Lemma (2.6), they constitute a cycle. Consequently P is also a prime ideal associated with $E(N)$. This shows that $E(N)$ is of periodicity n and is equivalent to $E(M)$ i.e., $E(N)$ and $E(M)$ have submodules F and F' respectively such that $F \neq E(N)$ and $E(N)/F \cong E(M)/F'$ [**10**, p. 1180]. Hence $E(M)$ is also of periodicity n and the prime ideals associated with $E(M)$, being same as with $E(N)$, constitute a cycle.

Since any indecomposable injective torsion R-module which is not completely faithful is an injective hull of a simple non-faithful R-module, we get the following.

THEOREM (2.8). (Periodicity Theorem) *If E is an indecomposable injective torsion module over an* (hnp)*-ring R with enough invertible ideals and if E is not completely faithful, then E is of finite periodicity n; the distinct prime ideals associated with E are members of a cycle of prime ideals in R and their intersection is a maximal invertible ideal.*

Since any bounded (hnp)-ring has enough invertible ideals [**8**] and it admits no torsion completely faithful module, we get the following.

COROLLARY (2.9). *Any indecomposable injective torsion module over a bounded* (hnp)*-ring is of finite periodicity.*

THEOREM (2.10). *Let E be an indecomposable injective torsion R-module, which is not completely faithful. If the periodicity of E is n and $(P_1, P_2, \ldots, P_n, \ldots)$ is the prime sequence of E, then $R/P_nP_{n-1} \ldots P_1$ is a generalized uniserial ring with homogeneous socle.*

Proof. Let $(0) = x_0R < x_1R < \ldots < x_mR < \ldots < E$ be the composition series of E. Consider x_nR. If $A = \text{ann}(x_nR)$, by Lemma (2.5) R/A is a generalized uniserial ring with homogeneous socle. Since $P_i = \text{ann}(x_iR/x_{i-1}R)$ for $1 \leqq i \leqq n$, and hence $P_nP_{n-1} \ldots P_1 \subset A$, the result will follow if we show that $A = P_nP_{n-1} \ldots P_1$. By [**10**, Theorem 2], $\bar{R} = R/A$ has a Kupisch series $\bar{e}_1\bar{R}$, $\bar{e}_2\bar{R}, \ldots, \bar{e}_n\bar{R}$, with $d(\bar{e}_i\bar{R}) = i$ for $1 \leqq i \leqq n$. Further $x_nR \cong \bar{e}_n\bar{R}$ by Lemma (2.4) and

$$\bar{e}_nP_n \ldots P_{i+1} = \bar{e}_i\bar{R} \text{ for } i \geqq 1.$$

Suppose that $A \neq P_nP_{n-1} \ldots P_1$. Then the composition length $d(R/A) < d(R/P_nP_{n-1} \ldots P_1)$. Since $J(R/A) = \bigcap_i P_i/A$ and $J(R/P_nP_{n-1} \ldots P_1) = \bigcap_i P_i/P_nP_{n-1} \ldots P_1$, the number of components in the expressions of R/A and $R/P_nP_{n-1} \ldots P_1$ as direct sums of indecomposable right ideals are the same. Hence there exists a primitive idempotent $e + P_nP_{n-1} \ldots P_1$ of $R/P_n \ldots P_1$ such that $d(e + A)R/A) < d(e + P_nP_{n-1} \ldots P_1)R/P_n \ldots P_1)$. Now $(e + A)R/A \cong \bar{e}_i\bar{R}$ for some i and by using Lemma (2.5) we have $\bar{e}_i\bar{R}P_n \ldots$

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$P_2 = \bar{e}_1 R$ and $\bar{e}_i \bar{R} P_n P_{n-1} \ldots P_1 = (0)$. On the other hand as $(e + A)R/A$ is a proper homomorphic image of $(e + P_n P_{n-1} \ldots P_1) \cdot R/P_n P_{n-1} \ldots P_1$, we get from Lemma (2.4) $(e + P_n P_{n-1} \ldots P_1) \cdot \tilde{R} P_n P_{n-1} \ldots P_1 \neq (0)$ where $\tilde{R} = R/P_n P_{n-1} \ldots P_1$. But obviously $(e + P_n P_{n-1} \ldots P_1) \tilde{R} P_n P_{n-1} \ldots P_1 = (0)$. Hence we get a contradiction.

