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OPTIMALITY CONDITIONS FOR VECTOR OPTIMISATION WITH SET-VALUED MAPS

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In this paper, we establish a Farkas-Minkowski type alternative theorem under the assumption of nearly semiconvexlike set-valued maps. Based on the alternative theorem and some other lemmas, we establish necessary optimality conditions and sufficient optimality conditions for set-valued vector optimisation problems with extended inequality constraints in a sense of weak E-minimisers.

1. INTRODUCTION

In recent years, vector optimisation with set-valued maps in infinite dimensional spaces has been received an increasing amount of attention. See [**6, 2, 5, 8, 4, 9**] and references therein, for its extensive applications in many fields such as mathematical programming, optimal control, management science. Vector optimisation with set-valued maps, sometimes called set-valued vector optimisation for short, essentially can be considered as an improvement on single-valued vector optimisation. Amongst research topics in optimisation problems, optimality conditions are especially important. For vector optimisation with set-valued maps, many authors have published interesting results on optimality conditions, and most of those results are obtained under different extended cone-convexity assumptions via alternative theorems. For instance, under the supposition of convexlikeness, Li and Chen [**6**] gave multiplier type and saddle point type optimality conditions for the existence of weak minimisers of set-valued vector optimisation with both inequality and equality constraints. Li [**5**], under the assumption of cone-subconvexlikeness of set-valued maps, established optimality conditions for set-valued vector optimisation by using the alternative theorem in ordered linear topological spaces.

In this paper, based on near cone-convexity, we introduce the notions of nearly cone-convexlike set-valued maps and nearly cone-semiconvexlike set-valued maps in infinite dimensional spaces, investigate the relationships between them, and give some

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characterisations of them. Then we establish a Farkas-Minkowski type alternative theorem for set-valued maps under the assumption of near cone-semiconvexlikeness. Finally, we obtain some necessary and sufficient optimality conditions for the existence of weak E-minimisers of set-valued vector optimisation with generalised inequality constraints.

The outline of this paper is as follows. In Section 2, some notation and preliminaries are given. In Section 3, the concepts of nearly cone-convexlike set-valued maps and nearly cone-semiconvexlike set-valued maps are defined, and a Farkas-Minkowski type alternative theorem is established under the supposition of nearly cone-semiconvexlike set-valued maps. In Section 4, weak minimisers for vector optimisation are extended to weak E-minimisers, and two main results of optimality conditions for vector optimisation with set-valued maps are obtained in the sense of weak E-minimisers.

2. Notations and Preliminaries

Throughout this paper, the scalars of topological vector spaces are always real. Denote by O the null element of every space. Let Z and W be two topological vector spaces with pointed convex cones Z_+ and W_+ respectively. Suppose that $\operatorname{int} Z_+$, the interior of Z_+, is nonempty, and let $\operatorname{int} Z_+ \neq Z_+$. However, the interior of W_+ is not required to be nonempty.

Denote by Z^* and W^* the dual spaces of Z and W, respectively. The dual cone Z_+^* of Z_+ is defined by $Z_+^* = \{z^* \in Z^* \mid \langle z, z^* \rangle \geqslant 0, \forall z \in Z_+\}$, where $\langle z, z^* \rangle$ denotes the value of the linear continuous functional z^* at the point z. We define W_+^* analoguously. Clearly, if $W_+ = \{O\}$, then we have $W_+^* = W^*$.

Let $B \subset Z$ be a nonempty subset. The closure of B is denoted by $\operatorname{cl} B$. The cone hull of B is defined by $\operatorname{cone}(B) = \{\alpha b \mid \alpha > 0, b \in B\}$. The relative interior of B is defined by $\operatorname{ri} B = \{y \in \operatorname{aff} B \mid \exists \text{ a neighbourhood of } N \text{ of } y \text{ such that } N \cap \operatorname{aff} B \subset B\}$, where aff denotes the affine hull operator. We recall the fact that if B is convex, then $\operatorname{ri} B$ is nonempty and $\operatorname{int} M$, the topological interior of B (interior for short), is not necessarily nonempty.

Denote by R the set of all real numbers. For $A \subset R, b \in R$, write $A \geqslant b$, if and only if $a \geqslant b, \forall a \in A$. Use $\leqslant, <$, and $>$ similarly.

Let D be a given nonempty abstract set, and $G : D \to 2^Z$, $H : D \to 2^W$ be set-valued maps such that $G(x) \neq \emptyset$, $H(x) \neq \emptyset, \forall x \in D$. Let

$$G(D) = \bigcup_{x \in D} G(x),$$
$$\langle G(x),\ z^* \rangle = \{\langle z, z^* \rangle \mid z \in G(x)\},$$
$$\langle G(D), z^* \rangle = \bigcup_{x \in D} \langle G(x), z^* \rangle.$$

Definition 1: A subset B in Z is called nearly convex, if there is $\alpha \in (0, 1)$ such that for each $z_1, z_2 \in B$, we have $\alpha z_1 + (1 - \alpha) z_2 \in B$.

Lemma 1. (See [**7**, Proposition 2.1]) *If $B \subset V$ is a nearly convex set, then the set $\Omega = \{\beta \in [0,1] \mid \forall y_1, y_2 \in B,\ \beta y_1 + (1-\beta)y_2 \in B\}$ is dense in $[0,1]$.*

