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GEOMETRIC AND FIXED POINT PROPERTIES IN PRODUCTS OF NORMED SPACES

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Abstract

Given two (real) normed (linear) spaces X and Y, let $X \otimes_1 Y = (X \otimes Y, \|\cdot\|)$, where $\|(x, y)\| = \|x\| + \|y\|$. It is known that $X \otimes_1 Y$ is 2-UR if and only if both X and Y are UR (where we use UR as an abbreviation for uniformly rotund). We prove that if X is m-dimensional and Y is k-UR, then $X \otimes_1 Y$ is $(m + k)$-UR. In the other direction, we observe that if $X \otimes_1 Y$ is k-UR, then both X and Y are $(k - 1)$-UR. Given a monotone norm $\|\cdot\|_E$ on $\mathbb{R}^2$, we let $X \otimes_E Y = (X \otimes Y, \|\cdot\|)$ where $\|(x, y)\| = \|(\|x\|_X, \|y\|_Y)\|_E$. It is known that if X is uniformly rotund in every direction, Y has the weak fixed point property for nonexpansive maps (WFPP) and $\|\cdot\|_E$ is strictly monotone, then $X \otimes_E Y$ has WFPP. Using the notion of k-uniform rotundity relative to every k-dimensional subspace we show that this result holds with a weaker condition on X.

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1. Introduction

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real normed linear spaces (normed spaces) and let $E = (\mathbb{R}^2, \|\cdot\|_E)$, where $\|\cdot\|_E$ is a monotone norm, that is, for any $s_1, s_2, t_1, t_2 \in \mathbb{R}$, $\|(s_1, t_1)\|_E \le \|(s_2, t_2)\|_E$ whenever $|s_1| \le |s_2|$ and $|t_1| \le |t_2|$. A straightforward way to define a norm on the product space $X \times Y$ is to set $\|(x, y)\| = \|(\|x\|_X, \|y\|_Y)\|_E$ for all $x \in X$ and $y \in Y$. We denote the resulting normed product space by $X \otimes_E Y$. The norm $\|\cdot\|_E$ is said to be strictly monotone if $\|(s_1, t_1)\|_E < \|(s_2, t_2)\|_E$ whenever $|s_1| < |s_2|$ and $|t_1| \le |t_2|$ or $|s_1| \le |s_2|$ and $|t_1| < |t_2|$. It is easy to see that for $1 \le p \le \infty$, the norm $\|\cdot\|_p$ on $\mathbb{R}^2$ is strictly monotone while, for $p = \infty$, it is monotone but not strictly monotone. If $\|\cdot\|_E$ is the standard p-norm $\|\cdot\|_p$ $(1 \le p \le \infty)$, then the product space normed as above is denoted by $X \otimes_p Y$ and is called the p-direct sum of X and Y. Throughout this article we assume that $\|\cdot\|_E$ is a monotone norm on $\mathbb{R}^2$. We shall avoid using the

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subscripts in the symbols $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_E$ when it is clear from the context which of these is meant.

A normed space X is said to be k-uniformly rotund (k-UR) [16] if, for every $\epsilon > 0$,

$$\inf\left\{1 - \frac{1}{k+1}\left\|\sum_{i=1}^{k+1} x_i\right\| : x_1, \ldots, x_{k+1} \in S_X, V(x_1, \ldots, x_{k+1}) \geq \epsilon\right\} > 0,$$

where

$$V(x_1, \ldots, x_{k+1}) = \sup\left\{\begin{vmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & \vdots & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{vmatrix} : f_1, \ldots, f_k \in B_{X^*}\right\}.$$

Here 1-UR is equivalent to uniform rotundity [4]. Given any $k \in \mathbb{Z}^+$, every k-dimensional subspace is trivially k-UR [19].

In [3], it is proved that $X \otimes_1 Y$ is 2-UR if and only if X, Y are uniformly rotund. We consider the following questions.

(1) If X is m-UR and Y is k-UR, does it imply that $X \otimes_1 Y$ is $(m + k)$-UR?
(2) If $X \otimes_1 Y$ is n-UR (or URE_n as defined below) for some $n \geq 2$, does it imply that there exist $m, k \in \mathbb{Z}^+$ such that X is m-UR (URE_m), Y is k-UR (URE_k) and $n = m + k$?

In this article, we make the following observations in connection with these questions.

(1) If X is m-dimensional and Y is k-UR, then $X \otimes_1 Y$ is $(m + k)$-UR.
(2) If the space $X \otimes_1 Y$ is k-UR for some positive integer $k > 1$, then both X and Y are $(k - 1)$-UR.

Let X be a normed space and let K be a nonempty subset of X. A map $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. A normed space X is said to have the weak fixed point property (WFPP) if for every nonempty weakly compact convex set K and nonexpansive map $T : K \to K$, there exists a point $x \in K$ such that $Tx = x$. A number of sufficient conditions for WFPP in normed spaces have been identified [9, 14]. Although it has been established that the Banach space $L_1[0, 1]$ does not have WFPP [1], it is still not known whether every reflexive Banach space has WFPP. Another interesting unsolved problem is whether WFPP is preserved in the p-direct sum of two normed spaces that have WFPP for any $1 \leq p \leq \infty$. Several conditions under which this happens have been identified (see, for example, [12, 17, 20]). We improve some existing results in the study of preservation of WFPP in products of normed spaces.

The approximation-theoretic notion of Chebyshev centres is closely related to fixed point theory for nonexpansive maps. For a nonempty bounded subset K and an element x of a normed space X, let $R(x, K) = \sup\{\|x - y\| : y \in K\}$. Also let $R(K) = \inf\{R(x, K) : x \in K\}$ and $C(K) = \{x \in K : R(x, K) = R(K)\}$. The set $C(K)$, called

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the Chebyshev centre of K, is convex whenever K is convex. If $C(K)$ is a proper subset of K for every nonempty weakly compact convex subset K of a normed space X, then X has WFPP [11]. Chebyshev centres are also closely connected with the geometry of normed spaces. This connection can be used to deduce WFPP from certain geometric properties.