By Theorem (2.7), $X = \bigcap_{i=1}^{n} P_i$, where $P_1, P_2, \ldots, P_n$ are distinct prime ideals associated with E, is a maximal invertible ideal. We say that E *belongs* to the maximal invertible ideal X. It can be easily seen that any indecomposable injective torsion R-module E' is equivalent to E if and only if E' is not completely faithful and it also belongs to X.

As defined in [**11**, § 3] any torsion module M over a bounded (hnp)-ring S is said to be a *primary module* if for every pair of uniform elements $x, y \in M$, $E(xS)$ and $E(yS)$ are equivalent. Here any torsion R-module M having no completely faithful submodule is said to be a primary module if for every pair of uniform elements x, y in M, $E(xR)$ and $E(yR)$ are equivalent. Now notice that given $x \in E$, $xX^{t(x)} = (0)$ for some $t(x) > 0$. This property holds for every E' equivalent to E. Using this fact we obtain the following.

LEMMA (2.11). *Let M be a torsion R-module having no completely faithful submodule. Then M is primary R-module if and only if there exists a maximal invertible ideal X such that for each $x \in M$, $xX^{t(x)} = 0$ for some $t(x) \geqq 1$.*

We say that M is an X-primary module. We give a few applications of the above results to quasi-projective and quasi-injective R-modules. Quasi-projective torsion modules over bounded (hnp)-rings were studied in [**11**]. For definitions of quasi-injective and quasi-projective modules refer to [**10**]. It was shown in [**11**, Theorem 14] that if a bounded (hnp)-ring R admits no indecomposable injective torsion module of zero periodicity, then any torsion quasi-projective R-module is reduced. This along with Corollary (2.9) yield the following.

THEOREM (2.12). *Any torsion quasi-projective module over a bounded* (hnp)-*ring is reduced.*

This result along with [**11**, Theorem 13] yields the following.

THEOREM (2.13). *A torsion module over a bounded* (hnp)-*ring R is quasi-projective if and only if each of its primary components N is projective as an $R/$*ann *(N)-module.*

Let us recall that given two indecomposable injective torsion modules over a bounded (hnp)-ring R, we defined in [**10**], $M(E, E')$ as the kernel of a homomorphism from E to E' such that the kernel of every homomorphism from E to E' contains $M(E, E')$. If E is of finite periodicity n and E' is equivalent to E, then $d(M(E, E')) \leqq n - 1$.

THEOREM (2.14). *Let N be a torsion primary module over a bounded* (hnp)-*ring R. N is quasi-injective if and only if N is injective as an $R/\text{ann}\ (N)$-module.*

Proof. Let N be quasi-injective. By [**10**] $N = \oplus \sum_{i \in I} N_i$ where N_i are uniform such that

$$|d(N_i) - d(N_j)| \leqq d(M(E_i, E_j))$$

for all $i, j \in I$, where $E_i = E(N_i)$. Since all these E_i are equivalent, they have the same finite periodicity n, by Theorem (2.8). Hence

(1) $|d(N_i) - d(N_j)| \leqq n - 1$

for all $i, j \in I$. Thus if any N_i is injective and hence of infinite length then every N_j is of infinite length and hence injective [**11**, Lemma 2(a)], in that case N is faithful and injective. Let no N_i be of infinite length. Then N is reduced and because of (1) there exists a positive integer k such that $d(N_i) \leqq k$ for all i. By similar arguments as in [**11**, Theorem 12] we get ann $(N) \neq (0)$. Consequently N is a quasi-injective faithful module over the artinian ring $R/\text{ann}\ (N)$. Hence N is injective as an $R/\text{ann}\ (N)$-module. The converse is obvious.

Since very torsion module over a bounded (hnp)-ring is a direct sum of primary modules [**11**, Lemma 9], the above theorem and [**10**, Theorem 7], give the following.

THEOREM (2.15). *Let M be a module over a bounded* (hnp)-*ring R. Then M is quasi-injective if and only if it satisfies one of the following:*

(1) *If M is not a torsion module, then M is injective;*

(2) *if M is a torsion module, then every primary component N of M is injective as an $R/\text{ann}\ (N)$-module.*

3. Quotient rings. Throughout this section R is an (hnp)-ring with enough invertible ideals. Eisenbud and Robson [**3**, Theorem (4.9)] proved that if R has only finitely many idempotent maximal ideals, then R is an intersection of Dedekind prime rings. By following the techniques of Kuzmanovitch [**7**] we prove that any (hnp)-ring with enough invertible ideals is an intersection of Dedekind prime rings (Theorem (3.10)).