Proposition 1. *If $B \subset Z$ is nearly convex and $\operatorname{ri} B \neq \emptyset$, then for every $t \in (0,1)$, we have*

$$t(\operatorname{ri} B) + (1-t)B \subset \operatorname{ri} B.$$

Proof: Let $t \in (0,1)$, $u_1 \in \operatorname{ri} B$, $u_2 \in B$. Then by definition there is an open neighbourhood N of u_1 such that $N \cap \operatorname{aff} B \subset B$. Set $u_0 = tu_1 + (1-t)u_2$. Since the map $\varphi : \lambda \to u_0/\lambda + u_2(1 - 1/\lambda)$ is continuous at t, hence noting $\varphi(t) = u_1$, we conclude from Lemma 1 that there is $\beta \in \Omega \setminus \{0\}$ such that $u' := u_0/\beta + u_2(1 - 1/\beta) \in N$. We notice that $u' \in \operatorname{aff} B$. Thus $u' \in B$, and hence $u_0 = \beta u' + (1-\beta)u_2 \in B$. Now we show $u_0 \in \operatorname{ri} B$. Define the map $r : Z \to Z$ by

$$r(x) = x/\beta + u_2(1 - 1/\beta).$$

Since the map r is continuous on Z, then $U := r^{-1}(N)$ is an open neighbourhood of u_0. Let $y \in U \cap \operatorname{aff} B$. Then we have $r(y) \in N$, and $r(y) \in \operatorname{aff} B$. Hence $y = \beta r(y) + (1-\beta)u_2 \in B$. Thus $U \cap \operatorname{aff} B \subset B$. Therefore, $u_0 \in \operatorname{ri} B$. ∎

Clearly, Proposition 2 and 3 below can be deduced directly by Proposition 1.

Proposition 2. *If $B \subset Z$ be a nearly convex set, then the set $\operatorname{ri} B$ is convex.*

Proposition 3. *If a nearly convex set $B \subset Z$ is relatively open, that is $\operatorname{ri} B = B$, then B is convex.*

Proposition 3 given here can be thought of as an extension of [**7**, Theorem 2.1].

Proposition 4. *Let $B \subset Z$ be a nearly convex set, and $\operatorname{ri} B \neq \emptyset$. Let $y^* \in Z^* \setminus \{O\}$. If $\langle u, y^* \rangle > 0, \forall u \in \operatorname{ri} B$, then $\langle u, y^* \rangle \geqslant 0, \forall u \in B$.*

Proof: Suppose the contrary. Then there is $u_0 \in B$ such that $\langle u_0, y^* \rangle < 0$. Fix $u_1 \in \operatorname{ri} B$. Since the function $s(t) = \langle tu_1 + (1-t)u_0, y^* \rangle$ is continuous on R, there is $\alpha \in (0,1)$ such that $s(\alpha) = \langle \alpha u_1 + (1-\alpha)u_0, y^* \rangle = 0$. On the other hand, from Proposition 1, we have $\alpha u_1 + (1-\alpha)u_0 \in \operatorname{ri} B$. This gives $\langle \alpha u_1 + (1-\alpha)u_0, y^* \rangle > 0$, a contradiction. ∎

We recall that $\operatorname{ri} B = \operatorname{int} B$ if and only if $\operatorname{int} B$ is nonempty(for example, see [**3**, Theorem 1.2.4]).

Lemma 2. *If $B \subset Z$ is a nearly convex set with nonempty interior, then for every $t \in (0,1)$ we have*

$$t(\operatorname{int} B) + (1-t)\operatorname{cl} B \subset \operatorname{int} B.$$

Proof: According to assumptions and Proposition 1, we obtain that for all $t \in (0,1)$, $t(\operatorname{int} B) + (1-t)B \subset \operatorname{int} B$. Since $\operatorname{int} B$ is nonempty, we suppose $b \in \operatorname{int} B$. Then

$$O \in (b - \operatorname{int} B), \quad \text{or} \quad \forall t \in (0,1),\ O \in t(b - \operatorname{int} B).$$

Hence for every $t \in (0,1)$ we get

$$\operatorname{cl}\big((1-t)B\big) \subset (1-t)B - t(b - \operatorname{int} B) \subset t(\operatorname{int} B) + (1-t)B - tb \subset (\operatorname{int} B) - tb.$$

It follows that

$$\forall t \in (0,1),\ tb + \operatorname{cl}\big((1-t)B\big) \subset \operatorname{int} B.$$

Since $tb + \operatorname{cl}\big((1-t)B\big) = tb + (1-t)\operatorname{cl} B$, and $b \in \operatorname{int} B$ can be arbitrarily chosen, hence we have $\forall t \in (0,1)$, $t(\operatorname{int} B) + (1-t)\operatorname{cl} B \subset \operatorname{int} B$. ∎

Lemma 3. *If $B \subset Z$ is a nearly convex set, then the set $\operatorname{int} B$ is convex.*

The following lemma is the same as Proposition 4 whenever the assumption of $\operatorname{int} B \neq \emptyset$ is imposed.

Lemma 4. *Let $B \subset Z$ be a nearly convex set, and $\operatorname{int} B \neq \emptyset$. Let $y^* \in Z^* \setminus \{O\}$. If $\langle u, y^* \rangle > 0, \forall u \in \operatorname{int} B$, then $\langle u, y^* \rangle \geqslant 0, \forall u \in B$.*

3. Nearly Cone-semiconvexlike Set-Valued Maps and Farkas-Minkowski Alternative Theorems

For simplicity, we put $U = Z \times W$, $U_+ = Z_+ \times W_+$, and $J = (G,H) : D \to 2^U$. The notation $J(x) = (G,H)(x)$ is used for $G(x) \times H(x)$ here. One can easily check that $U^* = Z^* \times W^*$, and $U_+^* = Z_+^* \times W_+^*$.

Definition 2: A set-valued map $J : D \to 2^U$ is called nearly U_+–convexlike, if there is an $\alpha \in (0,1)$ such that for any $x_1, x_2 \in D$, we have

$$\alpha J(x_1) + (1-\alpha) J(x_2) \subset J(D) + U_+.$$

Definition 3: A set-valued map $J : D \to 2^U$ is called nearly U_+–semiconvexlike, if there exists $u \in \operatorname{int} Z_+$, and $\alpha \in (0,1)$ such that for any $x_1, x_2 \in D$, and $\varepsilon > 0$, we have

$$\varepsilon(u, O) + \alpha J(x_1) + (1-\alpha) J(x_2) \subset J(D) + U_+.$$

Next, we give some important characterisations of nearly cone-semiconvexlike set-valued maps and nearly cone-convexlike set-valued maps, and state the relationships between them.