A normed space X is said to be uniformly rotund in the direction of $z \in X, z \neq 0$ if, for every $\epsilon > 0$,

$$\inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, \|x-y\| \geq \epsilon, x - y \in \operatorname{span}\{z\}\right\} > 0.$$

If X is uniformly rotund in the direction of each $z \in X, z \neq 0$, then X is said to be uniformly rotund in every direction (URED). It is well known that a normed space X is URED if and only if the Chebyshev centre of every nonempty bounded convex subset has at most one point [5, 6]. This shows that URED spaces have WFPP. By introducing the notion of k-uniform rotundity relative to a k-dimensional subspace, a natural generalisation of this characterisation is made in [19]. A normed space X is said to be k-uniformly rotund relative to a k-dimensional subspace Y if, for every $\epsilon > 0$,

$$\inf\left\{1 - \frac{1}{k+1}\left\|\sum_{i=1}^{k+1} x_i\right\| : \begin{array}{l} x_1, \ldots, x_{k+1} \in S_X, V(x_1, \ldots, x_{k+1}) \geq \epsilon, \\ \operatorname{span}\{x_1 - x_{k+1}, \ldots, x_k - x_{k+1}\} = Y \end{array}\right\} > 0.$$

If X is k-uniformly rotund relative to every k-dimensional subspace, then we write that X is URE$_k$. The approximation-theoretic characterisation of URE$_k$ spaces observed in [19] is that a normed space X is URE$_k$ if and only if the Chebyshev centre of every nonempty bounded convex subset is either empty or is a convex set of dimension at most $k-1$. This proves that URE$_k$ spaces have WFPP for every $k \in \mathbb{Z}^+$.

For a normed space X, let

$$\mathcal{A}_X = \left\{Z \subseteq X : \begin{array}{l} Z = \langle 0\rangle \text{ or } Z \text{ is a linear subspace of X such that, for} \\ \text{each } k \in \mathbb{Z}^+ \text{ and } k\text{-dimensional subspace } Y \text{ of } Z, \\ X \text{ is not } k\text{-uniformly rotund relative to } Y \end{array}\right\}.$$

We know from [18] that if a normed space X is URE$_k$, then each member of $\mathcal{A}_X$ has dimension at most $k-1$ and that if the members of $\mathcal{A}_X$ are all finite dimensional, then the Chebyshev centres of nonempty bounded convex sets are finite dimensional. So, if the members of $\mathcal{A}_X$ are finite dimensional, then X has WFPP. Given an m-dimensional space X and an arbitrary normed space Y, we prove the following statements.

(1) If the members of $\mathcal{A}_Y$ are all finite dimensional, then the members of $\mathcal{A}_{X\otimes_1 Y}$ are all finite dimensional.
(2) If Y is URE$_k$, then $X \otimes_1 Y$ is URE$_{m+k}$ for any $k \in \mathbb{Z}^+$.
(3) If Y is k-UR, then $X \otimes_1 Y$ is $(m+k)$-UR for any $k \in \mathbb{Z}^+$.

The problem of preservation of WFPP in $X \otimes_1 Y$ when X is URED was reduced to the case $X = \mathbb{R}$ in [12] with the observation that if X is URED, then $X \otimes_1 Y$ has WFPP

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if and only if $\mathbb{R} \otimes_1 Y$ has WFPP. In [17], a condition on Y sufficient for $X \otimes_p Y$ to have WFPP when X is finite-dimensional and $1 \leq p \leq \infty$, is identified. The question was finally settled in [20], where it is proved that if X is finite dimensional, Y has WFPP and $\|\cdot\|_E$ is a strictly monotone norm, then $X \otimes_E Y$ has WFPP and consequently that if X is URED, Y has WFPP and $\|\cdot\|_E$ is a strictly monotone norm, then $X \otimes_E Y$ has WFPP.

In this paper, we prove that if the members of $\mathcal{A}_X$ are all finite dimensional, Y has WFPP and $\|\cdot\|_E$ is a strictly monotone norm, then $X \otimes_E Y$ has WFPP. As a consequence, we show that if X is URE$_k$, Y has WFPP and $\|\cdot\|_E$ is a strictly monotone norm, then $X \otimes_E Y$ has WFPP for any $k \in \mathbb{Z}^+$.

2. Rotundity properties in 1-direct sums

In order to study the inheritance of rotundity properties in 1-direct sums of normed spaces, we introduce the notion of k-uniform rotundity relative to an arbitrary subspace. This is analogous to the notion of uniform rotundity relative to an arbitrary subspace considered in [2].

Definition 2.1. Let X be any normed space and let $k \in \mathbb{Z}^+$. Let W be any subspace of X of dimension at least k. We say that X is k-uniformly rotund relative to W if

$$\inf\left\{1 - \frac{1}{k+1}\left\|\sum_{i=1}^{k+1} x_i\right\| : \begin{array}{l} x_1, \ldots, x_{k+1} \in S_X, V(x_1, \ldots, x_{k+1}) \geq \epsilon, \\ \operatorname{span}\{x_1 - x_{k+1}, \ldots, x_k - x_{k+1}\} \subseteq W \end{array}\right\} > 0$$

for all $\epsilon > 0$.

The following result is straightforward.

Proposition 2.2. *Let W be any subspace of a normed space X of dimension at least k. If X is k-uniformly rotund relative to W, then X is k-uniformly rotund relative to every k-dimensional subspace of W.*

It is proved in [19] that a normed space X is k-uniformly rotund relative to a k-dimensional subspace Y if and only if, for each $\epsilon > 0$,

$$\inf\left\{1 - \frac{1}{k+1}\left\|\sum_{i=1}^{k+1} x_i\right\| : \begin{array}{l} x_1, \ldots, x_{k+1} \in B_X, V(x_1, \ldots, x_{k+1}) \geq \epsilon, \\ \operatorname{span}\{x_1 - x_{k+1}, \ldots, x_k - x_{k+1}\} = Y \end{array}\right\} > 0.$$

This is achieved by noting that for any $x_1, \ldots, x_{k+1} \in B_X$, there exist $y_1, \ldots, y_{k+1} \in S_X$ such that $x_1, \ldots, x_{k+1} \in \operatorname{co}\{y_1, \ldots, y_{k+1}\}$ and, if $w \in B_X \cap \operatorname{aff}\{x_1, \ldots, x_{k+1}\}$,

$$\frac{(1 - \|w\|)V(x_1, \ldots, x_{k+1})}{(k+1)^{(k+3)/2}} \leq 1 - \frac{1}{k+1}\left\|\sum_{i=1}^{k+1} x_i\right\|.$$

Similar arguments yield the following result.