To avoid the trivial case we suppose that R is not a simple artinian ring. Throughout let A be a maximal invertible ideal of R. As R satisfies the restricted minimum condition [**2**, Theorem (1.3)], the set

$$\begin{aligned}\mathscr{C}(A) &= \{c \in R : cx \in A \text{ implies } x \in A\}\\ &= \{c \in R : xc \in A \text{ implies } x \in A\}\\ &= \{c \in R : cR + A = R\}\\ &= \{c \in R : cR + A^m = Rc + A^m = R \text{ for any } m\}.\end{aligned}$$

LEMMA (3.1). *$\mathscr{C}(A)$ is a multiplicative set and any member of $\mathscr{C}(A)$ is regular.*

Proof. Let $c \in \mathscr{C}(A)$ and let the left annihilator $l(c) \neq (0)$. Since $\bigcap_m A^m = (0)$, for some m, $l(c) \not\subset A^m$, [**3**, Lemma (4.1)]. Then c is not a regular module A^m. This is a contradiction. The fact that $\mathscr{C}(A)$ is multiplicatively closed is obvious.

For any right ideal I of R, the largest two-sided ideal contained in I is called the *bound* of I. I is said to be *bounded*, in case its bound in nonzero. For any essential right ideal I of R we know that R/I is of finite length [**2**, Theorem (1.3)]. Now any two distinct maximal invertible ideals of R are comaximal. An ideal $C(\neq R)$ of R is said to *belong* to a maximal invertible ideal A if $A^t \subset C$ for some t. Clearly any proper ideal can belong to not more than one maximal invertible ideal. Let $\mathscr{B}$ be the family of all those proper ideals of R which belongs to some maximal invertible ideal.

LEMMA (3.2). *Let B be any proper ideal of R. Then*

(i) $B = \bigcap_{i=1}^{k} C_i$, *where each C_i belongs to a maximal invertible ideal, say A_i, and these A_i are distinct; and*

(ii) *the maximal invertible ideals A_i are uniquely determined by B.*

Proof. Now $R/B = \oplus \sum_{i=1}^{t} \bar{e}_i\bar{R}$; $\bar{R} = R/B$, $\bar{e}_i$ are primitive orthogonal idempotents of $\bar{R}$. Each $\bar{e}_i\bar{R}$ is a non-faithful uniserial R-module; hence its R-injective hull E_i cannot be completely faithful. Consequently E_i belongs to some maximal invertible ideal $I_i(1 \leqq i \leqq t)$. If $B_i = \text{ann }(\bar{e}_i\bar{R})$, then each B_i belongs to I_i and $B = \bigcap_i B_i$. Combining those B_i which belong to same maximal invertible ideal, we can write

$$E = \bigcap_{i=1}^{u} C_i,$$

where C_i belongs to some maximal invertible ideal, say A_i, and these A_i are distinct. This proves (i).

To prove (ii) let $B = \bigcap_{j=1}^{u} D_j$, D_j belongs to some maximal invertible ideal A_j' and these A_j' are distinct. Since the C_i are pairwise comaximal, we have

$$R/B \cong \oplus \sum_{i=1}^{u} R/C_i$$

similarly $R/B \cong \oplus \sum_{j=1} R/D_j$. Now each R/C_j can be expressed as the direct sum of uniserial right R-modules. For two distinct C_i's the corresponding uniserial modules are not equivalent. The same thing can be said about the R/D_j. Using the Krull-Schmidt, Azumaya Theorem we get that for each C_i, given a uniserial direct summand of R/C_i, there is a D_j such that this uniserial direct summand is isomorphic to a uniserial direct summand of R/D_j's. Hence, A_i is the same as A_j'. This proves (ii).

So given a proper ideal B of R, $B = \bigcap_{i=1}^{k} C_i$, where each C_i belongs to some

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maximal invertible ideal A_i and the A_i are distinct. These maximal invertible ideals $A_i (1 \leqq i \leqq k)$ are said to be *associated* with B; the expression $B = \bigcap_{i=1}^{k} C_i$ is called an *mi-decomposition* of B.