Proposition 5. *The set-valued map $J : D \to 2^U$ is nearly U_+–semiconvexlike, if and only if $M := J(D) + (\operatorname{int} Z_+) \times W_+$ is a nearly convex set.*

Proof: Sufficiency: Since $\operatorname{int} Z_+$ is nonempty, and M is nearly convex, hence, $\exists u \in \operatorname{int} Z_+$, $\exists \alpha \in (0,1)$, $\forall x_1, x_2 \in D$, $\forall \varepsilon > 0$, such that

$$\alpha\big(J(x_1) + \varepsilon(u, O)\big) + (1-\alpha)\big(J(x_2) + \varepsilon(u, O)\big) \subset M \subset J(D) + U_+.$$

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Therefore, $\varepsilon(u,O) + \alpha J(x_1) + (1-\alpha)J(x_2) \subset J(D) + U_+$, that is, J is nearly U_+–semiconvexlike.

Necessity: Let $m_1, m_2 \in M$; then $\exists x_i \in D$, $y_i \in (\operatorname{int} Z_+) \times W_+$, $i = 1, 2$, such that $m_i \in J(x_i) + y_i$. Since J is nearly U_+–semiconvexlike, there exist $u \in \operatorname{int} Z_+$, $\alpha \in (0,1)$, for the previous $x_1, x_2 \in D$, $\forall \varepsilon > 0$, we have

$$\varepsilon(u,O) + \alpha J(x_1) + (1-\alpha)J(x_2) \subset J(D) + U_+.$$

Thus $\varepsilon(u,O) + \alpha(m_1 - y_1) + (1-\alpha)(m_2 - y_2) \in J(D) + U_+$. Because the set $(\operatorname{int} Z_+) \times W_+$ is convex, we have $y_0 := \alpha y_1 + (1-\alpha)y_2 \in (\operatorname{int} Z_+) \times W_+$. Thereby,

$$m = \alpha m_1 + (1-\alpha)m_2 \in \alpha J(x_1) + (1-\alpha)J(x_2) + y_0. \tag{1}$$

Let $y_0 = (y_{01}, y_{02}) \in (\operatorname{int} Z_+) \times W_+$. Since $y_{01} \in \operatorname{int} Z_+$, there is $\varepsilon > 0$ such that $y_{01} - \varepsilon u \in \operatorname{int} Z_+$, Then, $y_0 - \varepsilon(u,O) = (y_{01} - \varepsilon u,\ \ y_{02}) \in (\operatorname{int} Z_+) \times W_+$. It follows by (1) that

$$\begin{aligned} m &\in \alpha J(x_1) + (1-\alpha)J(x_2) + \varepsilon(u,O) + y_0 - \varepsilon(u,O) \\ &\subset J(D) + U_+ + (\operatorname{int} Z_+) \times W_+ \subset J(D) + (\operatorname{int} Z_+) \times W_+ = M. \end{aligned}$$

Therefore M is nearly convex. $\square$

The following corollaries can be shown similarly.

Corollary 1. *The set-valued map $J : D \to 2^U$ is nearly U_+–convexlike, if and only if $M' = J(D) + Z_+ \times W_+$ is a nearly convex set.*

Corollary 2. *If $M' = J(D) + Z_+ \times W_+$ is nearly convex, then the set $M = J(D) + (\operatorname{int} Z_+) \times W_+$ is also nearly convex.*

It follows by Corollary 2 that nearly cone-convexlike set-valued maps imply nearly cone-semiconvexlike set-valued maps. However the example below shows that the converse implication is not always true.

Example 1. Let $D = \{0,1\}$, $Z = R^2$, $W = R$. Then $U = Z \times W = R^3$. Let

$$Z_+ = \{(y_1, y_2) \in R^2 \mid y_1 \geqslant 0, y_2 > 0\} \cup \{(0,0)\},\ W_+ = \{0\}.$$

Let

$$G(x) = (G_1(x),\ G_2(x)) : D \to 2^{R\times R},\ H(x) : D \to 2^R.$$

Define $J(x) = (G,H)(x) : D \to 2^U$ by

$$J(x) = \Big\{(G_1(x), G_2(x), H(x)) \in R \times R \times R \mid G_1(x) = x, G_2(x) \geqslant 0, H(x) = 0\Big\}, \forall x \in D.$$

It is easy to check that $M = J(D) + (\operatorname{int} Z_+) \times W_+$ is a convex set, so that it is nearly convex. But the set $M' = J(D) + Z_+ \times W_+$ is not nearly convex.

The following corollaries can be deduced directly by definition.

COROLLARY 3. *Let $J : D \to 2^U$ be a set-valued map. If we can find $\alpha \in (0,1)$ such that for any $x_i \in D$, $y_i \in J(x_i)$, $i = 1,2$, there is $x_3 \in D$ satisfying*

$$\alpha y_1 + (1-\alpha) y_2 \in J(x_3) + U_+.$$

Then J is nearly U_+–convexlike.

COROLLARY 4. *Let $J : D \to 2^U$ be a set-valued map. If we can find $u \in \operatorname{int} Z_+$, $\alpha \in (0,1)$ such that for any $x_i \in D$, $y_i \in J(x_i)$, $i = 1,2$, any $\varepsilon > 0$, there is $x_3 \in D$ satisfying*

$$\varepsilon(u, O) + \alpha y_1 + (1-\alpha) y_2 \in J(x_3) + U_+.$$

Then J is nearly U_+–semiconvexlike.

Next, we give some technical lemmas which will be used in the proof of the alternative theorem.