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Proposition 2.3. *A normed space X is k-uniformly rotund relative to a subspace W if and only if, for each $\epsilon > 0$,*

$$\inf\left\{1 - \frac{1}{k+1}\left\|\sum_{i=1}^{k+1} x_i\right\| : \begin{array}{l} x_1, \ldots, x_{k+1} \in B_X, V(x_1, \ldots, x_{k+1}) \geq \epsilon, \\ \operatorname{span}\{x_1 - x_{k+1}, \ldots, x_k - x_{k+1}\} \subseteq W \end{array}\right\} > 0.$$

Let $k \in \mathbb{Z}^+$ and let W be any subspace of a normed space X of dimension at least k. For any $\epsilon \geq 0$, let

$$\delta_X^{(k)}(\epsilon, W) = \inf\left\{1 - \frac{1}{k+1}\left\|\sum_{i=1}^{k+1} x_i\right\| : \begin{array}{l} x_1, \ldots, x_{k+1} \in B_X, V(x_1, \ldots, x_{k+1}) \geq \epsilon, \\ \operatorname{span}\{x_1 - x_{k+1}, \ldots, x_k - x_{k+1}\} \subseteq W \end{array}\right\}.$$

In [15], the following modulus of finite-dimensional uniform rotundity was introduced:

$$\Delta_X^{(k)}(\epsilon) = \inf_{Y \in \mathcal{S}_k(X)} \inf_{x \in S_X} \max_{\|y\|=\epsilon, y \in Y} \{\|x + y\|\} - 1, \quad \epsilon \geq 0,$$

where $\mathcal{S}_k(X)$ is the collection of all k-dimensional subspaces of X. A normed space X is k-uniformly rotund if and only if $\Delta_X^{(k)}(\epsilon) > 0$ for every $\epsilon > 0$ [13].

Given a subspace W of a normed space X of dimension at least k and $\epsilon \geq 0$, let

$$\Delta_X^{(k)}(\epsilon, W) = \inf_{Y \in \mathcal{S}_k(W)} \inf_{x \in S_X} \max_{\|y\|=\epsilon, y \in Y} \{\|x + y\|\} - 1,$$

where $\mathcal{S}_k(W)$ is the collection of all k-dimensional subspaces of W. From [19], if Y is a k-dimensional subspace of a normed space X, then

$$\delta_X^{(k)}\left(\frac{(k+1)\epsilon^k}{2^k(1+\epsilon)^k}, Y\right) \leq \Delta_X^{(k)}(\epsilon, Y)$$

for all $\epsilon > 0$ and

$$\left(\frac{1}{1+\epsilon}\right)\Delta_X^{(k)}\left(\frac{\epsilon}{(k+1)^{k+1}}, Y\right) \leq \delta_X^{(k)}(\epsilon, Y)$$

for all $\epsilon > 0$ such that $\delta_X^{(k)}(\epsilon, Y) < 1$.

Theorem 2.4. *Let X be a normed space and W a subspace of X of dimension at least $k \in \mathbb{Z}^+$. The space X is k-uniformly rotund relative to W if and only if $\Delta_X^{(k)}(\epsilon, W) > 0$ for every $\epsilon > 0$.*

Proof. For a subspace W of a normed space X of dimension at least k,

$$\Delta_X^{(k)}(\epsilon, W) = \inf\{\Delta_X^{(k)}(\epsilon, Y) : Y \text{ is a } k\text{-dimensional subspace of } W\}$$

and $\delta_X^{(k)}(\epsilon, W) = \inf\{\delta_X^{(k)}(\epsilon, Y) : Y \text{ is a } k\text{-dimensional subspace of } W\}$. Therefore,

$$\delta_X^{(k)}\left(\frac{(k+1)\epsilon^k}{2^k(1+\epsilon)^k}, W\right) \leq \Delta_X^{(k)}(\epsilon, W)$$

for all $\epsilon > 0$ and

$$\left(\frac{1}{1+\epsilon}\right)\Delta_X^{(k)}\left(\frac{\epsilon}{(k+1)^{k+1}}, W\right) \leq \delta_X^{(k)}(\epsilon, W)$$

for all $\epsilon > 0$ such that $\delta_X^{(k)}(\epsilon, W) < 1$. This proves that X is k-uniformly rotund relative to W if and only if $\Delta_X^{(k)}(\epsilon, W) > 0$ for every $\epsilon > 0$. □

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We give a proof of the continuity of the volume function, which we use later.

LEMMA 2.5. *Let X be a normed space. The map $(x_1, \dots, x_{k+1}) \to V(x_1, \dots, x_{k+1})$ is continuous with respect to the product topology on X^{k+1} for any $k \in \mathbb{Z}^+$.*

PROOF. For each $x_1, \dots, x_{k+1} \in X$, define $\Phi_{(x_1,\dots,x_{k+1})} : (B_{X^*})^k \to \mathbb{R}$ by

$$\Phi_{(x_1,\dots,x_{k+1})}(f_1, \dots, f_k) = \begin{vmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & \vdots & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{vmatrix}.$$

Then $\Phi_{(x_1,\dots,x_{k+1})}$ is a bounded and continuous real-valued function. Suppose that $C((B_{X^*})^k, \mathbb{R})$ is the Banach space of all bounded and continuous real-valued functions on $(B_{X^*})^k$ with the standard supremum norm $\|\cdot\|_\infty$. Then $\Phi_{(x_1,\dots,x_{k+1})} \in C((B_{X^*})^k, \mathbb{R})$ and $V(x_1, \dots, x_{k+1}) = \|\Phi_{(x_1,\dots,x_{k+1})}\|_\infty$. Hence, to prove that the volume map is continuous, it is enough to prove that the map $(x_1, \dots, x_{k+1}) \to \Phi_{(x_1,\dots,x_{k+1})}$ is continuous on X^{k+1}.

