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LETTERS TO THE EDITOR

SOME REMARKS ON THE RENEWAL FUNCTION OF THE UNIFORM DISTRIBUTION

U. JENSEN,* *Universität Stuttgart-Hohenheim*

Russell (1983) considers the distribution properties of $N(a)$, the minimum value of n for which the sum S_n of n random variables, which are i.i.d. uniformly on (0, 1), exceeds the constant a, $a \geqq 0$.

The main results of Russell's paper are rediscoveries of known results in renewal theory.

Proceeding from the n-fold convolution F_n of the uniform distribution

$$F_n(x) = \frac{1}{n!} \sum_{j=0}^{[x]} (-1)^j \binom{n}{j} (x-j)^n, \qquad x \geqq 0, \qquad F(x) = F_1(x)$$

the probabilities $H_n(a) := P[N(a) = n]$ are easily computed (for notation see Russell (1983)):

$$\begin{aligned} H_n(a) &= P[S_{n-1} \leqq a, S_n > a] \\ &= F_{n-1}(a) - F_n(a), \qquad n = 1, 2, \cdots \end{aligned}$$

where $F_0(a) = 1$ for $a \geqq 0$.

From this we get the following recursion formula between successive probabilities:

$$\begin{aligned} H_{n+1}(a) &= (H_n * F)(a) = \int_{a-1}^{a} H_n(y)\, dy, \qquad n = 1, 2, \cdots \\ H_1(a) &= 1 - F(a), \end{aligned}$$

where the asterisk denotes convolution. It follows obviously that $H_n = F_{n-1} * (1-F)$.

Let $U(a) = E[N(a)]$ denote the renewal function. Then, from Feller (1971), p. 372,

$$U(a) = \sum_{n=0}^{\infty} F_n(a).$$

The explicit form of the renewal function for the uniform distribution is given in Feller (1971), p. 385.

Since the expressions for $E[N(a)]$ and $\mathrm{Var}\,[N(a)]$ are not easy to simplify in case of the uniform distribution, the following linear bounds and asymptotic approximations are useful:

$$2a \leqq EN(a) = U(a) \leqq 2a + 1 \text{ for all } a \geqq 0$$

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* Postal address: Institut für Angewandte Mathematik und Statistik, Universität Hohenheim (110), Postfach 70 05 62, 7000 Stuttgart 70, W. Germany.

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(see Barlow and Proschan (1965) p. 54),

$$U(a) = 2a + \tfrac{2}{3} + o(1), \qquad a \to \infty, \quad \text{(see Feller (1971), p. 385)},$$

$$\text{Var}\,[N(a)] = \tfrac{2}{3}a + \tfrac{2}{9} + o(1), \qquad a \to \infty, \quad \text{(see Brown and Solomon (1975))}.$$

These results corresponding to the uniform distribution in (0, 1) may be carried over to the general case of uniform distributions in $(0, b)$, $b > 0$ by the relation $U_{(0,b)}(x) = U_{(0,1)}(x/b)$ between the two renewal functions.

It is of interest to obtain similar results for the discrete uniform distribution. As far as I know the renewal function corresponding to the discrete uniform distribution has not yet been determined.

Let S_n be the sum of i.i.d. random variables each assuming the values $1, 2, \dots, c$ with probability $1/c$. From Feller (1968), p. 285 we have

$$P[S_n \leqq j] = \frac{1}{c^n} \sum_{\nu=0}^{\infty} (-1)^\nu \binom{n}{\nu} \binom{j - c\nu}{n}.$$

After some elementary calculations, using

$$\binom{n}{\nu}\binom{j-c\nu}{n} = \binom{j-c\nu}{\nu}\binom{j-(c+1)\nu}{n-\nu},$$

the renewal function is seen to be

$$\begin{aligned} E[N(j)] &= 1 + \sum_{n=1}^{\infty} P[S_n \leqq j] \\ &= \sum_{\nu=0}^{\infty} (-1)^\nu \binom{j-c\nu}{\nu} \frac{1}{c^\nu} \left(1 + \frac{1}{c}\right)^{j-(c+1)\nu} \end{aligned}$$

The limit for $c \to \infty$ and $j \to \infty$, so that $j/c \to a$, leads to the renewal function for continuous uniform distribution in (0, 1).

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