LEMMA 5. *The set $\operatorname{int}\big(\operatorname{cone}\big(J(D)\big) + Z_+ \times W_+\big) \neq \emptyset$, if and only if the set $\operatorname{int}\big(\operatorname{cone}\big(J(D)\big) + (\operatorname{int} Z_+) \times W_+\big) \neq \emptyset$.*

PROOF: Sufficiency is trivial. Suppose that $\operatorname{int}\big(\operatorname{cone}\big(J(D)\big) + Z_+ \times W_+\big) \neq \emptyset$. Then there are $\alpha \geqslant 0$, $x_1 \in D$, $z \in Z_+$, $w \in W_+$, $p \in G(x_1)$, $q \in H(x_1)$, such that $(\alpha p + z, \alpha q + w) \in \operatorname{int}\big(\operatorname{cone}\big(J(D)\big) + Z_+ \times W_+\big)$. Hence, there are S and T, neighbourhoods of the origins in Z and W respectively such that

$$\big(\alpha p + z + (\operatorname{int} Z_+) \cap S\big) \times (\alpha q + w + T) \subset (\alpha p + z + S) \times (\alpha q + w + T) \subset \operatorname{cone}\big(J(D)\big) + Z_+ \times W_+.$$

Thus for each $s \in (\operatorname{int} Z_+) \cap S$, each $t \in T$, there exist $\beta \geqslant 0, x' \in D, z' \in Z_+, w' \in W_+$, such that $\alpha p + z + s \in \beta G(x') + z'$, and $\alpha q + w + t \in \beta H(x') + w'$. So, $\alpha p + z + 2s \in \beta G(x') + z' + s \subset \beta G(x') + \operatorname{int} Z_+$, and $\alpha q + w + t \in \beta H(x') + W_+$. Therefore,

$$\big(\alpha p + z + 2\big((\operatorname{int} Z_+) \cap S\big)\big) \times (\alpha q + w + T) \subset \operatorname{cone}\big(J(D)\big) + (\operatorname{int} Z_+) \times W_+.$$

Observing the set in the left-hand side of the inclusion is open, we know that

$$\operatorname{int}\big(\operatorname{cone}\big(J(D)\big) + (\operatorname{int} Z_+) \times W_+\big)$$

is nonempty. ∎

In a similar way, we can also show the following lemma.

LEMMA 6. *The set $\operatorname{int}\big(J(D) + Z_+ \times W_+\big) \neq \emptyset$, if and only if the set $\operatorname{int}\big(J(D) + (\operatorname{int} Z_+) \times W_+\big) \neq \emptyset$.*

LEMMA 7. *If $u^* = (z^*, w^*) \in U_+^* = Z_+^* \times W_+^*$, with $z^* \neq O$, $u = (z, w) \in (\operatorname{int} Z_+) \times W_+$, then $\langle u, u^* \rangle > 0$.*

PROOF: According to definition of U_+^*, we have $\langle u, u^* \rangle \geqslant 0$. Assume that there exists $u_0 = (z_0, w_0) \in (\operatorname{int} Z_+) \times W_+$ such that $\langle u_0, u^* \rangle = 0$, that is, $\langle z_0, z^* \rangle + \langle w_0, w^* \rangle = 0$. Since $z_0 \in \operatorname{int} Z_+$, then there is a neighbourhood S of the origin in Z, such that $z_0 + S \subset \operatorname{int} Z_+$. Noting that S is absorbing, we see that for every $v \in Z$, there is $\varepsilon > 0$ such that $z_0 \pm \varepsilon v \in \operatorname{int} Z_+$. Hence, $\langle z_0 \pm \varepsilon v, z^* \rangle + \langle w_0, w^* \rangle \geqslant 0$, or in other words,

$$\langle z_0, z^* \rangle + \langle w_0, w^* \rangle \geqslant \pm \varepsilon \langle v, z^* \rangle.$$

Thus $\langle v, z^* \rangle$=0. Therefore $z^* = O$. However, this contradicts the assumption. Thus the proof is complete. ∎

In the remainder of this section, we consider the following two systems,

SYSTEM 1. $\exists x_0 \in D$, such that $-G(x_0) \cap \operatorname{int} Z_+ \neq \emptyset, -H(x_0) \cap W_+ \neq \emptyset$.

SYSTEM 2. $\exists u^* = (z^*, w^*) \in Z_+^* \times W_+^* \setminus \{(O, O)\}$, such that

$$\big\langle G(x), z^* \big\rangle + \big\langle H(x), w^* \big\rangle \geqslant 0, \forall x \in D. \tag{2}$$

In what follows, we use the above two systems to describe the Farkas-Minkowski type alternative theorem under the assumption of nearly cone-semiconvexlike set-valued maps. The proof of this theorem is based on the separation theorems of convex sets in topological vector spaces (for instance, see [**10**, Theorem 3.8]).

THEOREM 1. *Suppose that the set-valued map $J = (G, H) : D \to 2^U$ is nearly U_+-semiconvexlike on D. Suppose that the interior of the set $J(D) + U_+$ is nonempty, Then,*

(i) *If* System 2 *has a solution $(z^*, w^*) \in Z_+^* \times W_+^*$, with $z^* \neq O$, then* System 1 *has no solution.*

(ii) *If* System 1 *has no solution, then* System 2 *has a solution (z^*, w^*).*

PROOF: (i) Assume that System 2 admits a solution $(z^*, w^*) \in Z_+^* \times W_+^*$, with $z^* \neq O$. If System 1 admits a solution $x_0 \in D$, then there are $p \in G(x_0)$, $q \in H(x_0)$ such that $-p \in \operatorname{int} Z_+$, $-q \in W_+$. It follows by Lemma 7 that $\langle p, z^* \rangle + \langle q, w^* \rangle < 0$. This contradicts (2).

(ii) Set $M = J(D) + (\operatorname{int} Z_+) \times W_+$. According to Lemma 6 and the assumption of $\operatorname{int}(J(D) + Z_+ \times W_+) \neq \emptyset$, we have $\operatorname{int} M \neq \emptyset$. Since J is nearly U_+-semiconvexlike on D, hence M is nearly convex. It follows by Lemma 3 that $\operatorname{int} M$ is convex.