We shall prove this using induction on k. For $k = 1$, let $x_1, x_2, y_1, y_2 \in X$ and let $f \in B_{X^*}$. Then

$$\begin{aligned} |\Phi_{(x_1,x_2)}(f) - \Phi_{(y_1,y_2)}(f)| &= |f(x_2 - x_1) - f(y_2 - y_1)| \\ &\le |f(x_1 - y_1)| + |f(x_2 - y_2)| \le \|x_1 - y_1\| + \|x_2 - y_2\|. \end{aligned}$$

Thus, the map $(x_1, x_2) \to \Phi_{(x_1,x_2)}$ is continuous on X^2.

Suppose that for some $k \in \mathbb{Z}^+$, the map $(x_1, \dots, x_{k+1}) \to \Phi_{(x_1,\dots,x_{k+1})}$ is continuous on X^{k+1}. This implies that the map $(x_1, \dots, x_{k+1}) \to V(x_1, \dots, x_{k+1})$ is also continuous on X^{k+1}. We shall use the continuity of both of these functions to prove that the map $(x_1, \dots, x_{k+2}) \to \Phi_{(x_1,\dots,x_{k+2})}$ is continuous on X^{k+2}. Let $f_1, \dots, f_{k+1} \in B_{X^*}$. Then

$$\begin{aligned} &\Phi_{(x_1,\dots,x_{k+2})}(f_1, \dots, f_{k+1}) - \Phi_{(y_1,\dots,y_{k+2})}(f_1, \dots, f_{k+1}) \\ &= \begin{vmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+2}) \\ \vdots & \vdots & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+2}) \\ f_{k+1}(x_1) & \cdots & f_{k+1}(x_{k+2}) \end{vmatrix} - \begin{vmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+2}) \\ \vdots & \vdots & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+2}) \\ f_{k+1}(y_1) & \cdots & f_{k+1}(y_{k+2}) \end{vmatrix} \\ &\quad + \begin{vmatrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+2}) \\ \vdots & \vdots & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+2}) \\ f_{k+1}(y_1) & \cdots & f_{k+1}(y_{k+2}) \end{vmatrix} - \begin{vmatrix} 1 & \cdots & 1 \\ f_1(y_1) & \cdots & f_1(y_{k+2}) \\ \vdots & \vdots & \vdots \\ f_k(y_1) & \cdots & f_k(y_{k+2}) \\ f_{k+1}(y_1) & \cdots & f_{k+1}(y_{k+2}) \end{vmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} &\|\Phi_{(x_1,\dots,x_{k+2})} - \Phi_{(y_1,\dots,y_{k+2})}\|_\infty \\ &\quad \le \|x_1 - y_1\| V(x_2, \dots, x_{k+2}) + \cdots + \|x_{k+2} - y_{k+2}\| V(x_1, \dots, x_{k+1}) \\ &\qquad + \|y_1\| \cdot \|\Phi_{(x_2,\dots,x_{k+2})} - \Phi_{(y_2,\dots,y_{k+2})}\|_\infty + \cdots + \|y_{k+2}\| \cdot \|\Phi_{(x_1,\dots,x_{k+1})} - \Phi_{(y_1,\dots,y_{k+1})}\|_\infty. \end{aligned}$$

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By the induction assumption, the right-hand side of the above inequality tends to 0 as $(y_1, \ldots, y_{k+2}) \to (x_1 \ldots, x_{k+2})$ and hence so does the left-hand side, that is, the map $(x_1, \ldots, x_{k+2}) \to \Phi_{(x_1,\ldots,x_{k+2})}$ is continuous on X^{k+2}. Thus, for every $k \in \mathbb{Z}^+$, the map $(x_1, \ldots, x_{k+1}) \to \Phi_{(x_1,\ldots,x_{k+1})}$ is continuous on X^{k+1} and therefore the map $(x_1, \ldots, x_{k+1}) \to V(x_1, \ldots, x_{k+1})$ is continuous on X^{k+1}. □

We need the following lemma to proceed further.

Lemma 2.6. *Let X, Y be normed spaces and let $k \in \mathbb{Z}^+$. If Z is a subspace of Y of dimension at least k, then, for each $\epsilon > 0$, there exists c_ϵ with $0 \le c_\epsilon < 1$ such that $\delta^k_{X\otimes_1 Y}(\epsilon, \langle 0\rangle \otimes Z) \ge (1 - c_\epsilon)\delta^k_Y(\epsilon, Z)$.*

Proof. It is not difficult to see that $(x_1, y_1), \ldots, (x_{k+1}, y_{k+1}) \in X \otimes Y$ satisfies

$$\operatorname{span}\{(x_1, y_1) - (x_{k+1} - y_{k+1}), \ldots, (x_k, y_k) - (x_{k+1}, y_{k+1})\} \subseteq \langle 0\rangle \otimes Y$$

and $V((x_1, y_1), \ldots, (x_{k+1}, y_{k+1})) \ge \epsilon$ if and only if

$$x_1 = \cdots = x_{k+1},\ \operatorname{span}\{y_1 - y_{k+1}, \ldots, y_k - y_{k+1}\} \subseteq Y \quad \text{and} \quad V(y_1, \ldots, y_{k+1}) \ge \epsilon.$$

Let

$$\sup\left\{\|x\| : \begin{array}{l}(x, y_1), \ldots, (x, y_{k+1}) \in B_{X\otimes_1 Y},\\ V(y_1, \ldots, y_{k+1}) \ge \epsilon\end{array}\right\} = c_\epsilon.$$

We claim that $c_\epsilon < 1$.

