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A new derivation of the inner product formula for the Macdonald symmetric polynomials

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Abstract. We give a short proof of the inner product conjecture for the symmetric Macdonald polynomials of type A_{n-1}. As a special case, the corresponding constant term conjecture is also proved.

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1. Introduction

Macdonald's inner product formula, conjectured in [4], was recently proved for arbitrary root systems by Cherednik [1], using the double affine Hecke algebras. In addition to Cherednik's proof, a combinatorial proof by Macdonald [4] and representation-theoretic proof by Etingof and Kirillov Jr. [2] have been given for the A_{n-1} case. The aim of the present note is to give a short proof for the A_{n-1} case by means of asymptotic analysis with q-Selberg type integrals. One of our motivations is in the argument on the integral representation of solutions of eigenvalue problems of the Macdonald type [7]. In that case, choice of cycles associated with the integral corresponds to the choice of different solutions. Such study on the cycles leads to the present argument, another proof of the inner product conjecture for the Macdonald symmetric polynomials of type A_{n-1}. Our argument includes a new proof of the corresponding constant term conjecture as a special case (see also [5]).

Throughout this note, we consider q as a real number satisfying $0 < q < 1$ and $t = q^k$, where $k \in \mathbb{N}$.

2. Inner product formula

We begin recalling some fundamental facts. For a basic reference, we refer the reader to [6].

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A partition λ is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of non-negative integers in decreasing order; $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. The number of nonzero elements λ_i is called the length of λ, denoted by $l(\lambda)$. The sum of the λ_i is the weight of λ denoted by $|\lambda|$. Given a partition λ, we define the conjugate partition $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_n)$ by $\lambda'_i = \mathrm{Card}\{j; \lambda_j \geqslant i\}$.

On partitions, the dominance (or natural) ordering is defined by

$$\lambda \geqslant \mu \Leftrightarrow |\lambda| = |\mu| \quad \text{and} \quad \lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i \quad \text{for all } i \geqslant 1.$$

We consider the ring $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ of polynomials in n variables $x = (x_1, \dots, x_n)$. The subring of all symmetric polynomials is denoted by $\mathbb{C}[x]^{\mathfrak{S}_n}$.

For $f = \Sigma_\beta f_\beta x^\beta \in \mathbb{C}[x]$, we define

$$\bar{f} = \sum_\beta f_\beta x^{-\beta}$$

and let $[f]_1$ denote the constant term of f.

The inner product is defined by

$$\langle f, g \rangle = \frac{1}{n!} [f \bar{g} \Delta]_1,$$

for $f, g \in \mathbb{C}[x]$, with

$$\Delta = \Delta(x) = \prod_{1 \leqslant i \neq j \leqslant n} \frac{(x_i/x_j; q)_\infty}{(t x_i/x_j; q)_\infty} = \prod_{1 \leqslant i \neq j \leqslant n} (x_i/x_j; q)_k,$$

where $(a; q)_\infty = \prod_{i \geqslant 0} (1 \Leftrightarrow a q^i)$ and $(a; q)_n = (a; q)_\infty / (q^n a; q)_\infty$.

Then there is a unique family of symmetric polynomials $P_\lambda(x) = P_\lambda(x; q, t) \in \mathbb{C}[x]^{\mathfrak{S}_n}$ indexed by the partition $\lambda = (\lambda_1, \dots, \lambda_n)$ such that

(1) $P_\lambda = m_\lambda + \Sigma_{\mu < \lambda} c_{\lambda\mu} m_\mu$,

(2) $\langle P_\lambda, P_\mu \rangle = 0$ if $\lambda \neq \mu$,

where each m_μ expresses the monomial symmetric polynomial indexed by μ. The polynomials P_λ are called *Macdonald symmetric polynomials* (associated with the root system of type A_{n-1}).

Our aim is to prove the following.

THEOREM. *We have*

$$\langle P_\lambda, P_\lambda \rangle = \prod_{1 \leqslant i < j \leqslant n} \prod_{r=1}^{k-1} \frac{1 \Leftrightarrow q^{\lambda_i - \lambda_j + r} t^{j-i}}{1 \Leftrightarrow q^{\lambda_i - \lambda_j - r} t^{j-i}}.$$

When $\lambda = 0$ (so that $P_\lambda = 1$), the formula gives the constant term of $\Delta(x)$. This is the constant term conjecture of type A_{n-1} (see [3]).