Since System 1 has no solution, then $O \notin M$ so that $O \notin \operatorname{int} M$. As a matter of fact, assume that $O \in M$; there are $\alpha \geqslant 0$ and $x' \in D$ such that $O \in \alpha G(x') + \operatorname{int} Z_+$, and $O \in \alpha H(x') + W_+$. Since $O \notin \operatorname{int} Z_+$, hence $\alpha > 0$. therefore $-G(x') \cap \operatorname{int} Z_+ \neq \emptyset$, $-H(x') \cap W_+ \neq \emptyset$. This is impossible since System 1 admits no solution.

Now using the separation theorem for convex sets in topological vector spaces, we know that there is a hyperplane H *properly* separating $\{O\}$ and $\operatorname{int} M$, that is,

$$\exists u^* = (z^*, w^*) \in Z^* \times W^* \setminus \{(O, O)\},$$

$a \in R$, such that

$$\langle u, u^* \rangle \geqslant a \geqslant 0, \forall u \in \operatorname{int} M, \tag{3}$$

where the hyperplane function can be written as $H = \{y \in U \mid \langle y, u^* \rangle = a\}$.

In the following, we shall prove that

$$\langle u, u^* \rangle > 0, \forall u \in \operatorname{int} M. \tag{4}$$

There are two cases to be considered. The first case is $a > 0$. But this is simple because it follows by (3) that the inequality (4) holds.

The second case is $a = 0$. Here it follows again by (3) that

$$\langle u, u^* \rangle \geqslant 0, \forall u \in \operatorname{int} M. \tag{5}$$

Comparing (4) with (5), we can see that it is sufficient to show $\langle u, u^* \rangle \neq 0$, $\forall u \in \operatorname{int} M$. Suppose the contrary; there is $u_0 \in \operatorname{int} M$ such that $\langle u_0, u^* \rangle = 0$. Let $v \in \operatorname{int} M$ be given arbitrarily. Thus there is $\varepsilon > 0$ such that $u_0 - \varepsilon v \in \operatorname{int} M$. Hence it follows by (5) that $\langle u_0 - \varepsilon v, u^* \rangle \geqslant 0$, that is, $\langle u_0, u^* \rangle \geqslant \varepsilon \langle v, u^* \rangle$. So, $\langle v, u^* \rangle \leqslant 0$. On the other hand, also by (5), we get $\langle v, u^* \rangle \geqslant 0$. Therefore,

$$\langle v, u^* \rangle = 0, \forall v \in \operatorname{int} M.$$

This illustrates that the hyperplane H does not separate $\{O\}$ and $\operatorname{int} M$ *properly*. Then a contradiction is introduced.

Thus the proof that the inequality (4) holds is complete.

It follows by Lemma 4 that

$$\langle u, u^* \rangle \geqslant 0, \forall u \in M. \tag{6}$$

Next, we check $u^* = (z^*, u^*) \in Z_+^* \times W_+^*$; indeed, assume $z^* \notin Z_+^*$. Then there exists $z_1 \in Z_+$ such that $\langle z_1, z^* \rangle < 0$. Thus, $\lambda \langle z_1, z^* \rangle = \langle \lambda z_1, z^* \rangle < 0$, $\forall \lambda > 0$. According to (6), for each $x \in D$, each $z' \in \operatorname{int} Z_+$, and each $w' \in W_+$, we have $\langle p + z', z^* \rangle + \langle q + w', w^* \rangle \geqslant 0$, $\forall p \in G(x)$, $\forall q \in H(x)$. Since $\lambda z_1 \in Z_+$, $\lambda z_1 + z' \in \operatorname{int} Z_+$. Again by (6), we have $\langle p + \lambda z_1 + z', z^* \rangle + \langle q + w', w^* \rangle \geqslant 0$, that is,

$$\lambda \langle z_1, z^* \rangle + \langle p + z', z^* \rangle + \langle q + w', w^* \rangle \geqslant 0, \forall \lambda > 0. \tag{7}$$

However, (7) does not hold when λ is too large. Hence we have $z^* \in Z_+^*$. We can analogously show $w^* \in W_+^*$. Thus, $\exists u^* = (z^*, w^*) \in Z_+^* \times W_+^* \setminus \{(O, O)\}$, such that $\langle u, u^* \rangle \geqslant 0$, $\forall u \in M$, that is,

$$\langle J(x) + t, u^* \rangle \geqslant 0, \forall x \in D, \forall t \in (\operatorname{int} Z_+) \times W_+.$$

Take $t_0 \in (\operatorname{int} Z_+) \times W_+$, and $\lambda_n > 0$ such that $\lambda_n \to 0 (n \to \infty)$; then we have $\langle J(x) + \lambda_n t_0, u^* \rangle \geqslant 0$, $\forall x \in D$, $n = 1, 2, \ldots$. Letting $n \to \infty$, we obtain

$$\langle G(x), z^* \rangle + \langle H(x), w^* \rangle \geqslant 0, \forall x \in D.$$

The proof is thus complete. ∎

In particular, if we set $W_+ = \{O\}$, the following result is derived directly by Theorem 1.

Corollary 5. *Suppose that the set-valued map $J : D \to 2^U$ is nearly U_+-semiconvexlike on D. Suppose that the interior of the set $J(D) + U_+$ is nonempty. If there is no $x \in D$ such that $-G(x) \cap \operatorname{int} Z_+ \neq \emptyset$, $O \in H(x)$. Then $\exists u^* = (z^*, w^*) \in Z_+^* \times W^* \setminus \{(O, O)\}$, such that*

$$\langle G(x), z^* \rangle + \langle H(x), w^* \rangle \geqslant 0, \forall x \in D.$$

4. Weak E-minimisers and Optimality Conditions

Let Y be a topological vector space with pointed convex cone Y_+ with a nonempty interior. Let $F : D \to 2^Y$ be a set-valued map such that $F(x) \neq \emptyset, \forall x \in D$. Let $E \subset Y$ be a nonempty subset, and let $\varepsilon \in Y_+$, $O \in E$.