If $c_\epsilon = 1$, there exist sequences $\{(x^{(n)}, y_1^{(n)})\}, \ldots, \{(x^{(n)}, y_{k+1}^{(n)})\}$ in $B_{X\otimes_1 Y}$ such that $\lim_{n\to\infty} \|x^{(n)}\| = 1$ and $V(y_1^{(n)}, \ldots, y_{k+1}^{(n)}) \ge \epsilon$ for all $n \in \mathbb{Z}^+$. But $\|x^{(n)}\| + \|y_i^{(n)}\| \le 1$ for $i \in \{1, \ldots, k + 1\}$, so $\lim_{n\to\infty} \|y_i^{(n)}\| = 0$ and, by Lemma 2.5, $\lim_{n\to\infty} V(y_1^{(n)}, \ldots, y_{k+1}^{(n)}) = 0$. This is not possible, because $V(y_1^{(n)}, \ldots, y_{k+1}^{(n)}) \ge \epsilon$ for all $n \in \mathbb{Z}^+$. Hence, $0 \le c_\epsilon < 1$.

Let Z be a subspace of Y, $(x, y_1), \ldots, (x, y_{k+1}) \in B_{X\otimes_1 Y}$, $V(y_1, \ldots, y_{k+1}) \ge \epsilon$ and $\operatorname{span}\{y_1 - y_{k+1}, \ldots, y_k - y_{k+1}\} \subseteq Z$. Then $0 < 1 - c_\epsilon \le 1 - \|x\| \le 1$, because $\|x\| \le c_\epsilon < 1$. Also $\|y_i\|/(1 - \|x\|) \le 1$ for $i \in \{1, \ldots, k + 1\}$. Now

$$\begin{aligned}
1 - \Big\|\frac{1}{k+1}\sum_{i=1}^{k+1}(x, y_i)\Big\| &= 1 - \|x\| - \Big(\frac{1}{k+1}\Big\|\sum_{i=1}^{k+1} y_i\Big\|\Big)\\
&= (1 - \|x\|)\Big(1 - \frac{1}{(k+1)(1 - \|x\|)}\Big\|\sum_{i=1}^{k+1} y_i\Big\|\Big) \ge (1 - c_\epsilon)\Big(1 - \frac{1}{(k+1)(1 - \|x\|)}\Big\|\sum_{i=1}^{k+1} y_i\Big\|\Big)\\
&\ge (1 - c_\epsilon)\delta^k_Y\Big(\frac{\epsilon}{(1 - \|x\|)^k}, Z\Big) \ge (1 - c_\epsilon)\delta^k_Y(\epsilon, Z).
\end{aligned}$$

Thus, $\delta^k_{X\otimes_1 Y}(\epsilon, \langle 0\rangle \otimes Z) \ge (1 - c_\epsilon)\delta^k_Y(\epsilon, Z)$. □

Theorem 2.7. *Let $X = \mathbb{R}^m$ with an arbitrary norm and let Y be any normed space.*

(1) *If the members of $\mathcal{A}_Y$ are all finite dimensional, then the members of $\mathcal{A}_{X\otimes_1 Y}$ are all finite dimensional.*

(2) *If Y is URE$_k$, then $X \otimes_1 Y$ is URE$_{m+k}$.*

(3) *If Y is k-UR, then $X \otimes_1 Y$ is $(m + k)$-UR.*

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Proof. If $\dim(Y) \le k$, then $\dim(X \otimes_1 Y) \le m + k$ and $X \otimes_1 Y$ must be $(m + k)$-UR.

Suppose that $\dim(Y) > k$. Let $W = \operatorname{span}\{w_1, \dots, w_{m+k}\}$ be an $(m + k)$-dimensional subspace of $X \otimes_1 Y$. For each $j \in \{1, \dots, m + k\}$, choose $x_j \in X$, $y_j \in Y$ such that $w_j = (x_j, y_j)$. Choose $\alpha_{(i,j)} \in \mathbb{R}$ for $i, j \in \{1, \dots, m\}$ such $x_j = \alpha_{(1,j)}e_1 + \cdots + \alpha_{(m,j)}e_m$.

Suppose that $\lambda_1, \dots, \lambda_{m+k} \in \mathbb{R}$. Then $\lambda_1 w_1 + \cdots + \lambda_{m+k} w_{m+k} \in \langle 0\rangle \otimes Y$ if and only if $\lambda_1 x_1 + \cdots + \lambda_{m+k} x_{m+k} = 0$. This is equivalent to

$$\begin{bmatrix} \alpha_{(1,1)} & \cdots & \alpha_{(1,m+k)} \\ \vdots & \vdots & \vdots \\ \alpha_{(m,1)} & \cdots & \alpha_{(m,m+k)} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{m+k} \end{bmatrix} = 0.$$

Hence, $\Lambda = \{(\lambda_1, \dots, \lambda_{m+k}) \in \mathbb{R}^{m+k} : \lambda_1 w_1 + \cdots + \lambda_{m+k} w_{m+k} \in \langle 0\rangle \otimes Y\}$ is a subspace of $\mathbb{R}^{m+k}$ of dimension at least k. The space Λ is isomorphic to $W \cap (\langle 0\rangle \otimes Y)$ under the linear map $(\lambda_1, \dots, \lambda_{m+k}) \to \lambda_1 w_1 + \cdots + \lambda_{m+k} w_{m+k}$ from Λ onto $W \cap (\langle 0\rangle \otimes Y)$. Hence, $\dim(W \cap (\langle 0\rangle \otimes Y))$ is at least k.

(1) Suppose that the members of $\mathcal{A}_Y$ are all finite dimensional. Let W be an infinite-dimensional subspace of $X \otimes_1 Y$. If $w_1, w_2, w_3, \dots \in W$ are linearly independent, then $\operatorname{span}\{w_1, \dots, w_{m+k}\} \cap \langle 0\rangle \otimes Y$ has dimension at least k for each $k \in \mathbb{Z}^+$. Consequently, $W \cap \langle 0\rangle \otimes Y$ is infinite dimensional. Since the members of $\mathcal{A}_Y$ are all finite dimensional, by isometry, the members of $\mathcal{A}_{\langle 0\rangle \otimes Y}$ are all finite dimensional. So, we can choose a k-dimensional subspace Z of Y such that $\langle 0\rangle \otimes Z \subset W$ and Y is k-uniformly rotund relative to Z. By Lemma 2.6, it follows that $X \otimes_1 Y$ is k-uniformly rotund relative to $\langle 0\rangle \otimes Z$, which proves that $W \notin \mathcal{A}_{X \otimes_1 Y}$. Thus, the members of $\mathcal{A}_{X \otimes_1 Y}$ are all finite dimensional.