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3. Proof of theorem

LEMMA. *If $m \geqslant n$, for a polynomial $\psi(x) = \psi(x_1, \ldots, x_n)$, we have*

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{T^n} \prod_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} \frac{1}{(y_i/x_j;q)_k} \Delta(x)\psi(x)\frac{\mathrm{d}x_1\ldots \mathrm{d}x_n}{x_1\ldots x_n}$$

$$= \sum_{\substack{\{i_1,\ldots,i_n\}\\ \subset\{1,\ldots,n\}}} \sum_{0\leqslant l_1,\ldots,l_n\leqslant k-1}$$

$$\times \operatorname{Res}_{x=(y_{i_1}q^{l_1},\ldots,y_{i_n}q^{l_n})}\left\{\prod_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} \frac{1}{(y_i/x_j;q)_k}\Delta(x)\psi(x)\frac{\mathrm{d}x_1\ldots \mathrm{d}x_n}{x_1\ldots x_n}\right\},$$

where $i_1, \ldots, i_n$ are distinct, and $T^n = \{(t_1, \ldots, t_n) \in \mathbb{C}^n; |t_i| = 1(1 \leqslant i \leqslant n)\}$ with the standard orientation.

Proof. For a polynomial $\psi(x_1, x_2)$ and $0 \leqslant l \leqslant k - 1$, we have the equality

$$\operatorname{Res}_{x_1=yq^l}\frac{(x_1/x_2;q)_k(x_2/x_1;q)_k}{(y/x_1;q)_k(y/x_2;q)_k}\psi(x_1,x_2)\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}$$

$$= \frac{(yq^l/x_2;q)_k(x_2q^{-l}/y;q)_k}{(q^{-l};q)_l(q;q)_{k-1-l}(y/x_2;q)_k}\psi(yq^l,x_2)\frac{\mathrm{d}x_2}{x_2}. \tag{3.1}$$

Because $(y/x_2;q)_k$ divides $(yq^l/x_2;q)_k(x_2q^{-l}/y;q)_k$, the 1-form (3.1) has no poles on the x_2-plane. This shows that the set of poles of

$$\prod_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant 2}} \frac{1}{(y_i/x_j;q)_k}\Delta(x_1,x_2)\psi(x_1,x_2)\frac{\mathrm{d}x_1\,\mathrm{d}x_2}{x_1x_2},$$

is the union of $(x_1, x_2) = (y_{i_1}q^l, y_{i_2}q^l)$ for $1 \leqslant i_1 \neq i_2 \leqslant m$ and $0 \leqslant l \leqslant k - 1$, which implies the assertion of the above Lemma in the $n = 2$ case. Repeating this procedure, we have the desired result in case of general n. □

It is known ((3.11) in [4]) that

$$\sum_\lambda b_\lambda P_\lambda(y)P_\lambda(x) = \prod_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} \frac{(ty_ix_j;q)_\infty}{(y_ix_j;q)_\infty} = \prod_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} \frac{1}{(y_ix_j;q)_k} \tag{3.2}$$

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with

$$b_\lambda = b_\lambda(q,t) = \prod_{s\in\lambda} \frac{1 \Leftrightarrow q^{a(s)}t^{l(s)+1}}{1 \Leftrightarrow q^{a(s)+1}t^{l(s)}}.$$

Here the sum is taken over all partitions λ such that $l(\lambda) \leqslant \min\{m,n\}$, and the arm-length $a(s)$ (resp. the leg-length $l(s)$) is defined by $a(s) = \lambda_i \Leftrightarrow j$ (resp. $l(s) = \lambda'_j \Leftrightarrow i$) for a square $s = (i,j)$ in the diagram λ.

The formula (3.2) in the $m = n$ case with the orthogonality relation gives

$$\begin{aligned}
&b_\lambda P_\lambda(y)\langle P_\lambda, P_\lambda\rangle \\
&\quad = \frac{1}{n!}\left(\frac{1}{2\pi\sqrt{\Leftrightarrow 1}}\right)^n \int_{T^n} \prod_{1\leqslant i,j\leqslant n} \frac{1}{(y_i/x_j;q)_k} P_\lambda(x)\Delta(x)\frac{\mathrm{d}x_1\ldots \mathrm{d}x_n}{x_1\ldots x_n} \\
&\quad = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sum_{0\leqslant l_1,\ldots,l_n\leqslant k-1} \mathrm{Res}_{x=(y_{\sigma(1)}q^{l_1},\ldots,y_{\sigma(n)}q^{l_n})} \\
&\qquad \times\left\{\prod_{1\leqslant i,j\leqslant n}\frac{1}{(y_i/x_j;q)_k}P_\lambda(x)\Delta(x)\frac{\mathrm{d}x_1\ldots \mathrm{d}x_n}{x_1\ldots x_n}\right\} \\
&\quad = \sum_{0\leqslant l_1,\ldots,l_n\leqslant k-1} \mathrm{Res}_{x=(y_1q^{l_1},\ldots,y_nq^{l_n})} \\
&\qquad \times\left\{\prod_{1\leqslant i,j\leqslant n}\frac{1}{(y_i/x_j;q)_k}P_\lambda(x)\Delta(x)\frac{\mathrm{d}x_1\ldots \mathrm{d}x_n}{x_1\ldots x_n}\right\}. \qquad (3.3)
\end{aligned}$$