We consider the following set-valued vector optimisation (P),

$$\begin{aligned} &\min F(x), \\ &\text{such that } -G(x) \cap Z_+ \neq \emptyset, \\ &\qquad\qquad\quad -H(x) \cap W_+ \neq \emptyset. \end{aligned}$$

Whenever we set $W_+ = \{O\}$, (P) reduces to (P′),

$$\begin{aligned} &\min F(x), \\ &\text{such that } -G(x) \cap Z_+ \neq \emptyset, \\ &\qquad\qquad\quad O \in H(x). \end{aligned}$$

In this section, we work at the optimality conditions for (P). The feasible set of (P) is defined by $K = \{x \in D \mid -G(x) \cap Z_+ \neq \emptyset, -H(x) \cap W_+ \neq \emptyset\}$.

Definition 4:

(i) $x_0 \in K$ is called a weakly efficient solution of (P), if there is $y_0 \in F(x_0)$ such that $(y_0 - F(K)) \cap \operatorname{int} Y_+ = \emptyset$. The pair (x_0, y_0) is called a weak minimiser of (P).

(ii) $x_0 \in K$ is called a weakly ε-efficient solution of (P), if there is $y_0 \in F(x_0)$ such that $(y_0 - F(K) - \varepsilon) \cap \operatorname{int} Y_+ = \emptyset$. The pair (x_0, y_0) is called a weak ε-minimiser of (P).

In [1], the authors defined an H near the minimum solution of vector optimisation. In this section, we use their idea to define weakly E-efficient solutions of set-valued vector optimisation, and then discuss the existence of weakly E-efficient solutions and weak E-minimisers of set-valued vector optimisation.

DEFINITION 5: A point $x_0 \in K$ is called a weakly E-efficient solution of (P), if and only if $\exists y_0 \in F(x_0)$ such that $(y_0 - F(K) - E) \cap \operatorname{int} Y_+ = \emptyset$. The pair (x_0, y_0) is called a weak E-minimiser of (P).

It is clear that the set of weakly efficient solutions contains the set of weakly ε-efficient solutions, or the set of E-efficient solutions. Now we investigate the relationships between weakly ε-efficient solutions and weakly E-efficient solutions.

THEOREM 2.

(i) *If $E = \{\varepsilon\}$, then weakly E-efficient solutions are equivalent to weakly ε-efficient solutions.*

(ii) *If there is $\varepsilon' \in E$ such that $\varepsilon - \varepsilon' \in Y_+$, then weakly E-efficient solutions imply ε-efficient solutions.*

(iii) *If $E - \varepsilon \subset Y_+$, then weakly ε-efficient solutions imply weakly E-efficient solutions.*

PROOF: We only show (ii) as (iii) can be proved similarly. Assume there is $\varepsilon' \in E$ such that $\varepsilon - \varepsilon' \in Y_+$. Thus, we have $\varepsilon + \operatorname{int} Y_+ \subset \varepsilon' + Y_+ + \operatorname{int} Y_+ \subset \varepsilon' + \operatorname{int} Y_+ \subset E + \operatorname{int} Y_+$. Suppose that $x_0 \in K$ is a weakly E-efficient solution. Then $\big(y_0 - F(K)\big) \cap (E + \operatorname{int} Y_+) = \emptyset$. Hence, $\big(y_0 - F(K)\big) \cap (\varepsilon + \operatorname{int} Y_+) = \emptyset$. Therefore x_0 is also a weakly ε-efficient solution. $\square$

Set $I(x) = F(x) \times G(x) \times H(x) = (F, G, H)(x)$, $\forall x \in D$, $V = Y \times Z \times W$. Hence we have $V_+ = Y_+ \times Z_+ \times W_+$, $V^* = Y^* \times Z^* \times W^*$, and $V_+^* = Y_+^* \times Z_+^* \times W_+^*$. The definition below coincides with Definition 3 when we consider V as the product of $(Y \times Z)$ and W.

DEFINITION 6: The set-valued map $I = (F, G, H) : D \to 2^V$ is called nearly V_+-semiconvexlike on D, if and only if $\exists t \in \operatorname{int} Y_+$, $\exists u \in \operatorname{int} Z_+$, $\exists \alpha \in (0,1)$ such that $\forall x_1, x_2 \in D$, $\forall \varepsilon > 0$, we have $\varepsilon(t, u, O) + \alpha I(x_1) + (1 - \alpha) I(x_2) \subset I(D) + V_+$.

In view of Proposition 5, we can find that the set-valued map $I : D \to 2^V$ is nearly V_+-semiconvexlike on D if and only if the set $I(D) + (\operatorname{int} Y_+ \times \operatorname{int} Z_+) \times W_+$ is nearly convex.

A set-valued Lagrangian function $L : D \times Y_+^* \times Z_+^* \times W_+^* \to 2^R$ for (P) is defined as,

$$L(x, y^*, z^*, w^*) = \langle F(x), y^* \rangle + \langle G(x), z^* \rangle + \langle H(x), w^* \rangle, \quad (x, y^*, x^*, w^*) \in D \times Y_+^* \times Z_+^* \times W_+^*.$$

We consider the following unconstrained scalar optimisation problem (UP) with set-valued functions induced by (P),

$$\min_{x \in D} L(x, y^*, z^*, w^*), \qquad (y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^*.$$

DEFINITION 7: A point $x_0 \in D$ is called $\langle E, y^* \rangle$-optimal solution of (UP), if and only if $\exists r_0 \in L(x_0, y^*, z^*, w^*)$ such that $r_0 \leqslant L(x, y^*, z^*, w^*) + \langle E, y^* \rangle$, $\forall x \in D$. The pair (x_0, r_0) is called an $\langle E, y^* \rangle$-optimiser of (UP).