(2) Let W be an $(m + k)$-dimensional subspace of $X \otimes_1 Y$. Then there exists a k-dimensional subspace Z of Y such that $\langle 0\rangle \otimes Z \subset W$. If Y is k-uniformly rotund relative to Z, then, by Lemma 2.6, $X \otimes_1 Y$ is k-uniformly rotund relative to $\langle 0\rangle \otimes Z$. From [19], if a normed space N is not n-uniformly rotund relative to an n-dimensional subspace Y and Z is a subspace of Y of dimension q, then N is not q-uniformly rotund relative to Z. Thus, $X \otimes_1 Y$ is $(m + k)$-uniformly rotund relative to W. It follows that if Y is URE_k, then $X \otimes_1 Y$ is URE_{m+k}.

(3) Suppose that Y is k-UR and $\epsilon > 0$. Then $\delta^k_{X \otimes_1 Y}(\epsilon, \langle 0\rangle \otimes Y) \ge (1 - c_\epsilon)\delta^k_Y(\epsilon) > 0$ by Lemma 2.6, so $\Delta^{(k)}_{X \otimes_1 Y}(\epsilon, \langle 0\rangle \otimes Y) > 0$ by Theorem 2.4. For an $(m + k)$-dimensional subspace W of $X \otimes_1 Y$, if Z is a k-dimensional subspace of Y such that $\langle 0\rangle \otimes Z \subseteq W$, then $\Delta^{(m+k)}_{X \otimes_1 Y}(\epsilon, W) \ge \Delta^{(k)}_{X \otimes_1 Y}(\epsilon, \langle 0\rangle \otimes Z)$. So,

$$\begin{aligned} \Delta^{(m+k)}_{X \otimes_1 Y}(\epsilon, W) &\ge \inf\{\Delta^{(k)}_{X \otimes_1 Y}(\epsilon, \langle 0\rangle \otimes Z) : Z \text{ is a subspace of } Y,\ \dim(Z) = k, \langle 0\rangle \otimes Z \subseteq W\} \\ &\ge \inf\{\Delta^{(k)}_{X \otimes_1 Y}(\epsilon, \langle 0\rangle \otimes Z) : Z \text{ is a subspace of } Y,\ \dim(Z) = k\} \\ &= \Delta^{(k)}_{X \otimes_1 Y}(\epsilon, \langle 0\rangle \otimes Y). \end{aligned}$$

But now $\Delta^{(m+k)}_{X \otimes_1 Y}(\epsilon) = \inf\{\Delta^{(m+k)}_{X \otimes_1 Y}(\epsilon, W) : W \text{ is a subspace of } X \otimes_1 Y,\ \dim(W) = m + k\}$ is at least $\Delta^{(k)}_{X \otimes_1 Y}(\epsilon, \langle 0\rangle \otimes Y)$. Thus, $\Delta^{(m+k)}_{X \otimes_1 Y}(\epsilon) > 0$, which proves that $X \otimes_1 Y$ is $(m + k)$-UR. □

We illustrate with an example showing how the above result can be used to obtain k-UR (URE$_k$) renormings of spaces which are UR (or URED) for arbitrary $k \in \mathbb{Z}^+$.

Example 2.8. For $k \in \mathbb{Z}^+$, $x \in l_2$, let $x' = (x(1), \ldots, x(k))$, $x'' = (x(k+1), x(k+2), \ldots)$. Define $\|x\|_1^{(k)} = \|x'\|_1 + \|x''\|_2$ for all $x \in l_2$. Then $(l_2, \|\cdot\|_1^{(k)})$ is $(k+1)$-UR but not k-UR.

To see this, note that from the definition of $\|x\|_1^{(k)}$, the space $(l_2, \|\cdot\|_1^{(k)})$ is isometrically isomorphic to $(\mathbb{R}^k, \|\cdot\|_1) \otimes_1 (l_2, \|\cdot\|_2)$. Therefore, by Theorem 2.7, this space is $(k+1)$-UR. It is not k-UR, because it contains $(\mathbb{R}^k, \|\cdot\|_1)$ as a subspace.

We now give a necessary condition for the 1-direct sum of two normed spaces to be k-UR for $k > 1$. The case $k = 2$ was solved in [3] and we adapt the same technique for the general case.

Theorem 2.9. *Let X and Y be normed spaces. For any $k \geq 2$, if $X \otimes_1 Y$ is k-UR, then both X and Y are $(k-1)$-UR.*

Proof. Let $x_1, \ldots, x_k \in B_X$ and $y \in Y$ with $\|y\| = (k-1)/k$. If $z_i = (x_i/k, y)$ for $i \in \{1, \ldots, k\}$ and $z_{k+1} = (0, ky/(k-1))$, then $\|z_i\| \leq 1$ for $i \in \{1, \ldots, k\}$ and $\|z_{(k+1)}\| = 1$. Also,

$$\frac{1}{k+1}\Big\|\sum_{i=1}^{k+1} z_i\Big\| = \frac{1}{k+1}\Big(\frac{1}{k}\Big\|\sum_{i=1}^{k} x_i\Big\| + \Big(k + \frac{k}{k-1}\Big)\|y\|\Big) = \frac{1}{k+1}\Big(\frac{1}{k}\Big\|\sum_{i=1}^{k} x_i\Big\| + k\Big)$$

and $V(z_1, \ldots, z_{k+1}) \geq V(z_1, \ldots, z_k)\operatorname{dist}(z_{k+1}, \operatorname{aff}\{z_1, \ldots, z_k\})$ [7]. Here

$$V(z_1, \ldots, z_k) = V((x_1/k, y), \ldots) = V((x_1/k, 0), \ldots) = \frac{1}{k^{k-1}} V(x_1, \ldots, x_k)$$

while, for any $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $\sum_{i=1}^k \lambda_i = 1$,

$$\begin{aligned}\|z_{k+1} - (\lambda_1 z_1 + \cdots + \lambda_k z_k)\| &= \frac{\|\lambda_1 x_1 + \cdots + \lambda_k x_k\|}{k} + \frac{\|y\|}{k-1}\\ &\geq \frac{1}{k}(\operatorname{dist}(0, \operatorname{aff}\{x_1, \ldots, x_k\}) + 1).\end{aligned}$$