Here the second equality is given by Lemma above and the third equality by the symmetry of the summand with respect to the variables $x = (x_1,\ldots,x_n)$.

Next, by changing the integration variables on the right-hand side according to $x_i \to y_i x_i$, we have

$$\begin{aligned}
&\sum_{0\leqslant l_1,\ldots,l_n\leqslant k-1} \mathrm{Res}_{x=(q^{l_1},\ldots,q^{l_n})} \left\{\prod_{1\leqslant i,j\leqslant n}\frac{1}{\left(\frac{y_i}{y_jx_j};q\right)_k}\prod_{1\leqslant i\neq j\leqslant n}\left(\frac{y_ix_i}{y_jx_j};q\right)_k\right. \\
&\quad \left.\times P_\lambda(y_1x_1,\ldots,y_nx_n)\frac{\mathrm{d}x_1\ldots \mathrm{d}x_n}{x_1\ldots x_n}\right\},
\end{aligned}$$

which tends to

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$$\sum_{0\leqslant l_1,\dots,l_n\leqslant k-1} \mathrm{Res}_{x=(q^{l_1},\dots,q^{l_n})} \left\{ \frac{x_1^k(x_1x_2)^k\dots(x_1\dots x_{n-1})^k}{\prod_{i=1}^n(1/x_i)_k} \right.$$

$$\left. \times \{(y_1x_1)^{\lambda_1}\dots(y_nx_n)^{\lambda_n} + \text{lower order terms}\} \frac{\mathrm{d}x_1\dots\mathrm{d}x_n}{x_1\dots x_n} \right\}$$

$$= y^\lambda \sum_{0\leqslant l_1,\dots,l_n\leqslant k-1} \mathrm{Res}_{x=(q^{l_1},\dots,q^{l_n})} \left\{ \prod_{i=1}^n \frac{(x_i)^{\lambda_i+(n-i)k}}{(1/x_i;q)_k} \frac{\mathrm{d}x_1\dots\mathrm{d}x_n}{x_1\dots x_n} \right\}$$

$$+ \text{lower order terms}$$

$$= y^\lambda \prod_{i=1}^n \frac{(q^{\lambda_i+(n-i)k+1};q)_{k-1}}{(q;q)_{k-1}} + \text{lower order terms},$$

if

$$1 > |y_1| \gg |y_2| \gg \dots \gg |y_n|. \tag{3.4}$$

Here we used the q-binomial theorem

$$\sum_{l\geqslant 0} \frac{(a;q)_l}{(q;q)_l} z^l = \frac{(az;q)_\infty}{(z;q)_\infty} \quad (|z| < 1), \tag{3.5}$$

to derive the last equality above.

Comparing the coefficients of y^λ of (3.3) in the region (3.4) leads to

$$b_\lambda \langle P_\lambda, P_\lambda \rangle = \prod_{i=1}^n \frac{(q^{\lambda_i+(n-i)k+1};q)_{k-1}}{(q;q)_{k-1}},$$

which is equivalent to

$$\langle P_\lambda, P_\lambda \rangle = \prod_{1\leqslant i<j\leqslant n} \frac{(q^{\lambda_i-\lambda_j+1+(j-i)k};q)_{k-1}}{(q^{\lambda_i-\lambda_j+1+(j-i-1)k};q)_{k-1}}.$$

Here we used the equality

$$b_\lambda = \prod_{1\leqslant i<j\leqslant n} \frac{(q^{\lambda_i-\lambda_j+1+(j-i-1)k};q)_{k-1}}{(q^{\lambda_i-\lambda_j+1+(j-i)k};q)_{k-1}} \prod_{i=1}^n \frac{(q^{\lambda_i+1+k(n-i)};q)_{k-1}}{(q;q)_{k-1}}.$$

This completes the proof of our Theorem.

Remark. When we would like to consider the $q = 1$ case directly, we need only modify the proof of Lemma and the calculation of the residue at the final step.

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