LEMMA (3.3). *Let B be a proper ideal of R. R/B admits a simple module, which as an R-module belongs to a maximal invertible ideal A if and only if A is associated with B.*

Proof. Let $B = \bigcap_{i=1}^{t} C_i$ be an mi-decomposition of B and $A_1, A_2, \ldots\ldots, A_t$ be the respective maximal ideals associated with B. Let M be a simple R/B-module which as an R-module belongs to A. Let $P = \text{ann}_R(M)$; then $P \supset B$. Hence $P \supset C_i$ for some i. However $A_i^k \subset C_i$ for some i. We get $A_i \subset P$. Also $A \subset P$. Since any two distinct maximal invertible ideals are comaximal we get $A = A_i$. This proves necessity.

To prove sufficiency, let $A = A_i$. Let P be a prime ideal containing C_i. Then $A_i \subset P$, that is $A \subset P$. This yields that P is a member of the cycle of which A is an intersection. Clearly then any R/P-simple module, as an R-module, belongs to A; this is also a simple R/B module.

LEMMA (3.4). *Let K be an essential right ideal of R and A be a maximal invertible ideal of R. Then a composition factor of R/K belongs to A if and only if there exists a bounded right ideal I containing K such that A is associated with the bound of I.*

Proof. If B is the bound of I then R/I is a faithful R/B-module. Since R/I is a homomorphic image of R/K, the sufficiency follows from Lemma (3.3) and the fact that R/B is embeddable in a direct sum of finitely many copies of R/I.

By Eisenbud and Robson [**3**, Theorem (3.1)] R/K is a direct sum of a completely faithful and an unfaithful module. Since R/K has a composition factor belonging to A, R/K is not completely faithful. Consequently

$$R/K = I/K \oplus J/K$$

for some right ideals I and J containing K such that J/K is nonzero and unfaithful as an R-module, and I/K, if non-zero, is completely faithful. Then J/K has a composition factor belonging to A. However $R/I \cong J/K$. Consequently R/I is unfaithful and by Lemma (3.3) A is associated with the bound of I. This proves necessity and completes the proof of the lemma.

Let $\mathscr{C}'(A)$ be the set of those regular elements b of R such that R/bR has no composition factor belonging to A. Proofs of the next two lemmas are essentially on the same lines as respective proofs of Lemmas (2.3) and (2.4) in [**7**]; in the proofs replace M by maximal invertible ideals A and the expression of an ideal as a product of prime ideals by its mi-decomposition.

LEMMA (3.5). $\mathscr{C}(A) = \mathscr{C}'(A)$

LEMMA (3.6). *Let $K_R \subset aR \subset R$ and $a^{-1}K = \{r \in R : ar \in K\}$. Then $aR/K \cong R/a^{-1}K$.*

LEMMA (3.7). *R satisfies Ore conditions with respect to $\mathscr{C}(A)$.*

Proof. Take $a \in R$, $b \in \mathscr{C}(A)$. Take $K = a^{-1}(aR \cap bR)$. Then $R/K \cong aR/(aR \cap bR) \cong (aR + bR)/bR$. Consequently as R/bR is artinian [**2**, Theorem (1.3)], R/K is artinian. Hence by [**2**, Theorem 1.3)] K is an essential right ideal of R. As $b \in \mathscr{C}(A)$, by Lemma (3.5) R/K has no composition factor belonging to A. If $K + A \neq R$, then $K + A$ is a bounded right ideal whose bound B contains A, and consequently B belongs to A. This will contradict Lemma (3.4). Hence $K + A = R$. Then $1 = d + x$ for some $d \in K$, $x \in A$. Then $d \in \mathscr{C}(A) \cap K$ and $ad = br$ for some $r \in R$. Hence R satisfies the right Ore condition with respect to $\mathscr{C}(A)$. Similarly R satisfies the left Ore condition with respect to $\mathscr{C}(A)$.

Let Q be the classical quotient ring of R and R_A be the set of all those elements in Q which are of the form ab^{-1}, $a \in R$, $b \in \mathscr{C}(A)$. Then R_A is an over-ring of R, and hence by [**7**, Proposition (1.6)], R_A is an (hnp)-ring.

LEMMA (3.8). (i) $J(R_A) = AR_A = R_AA$.

(ii) *For all* $k \geqq 1$, $(AR_A)^k = A^kR_A$, $A^kR_A \cap R = A^k$ *and* $R/A^k = R_A/(AR_A)^k$ *under the canonical map* $\lambda : R/A^k \rightarrow R_A/A^kR_A$ *given by* $\lambda(r + A^k) = x + A^kR_A$.