Since $x_1, \ldots, x_k \in B_X$,

$$V(z_1, \ldots, z_{k+1}) \geq \frac{1}{k^k} V(x_1, \ldots, x_k)\,(\operatorname{dist}(0, B_X \cap \operatorname{aff}\{x_1, \ldots, x_k\}) + 1).$$

Suppose that X is not $(k-1)$-uniformly rotund. Choose $\epsilon > 0$ with $\delta_X^{(k-1)}(\epsilon) = 0$. There exist sequences $\{x_1^{(n)}\}, \ldots, \{x_k^{(n)}\}$ in B_X such that $V(x_1^{(n)}, \ldots, x_k^{(n)}) \geq \epsilon$ for all n while $\lim_{n\to\infty}(1/k)\|\sum_{i=1}^k x_i^{(n)}\| = 1$. Thus, using the remark which precedes Proposition 2.3, $\lim_{n\to\infty}\operatorname{dist}(0, B_X \cap \operatorname{aff}\{x_1^{(n)}, \ldots, x_k^{(n)}\}) = 1$. So, we may assume that $\operatorname{dist}(0, \operatorname{aff}\{x_1^{(n)}, \ldots, x_k^{(n)}\}) \geq \frac{1}{2}$ for all $n \in \mathbb{Z}^+$.

Choose $y \in Y$ with $\|y\| = (k-1)/k$. Let $z_i^{(n)} = (x_i^{(n)}/k, y)$ for $n \in \mathbb{Z}^+$ and $i \in \{1, \ldots, k\}$ and $z_{k+1}^{(n)} = (0, ky/(k-1))$. Then

$$\lim_{n\to\infty}\frac{1}{k+1}\Big\|\sum_{i=1}^{k+1} z_i^{(n)}\Big\| = \lim_{n\to\infty}\frac{1}{k+1}\Big(\frac{1}{k}\Big\|\sum_{i=1}^{k} x_i^{(n)}\Big\| + k\Big) = 1.$$

But $V(z_1^{(n)}, \ldots, z_{k+1}^{(n)}) \geq 3V(x_1^{(n)}, \ldots, x_k^{(n)})/2k^k \geq 3\epsilon/2k^k$ for $n \in \mathbb{Z}^+$. By a sequential characterisation of k-uniform rotundity, it follows that $X \otimes_1 Y$ is not k-UR. □

3. Preservation of WFPP in some product spaces

To prove that a normed space X has WFPP, it is enough to prove that every nonempty separable weakly compact convex subset of X has the fixed point property for nonexpansive maps [9]. We use the following well-known result.

THEOREM 3.1 [8, 10]. *Let K be a nonempty weakly compact convex subset of a normed space. Suppose that $T : K \to K$ is a nonexpansive map and K is minimal with respect to being a closed convex subset of K that is invariant under T. Then there exists a sequence $\{x_n\}$ in K such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ and, for such a sequence (called an approximate fixed point sequence), $\lim_{n\to\infty} \|x - x_n\| = \operatorname{diam}(K)$ for all $x \in K$.*

Throughout this section, the following assumptions hold:

(1) X, Y are normed spaces and $E = (\mathbb{R}^2, \|\cdot\|_E)$, where $\|\cdot\|_E$ is strictly monotone;
(2) K is a nonempty separable weakly compact convex subset of $X \otimes_E Y$;
(3) $K_X = \{x \in X : (x, y) \in K\}$ and $K_Y = \{y \in Y : (x, y) \in K\}$;
(4) $T : K \to K$ is a nonexpansive map;
(5) K is minimal as a closed convex subset of K that is invariant under T;
(6) $\{(a_n, b_n)\}$ is an approximate fixed point sequence for T in K such that, for every $(x, y) \in K$, the limits $\lim_{n\to\infty} \|x - a_n\|$ and $\lim_{n\to\infty} \|y - b_n\|$ exist.

The existence of a sequence $\{(a_n, b_n)\}$ as described in assumption (6) is observed in [12] and can be verified easily by using the separability of K and a standard diagonal sequence argument.

LEMMA 3.2. *For any $c \geq 0$, the set $\{x \in K_X : \lim_{n\to\infty} \|x - a_n\| = c\}$ is convex and therefore contained in a translation of some $Y \in \mathcal{A}_X$.*

PROOF. By Lemma 3.1, $\lim_{n\to\infty} \|(x - a_n, y - b_n)\| = \lim_{n\to\infty} \|(\|x - a_n\|, \|y - b_n\|)\| = \|(\lim_{n\to\infty} \|x - a_n\|, \lim_{n\to\infty} \|y - b_n\|)\| = \operatorname{diam}(K)$ for all $(x, y) \in K$. Since K is convex, K_X is convex. Let $c \geq 0$. By the strict monotonicity of $\|\cdot\|_E$, if $d_1, d_2 \geq 0$ and $\|(c, d_1)\|_E = \|(c, d_2)\|_E$, then we must have $d_1 = d_2$. Let $x_1, x_2 \in K_X$ be such that $\lim_{n\to\infty} \|x_1 - a_n\| = c = \lim_{n\to\infty} \|x_2 - a_n\|$. Choose $y_1, y_2 \in Y$ such that $(x_1, y_1), (x_2, y_2) \in K$. Then $\|(c, \lim_{n\to\infty} \|y_1 - b_n\|)\| = \operatorname{diam}(K) = \|(c, \lim_{n\to\infty} \|y_2 - b_n\|)\|$. So, there exists $d_c \geq 0$ such that $\lim_{n\to\infty} \|y_1 - b_n\| = d_c = \lim_{n\to\infty} \|y_2 - b_n\|$. Let $\lambda \in (0, 1)$. Now $\lim_{n\to\infty} \|(1 - \lambda)x_1 + \lambda x_2 - a_n\| \leq c$, $\lim_{n\to\infty} \|(1 - \lambda)y_1 + \lambda y_2 - b_n\| \leq d_c$ and also $\|(\lim_{n\to\infty} \|(1 - \lambda)x_1 + \lambda x_2 - a_n\|, \lim_{n\to\infty} \|(1 - \lambda)y_1 + \lambda y_2 - b_n\|)\| = \operatorname{diam}(K) = \|(c, d_c)\|$. Again by strict monotonicity, we have $\lim_{n\to\infty} \|(1 - \lambda)x_1 + \lambda x_2 - a_n\| = c$ and so $\{x \in K_X : \lim_{n\to\infty} \|x - a_n\| = c\}$ is a convex set.