Now, we establish the optimality conditions in terms of (P) and (UP). For the simplicity, we suppose that the set E, satisfying $O \in E \subset Y$, is convex. It is easy to verify that if the set-valued map H is nearly V_+-semiconvexlike on D, $y_0 \in Y$, then $\bigl(F(x) + E - y_0\bigr) \times G(x) \times H(x)$ is also nearly V_+-semiconvexlike on D.

THEOREM 3. *Let (x_0, y_0) be a weak E-minimiser of (P); assume that*

(i) *$I(x) = F(x) \times G(x) \times H(x)$ is nearly V_+-semiconvexlike on D;*

(ii) *$\exists z_0 \in Y$, such that $(z_0, O, O) \in \operatorname{int}\bigl(I(D) + V_+\bigr)$.*

Then $\exists (y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^*$, with $y^* \neq O$ such that $\bigl(x_0, \langle y_0, y^* \rangle\bigr)$ is an $\langle E, y^* \rangle$-optimiser of (UP), and $\inf\bigl\langle G(x_0), z^* \bigr\rangle = 0$.

PROOF: Let $P(x) = \bigl(F(x) + E - y_0\bigr) \times G(x) \times H(x)$. It follows by assumption (i) that $P(x)$ is also nearly V_+-semiconvexlike on D. Since (x_0, y_0) is a weak E-minimiser of (P), we have $-\bigl(F(K) - y_0 + E\bigr) \cap \operatorname{int} Y_+ = \emptyset$. It is obvious that $-\bigl(G(x) \times H(x)\bigr) \cap (\operatorname{int} Z_+) \times W_+ = \emptyset$, $\forall x \in D \setminus K$. Thus

$$-P(x) \cap \bigl((\operatorname{int} Y_+) \times (\operatorname{int} Z_+) \times W_+\bigr) = \emptyset, \quad \forall x \in D.$$

Since $(z_0, O, O) \in \operatorname{int}\bigl(I(D) + V_+\bigr)$, hence $\exists x' \in D$ such that $z_0 \in \operatorname{int}\bigl(F(x') + Y_+\bigr)$, $(O, O) \in \operatorname{int}\bigl(G(x') \times H(x') + Z_+ \times W_+\bigr)$. Thus $z_0 - y + E \subset -y + E + \operatorname{int}\bigl(F(x') + Y_+\bigr) \subset \operatorname{int}\bigl(F(x') - y + E + Y_+\bigr)$. So, $\operatorname{int}\bigl(P(D) + V_+\bigr) \neq \emptyset$.

By applying (ii) in Theorem 1, we have that $\exists (y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^* \setminus \bigl\{(O, O, O)\bigr\}$, such that $\bigl\langle P(x), (y^*, z^*, w^*) \bigr\rangle \geqslant 0$, $x \in D$. That is

$$\langle E, y^* \rangle + \bigl\langle F(x), y^* \bigr\rangle + \bigl\langle G(x), z^* \bigr\rangle + \bigl\langle H(x), w^* \bigr\rangle \geqslant \langle y_0, y^* \rangle, \forall x \in D. \tag{8}$$

Next, we show $y^* \neq O$. Assume the contrary. Then $(z^*, w^*) \neq (O, O)$, and (8) can be rewritten as

$$\bigl\langle G(x), z^* \bigr\rangle + \bigl\langle H(x), w^* \bigr\rangle \geqslant 0, \forall x \in D. \tag{9}$$

Hence

$$\bigl\langle G(x) + Z_+, z^* \bigr\rangle + \bigl\langle H(x) + W_+, w^* \bigr\rangle \geqslant 0, \forall x \in D. \tag{10}$$

We have two cases to be discussed. One case is $z^* \neq O$. Since $(O, O) \in \operatorname{int}\bigl(G(x') \times H(x') + Z_+ \times W_+\bigr)$, then we can take $x_1 \in D$ arbitrarily, and for any $v_1 \in G(x_1)$, $v_2 \in H(x_1)$, $k_1 \in \operatorname{int} Z_+$, $k_2 \in W_+$, satisfying $(v_1 + k_1, v_2 + k_2) \in Z \times W$, there is $\varepsilon > 0$ such that $\pm\varepsilon(v_1 + k_1, v_2 + k_2) \in \operatorname{int}\bigl(G(x') \times H(x') + Z_+ \times W_+\bigr)$. It follows by (10) that

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$\langle v_1 + k_1, z^* \rangle + \langle v_2 + k_2, w^* \rangle = 0$. Observing (9), we obtain $\langle k_1, z^* \rangle + \langle k_2, w^* \rangle \leqslant 0$. This is in contradiction to Lemma 7.

The other case is $z^* = O$. Then (10) can be rewritten as $\langle H(x) + W_+, w^* \rangle \geqslant 0$, $\forall x \in D$. Because of $O \in \operatorname{int}(H(x') + W_+)$, we have that for each $v \in W$, there is $\varepsilon_0 > 0$ such that $\pm\varepsilon_0 v \in \operatorname{int}(H(x') + W_+)$. Thus, $\varepsilon_0 \langle v, w^* \rangle = 0$, $\forall v \in W$, which implies $w^* = O$. This is also a contradiction.

Thus the proof of $y^* \neq O$ is complete.

Observing $O \in E$, we rewrite (8) as

$$\langle F(x), y^* \rangle + \langle G(x), z^* \rangle + \langle H(x), w^* \rangle \geqslant \langle y_0, y^* \rangle, \forall x \in D. \tag{11}$$

Since $x_0 \in K$, there are $p \in G(x_0)$, $q \in H(x_0)$ such that $p \in -Z_+$, $-q \in W_+$. It follows that $\langle p, z^* \rangle + \langle q, w^* \rangle \leqslant 0$. On the other hand, setting $x = x_0$ in (11), we get

$$\langle y_0, y^* \rangle + \langle p, z^* \rangle + \langle q, w^* \rangle \geqslant \langle y_0, y^* \rangle.$$

That is $\langle p, z^* \rangle + \langle q, w^* \rangle \geqslant 0$. Thus

$$\langle p, z^* \rangle + \langle q, w^* \rangle = 0. \tag{12}$$

Hence $\langle y_0, y^* \rangle \in \langle F(x_0), y^* \rangle + \langle G(x_0), z^* \rangle + \langle H(x_0), w^* \rangle = L(x_0, y^*, z^*, w^*)$. Observing (8), we know that $(x_0, \langle y_0, y^* \rangle)$ is an $\langle E, y^* \rangle$–optimiser of (UP).