(iii) *For any right ideal* I *of* R, $I + A^kR_A = IR_A + A^kR_A$.

Proof. Since for any $c \in \mathscr{C}(A)$, $\bar{x} \in R/A$ $\bar{x}c = \bar{o}$ implies $\bar{x} = \bar{o}$ and further as $Rc + A = R$, we get R/A is a right R_A-module and all its R-submodules are R_A-submodules. Now any elements of AR_A is of the type ad^{-1}, $a \in A$, $d \in \mathscr{C}(A)$; $dR + A = R$ yield $(d + a)R + A = R$. Consequently $d + a \in \mathscr{C}(A)$. Hence $1 + ad^{-1} = (d + a)d^{-1}$ is invertible in R_A. This proves that $AR_A \subset J(R_A)$.

R/A being semi-simple artinian is completely reducible as an R-module hence also as R_A-module. The mapping

$$\eta : R/A \rightarrow R_A/AR_A$$

given by $\eta(x + A) = x + AR_A$ is an R_A-homomorphism. Given any $d \in \mathscr{C}(A)$, since $ud + v = 1$ for some $u \in R$, $v \in A$, we get $d^{-1} + AR_A = u + AR_A$. This shows η is onto. Hence R_A/AR_A is a completely reducible R_A-module. This yields $J(R_A) \subset AR_A$. Hence $J(R_A) = AR_A = R_AA$. This immediately yields $(J(R_A))^k = A^kR_A$ for every k. Now $x \in A^kR_A \cap R$ yields $x = ad^{-1}$ for some $a \in A^k$, $d \in \mathscr{C}(A)$. This yields $xd \in A^k$. Consequently $x \in A^k$, as d is a regular module A^k. Hence $A^kR_A \cap R = A^k$.

Now the mapping $\lambda : R/A^k \rightarrow R_A/A^kR_A$ given by $\lambda(x + A^k) = x + A^kR_A$ is a ring homomorphism. On the same lines as for the mapping η, λ is onto. λ is also one-to-one, since $A^kR_A \cap R = A^k$. Hence $R/A^k \cong R_A/A^kR_A$. This proves (i) and (ii); (iii) is immediate from (ii).

Let $S = \{q \in Q : qB \subset R$ for some non-zero ideal B of $R\}$.

LEMMA (3.9). *S is an overring of R and is a simple Dedekind prime ring.*

Proof. Obviously S is an overring of R. Let I be any non-zero two-sided ideal

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of S. Then $I \cap R \neq (0)$. As R has enough invertible ideals, $I \cap R$ contains an invertible ideal B; then $1 \in B^{-1}B \subset I$ yields $I = S$; hence S is simple. Also by [**7**, Proposition (1.6)] S is an (hnp)-ring. Since S contains no proper idempotent ideal, S is a Dedekind prime ring [**3**, Theorem (1.2)].

THEOREM (3.10). *Any* (hnp)*-ring R with enough invertible ideals is an intersection of Dedekind prime rings.*

Proof. Let A be a maximal invertible ideal of R. By Lemma (3.9), $J(R_A) = AR_A \neq (0)$. Hence R_A has only finitely many maximal ideals and is bounded [**3**, Theorem (4.10)]. By Lenagan [**9**], R_A has enough invertible ideals. Consequently by [**3**, Theorem (4.9)], R_A is an intersection of Dedekind prime rings Thus if we show that R is an intersection of rings R_A, where A ranges over all maximal invertible ideals and the ring S in Lemma (3.9), the result follows.

Let T be the intersection. Clearly $R \subset T$. Consider $x \in T$. Let $C = \{a \in R | xa \in R\}$. There exists a non-zero two sided ideal B of R, such that $xB \subset R$. Let B' be the largest ideal of R satisfying $xB' \subset R$. Clearly $B' \neq (0)$. Suppose $B' \neq R$. Let $B' = \cap_{i=1}^{t} C_i$ be an mi-decomposition of B' and let C_i belong to the maximal invertible ideal A_i. Then $A_i^k \subset C_i$ for all i and some fixed k. Now for any maximal invertible ideal A of R, $x \in R_A$ implies that there exists $d \in R$ such that $xd \in R$ and $dR + A = R$. Consequently $d \in C$ and $C + A = R$. This yields $C + \cap_{i=1}^{t} A_i^k = R$. But $\cap_i A_i^k \subset B \subset C$. Hence $C = R$. This proves that $x \in R$. Hence $R = T$. This proves the theorem.

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