Let $k \in \mathbb{Z}^+$. If $x_1, \ldots, x_{k+1} \in K_X$ with $\lim_{n\to\infty} \|x_i - a_n\| = c$ for each i, then the convexity of $\{x \in K_X : \lim_{n\to\infty} \|x - a_n\| = c\}$ gives

$$\lim_{n\to\infty} \left\| \frac{1}{k+1} \sum_{i=1}^{k+1} x_i - a_n \right\| = c.$$

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Consequently, if $V(x_1, \dots, x_{k+1}) > 0$, then, by a sequential characterisation of k-uniform rotundity relative to a k-dimensional subspace [19], it follows that X is not k-uniformly rotund relative to $\operatorname{span}\{x_1 - x_{k+1}, \dots, x_k - x_{k+1}\}$. From [18], if K is a nonempty subset of a normed space X such that X is not k-uniformly rotund relative to $\operatorname{span}\{u_1, \dots, u_k\}$ for all $k \in \mathbb{Z}^+$ and linearly independent $u_1, \dots, u_k \in K$, then $\operatorname{span}(K) \in \mathcal{A}_X$. Thus, $\{x \in K_X : \lim_{n\to\infty} \|x - a_n\| = c\}$ is contained in a translation of Y for some $Y \in \mathcal{A}_X$. □

Lemma 3.3.

(1) *If* $\dim K_X \geq k+1$, *then, for some* $c \geq 0$, *the set* $\{x \in K_X : \lim_{n\to\infty} \|x - a_n\| = c\}$ *is of dimension at least* k.

(2) *If* $\dim K_X = \infty$, *then the set* $\{x \in K_X : \lim_{n\to\infty} \|x - a_n\| = c\}$ *is infinite dimensional for some* $c \geq 0$.

Proof. (1) Suppose that K_X is of dimension at least $k+1$. Choose affinely independent $x_0, \dots, x_{k+1} \in K_X$ such that $\lim_{n\to\infty} \|x_0 - a_n\| \leq \cdots \leq \lim_{n\to\infty} \|x_{k+1} - a_n\|$. If the first k or the last k of the above inequalities are equations, then we are done. If this is not the case, then $\lim_{n\to\infty} \|x_p - a_n\| < c < \lim_{n\to\infty} \|x_{p+1} - a_n\|$ for some p and some $c > 0$. The map $x \to \lim_{n\to\infty} \|x - a_n\|$ on the convex set K_X is continuous. So, there exist $\lambda_1, \dots, \lambda_{k+1-p}, \mu_1, \dots, \mu_p \in (0,1)$ such that $\lim_{n\to\infty} \|(1-\lambda_i)x_0 + \lambda_i x_{p+i} - a_n\| = c$ for $i \in \{1, \dots, k+1-p\}$ and $\lim_{n\to\infty} \|(1-\mu_i)x_{p+1} + \mu_i x_i - a_n\| = c$ for $i \in \{1, \dots, p\}$. Let $u_i = (1-\lambda_i)x_0 + \lambda_i x_{p+i}$ for $i \in \{1, \dots, k+1-p\}$ and $u_{k+1-p+i} = (1-\mu_i)x_{p+1} + \mu_i x_i$ for $i \in \{1, \dots, p\}$. The affine independence of $x_0, \dots, x_{k+1}$ implies the affine independence of $u_1, \dots, u_{k+1}$. Thus, the set $\{x \in K_X : \lim_{n\to\infty} \|x - a_n\| = c\}$ is of dimension at least k.

(2) Suppose that K_X is infinite dimensional. Then there exist affinely independent $x_0, x_1, x_2, \dots \in K_X$ such that $\lim_{n\to\infty} \|x_0 - a_n\| \leq \lim_{n\to\infty} \|x_1 - a_n\| \leq \cdots$. If infinitely many of the above inequalities are equations, then we have nothing to prove. If this is not the case, then we may assume without loss of generality that for some $p \geq 1$, $\lim_{n\to\infty} \|x_p - a_n\| < c < \lim_{n\to\infty} \|x_{p+1} - a_n\|$. By the same arguments as in (1), for each $k \geq p$, there exist $u_1, \dots, u_{k+1} \in \{x \in K_X : \lim_{n\to\infty} \|x - a_n\| = c\}$ which are affinely independent. Thus, $\{x \in K_X : \lim_{n\to\infty} \|x - a_n\| = c\}$ is infinite dimensional. □

Theorem 3.4. *If the members of* $\mathcal{A}_X$ *are finite dimensional and* Y *has WFPP, then* $X \otimes_E Y$ *has WFPP.*

Proof. By Lemma 3.2, $\{x \in K_X : \lim_{n\to\infty} \|x - a_n\| = c\}$ is finite dimensional for any $c \geq 0$ and so, by Lemma 3.3(2), K_X is finite dimensional, that is, $\operatorname{span}(K_X - x_0)$ is finite dimensional for $(x_0, y_0) \in K$. Now $K - (x_0, y_0) \subseteq (K_X - x_0) \times (K_Y - y_0) \subseteq Z \times Y$, where Z is a finite-dimensional subspace of X. Thus, K is contained in a translation of $Z \otimes_E Y$ for some finite-dimensional subspace Z of X. From [20], $Z \otimes_E Y$ has WFPP and so $X \otimes_E Y$ has WFPP. □

Corollary 3.5. *If* X *is* URE_k *and* Y *has WFPP, then* $X \otimes_E Y$ *has WFPP.*

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