Because of $p \in -Z_+$, and $q \in -W_+$, we get $\langle p, z^* \rangle \leqslant 0$, and $\langle q, w^* \rangle \leqslant 0$. Noticing (12), we have $\langle p, z^* \rangle = \langle q, w^* \rangle = 0$.

Take $x = x_0$ in (11) again. We obtain

$$\langle y_0, y^* \rangle + \langle G(x_0), z^* \rangle + \langle q, w^* \rangle \geqslant \langle y_0, y^* \rangle.$$

That is $\langle G(x_0), z^* \rangle \geqslant 0$. Due to $0 = \langle q, z^* \rangle \in \langle G(x_0), z^* \rangle$, consequently, we have $\inf \langle G(x_0), z^* \rangle = 0$. ∎

Corollary 6. *Let (x_0, y_0) be a weak E-minimiser of* (P)*; assume that*

(i) *$I(x) = F(x) \times G(x) \times H(x)$ is nearly V_+–semiconvexlike on D;*

(ii) *$\exists x' \in D$, such that $-G(x') \cap \operatorname{int} Z_+ \neq \emptyset$, $-\operatorname{int} H(x') \cap W_+ \neq \emptyset$.*

Then $\exists (y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^*$, with $y^* \neq O$ such that $(x_0, \langle y_0, y^* \rangle)$ is an $\langle E, y^* \rangle$–optimiser of (UP)*, and $\inf \langle G(x_0), z^* \rangle = 0$.*

In practice, from assumption (ii) in Corollary 6, one can readily deduce condition (ii) in Theorem 3, thus the proof of Corollary 6 is similar to that of Theorem 3. In the rest of this section, we give some sufficient optimality conditions for Problem (P) under the supposition of generalised constraint qualifications, without any convexity assumptions.

Theorem 4. *Let $x_0 \in K$; assume that,*

(i) $\exists y_0 \in F(x_0)$, $\exists (y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^* \setminus \{(O, O, O)\}$ *such that*

$$\min_{x \in D}\Big(\big\langle F(x), y^*\big\rangle + \big\langle G(x), z^*\big\rangle + \big\langle H(x), w^*\big\rangle\Big) \geqslant \langle y_0, y^*\rangle;$$

(ii) $-\operatorname{int}\big(H(D)\big) \cap W_+ \neq \emptyset$; $\exists x' \in D$, *such that* $-G(x') \cap \operatorname{int} Z_+ \neq \emptyset$, $-H(x') \cap W_+ \neq \emptyset$.

Then (x_0, y_0) *is a weak E-minimiser of (P).*

Proof: According to assumption (i), we have

$$\big\langle F(x) - y_0, y^*\big\rangle + \big\langle G(x), z^*\big\rangle + \big\langle H(x), w^*\big\rangle \geqslant 0, \forall x \in D. \tag{13}$$

We show $y^* \neq O$ below. Suppose that $y^* = O$. Then

$$\big\langle G(x), z^*\big\rangle + \big\langle H(x), w^*\big\rangle \geqslant 0, \forall x \in D. \tag{14}$$

In order to derive a contradiction, we consider the following two cases respectively. One case is $z^* \neq O$. By assumption (ii), there are $x' \in D$, $u_1 \in G(x')$, $u_2 \in H(x')$ such that $-u_1 \in \operatorname{int} Z_+$, $-u_2 \in W_+$. Hence $\langle u_1, z^*\rangle + \langle u_2, w^*\rangle < 0$. This contradicts (14).

The other case is $z^* = O$. It follows by assumption (i) that $w^* \neq O$. From assumption (ii), there is $y' \in W_+$ such that $-y' \in \operatorname{int} H(D)$. For each $v \in W$, it is not difficult to check $\langle v, w^*\rangle = 0$. This implies $w^* = O$, which is exactly in contradiction.

Therefore the proof of $y^* \neq O$ is complete.

Next we show (x_0, y_0) is a weak E-minimiser of (P). Otherwise, there are $x_1 \in K$, $t \in F(x_1)$, $e \in E$ such that $y_0 - t - e \in \operatorname{int} Y_+$. By [5, Lemma 1.1], we have

$$\langle t - y_0 + e, y^*\rangle < 0. \tag{15}$$

Since $x_1 \in K$, there are $p \in G(x_1)$, $q \in H(x_1)$ such that $-p \in Z_+$, $-q \in W_+$. Taking (15) into account, we obtain $\langle t - y_0 + e, y^*\rangle + \langle p, z^*\rangle + \langle q, w^*\rangle < 0$. Seeing the fact $\langle e, y^*\rangle \geqslant 0$, we again obtain

$$\langle t - y_0, y^*\rangle + \langle p, z^*\rangle + \langle q, w^*\rangle < 0.$$

This conflicts with (13). Thus (x_0, y_0) is a weak E-minimiser of (P). ∎

The following corollary is very natural.

Corollary 7. *Let* $x_0 \in K$; *assume that there are* $y_0 \in F(x_0)$, $(y^*, z^*, w^*) \in Y_+^* \times Z_+^* \times W_+^* \setminus \{(O, O, O)\}$, *with* $y^* \neq O$, *such that*

$$\min_{x \in D}\Big(\big\langle F(x), y^*\big\rangle + \big\langle G(x), z^*\big\rangle + \big\langle H(x), w^*\big\rangle\Big) \geqslant \langle y_0, y^*\rangle.$$

Then (x_0, y_0) *is a weak E-minimiser of (P).*

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