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THE UNITAL EXT-GROUPS AND CLASSIFICATION OF C^*-ALGEBRAS

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Abstract. The semigroups of unital extensions of separable C^*-algebras come in two flavours: a strong and a weak version. By the unital Ext-groups, we mean the groups of invertible elements in these semigroups. We use the unital Ext-groups to obtain K-theoretic classification of both unital and non-unital extensions of C^*-algebras, and in particular we obtain a complete K-theoretic classification of full extensions of UCT Kirchberg algebras by stable AF algebras.

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1. Introduction. Elliott's programme of classifying nuclear C^*-algebras has seen great recent success in the case of finite, simple C^*-algebras due to the work of many hands, most prominently by work of Elliott [**14**] and Gong et al. [**17**], as well as the Quasidiagonality Theorem of Tikuisis et al. [**35**]. This crowning achievement together with the groundbreaking Kirchberg–Phillips classification of purely infinite, simple C^*-algebras [**19**, **26**] completes the classification of separable, unital, simple C^*-algebras with finite nuclear dimension which satisfy the universal coefficient theorem (UCT). The main focus of this paper is the classification of non-simple C^*-algebras. The non-simple classification is especially convoluted due to the lack of a dichotomy between the purely infinite and the stably finite case. A rich class of non-simple C^*-algebras failing this dichotomy is the class of graph C^*-algebras. Great progress was made recently in [**12**], where all unital graph C^*-algebras were classified by a K-theoretic invariant.

The classification of unital graph C^*-algebras was an internal classification result, meaning that it can only be used to compare objects which are already known to be unital graph C^*-algebras. The lack of external classification prevents the result from being applicable in the study of permanence properties for the class of graph C^*-algebras. For instance, it is an open problem whether extensions of graph C^*-algebras are again graph C^*-algebras, subordinate to K-theoretic obstructions. The main results of this paper will be used to solve this question for extensions of simple graph C^*-algebras in [**9**].

The focal point for us is the classification of extensions of classifiable C^*-algebras. In seminal work of Rørdam [**29**], a Weyl–von Neumann–Voiculescu-type absorption theorem of Kirchberg was applied to obtain classification of extensions of non-unital UCT

Kirchberg algebras.[1] This absorption theorem was generalised by Elliott and Kucerovsky [15], thus making the techniques of Rørdam applicable for much more general classification results, as explored by Eilers, Restorff, and Ruiz in [11].

These methods relied heavily on the *non-unital* Ext-group, which is known to be isomorphic to Kasparov's group KK^1. It is not hard to observe that similar methods should apply to *unital* extensions if one applies the strong unital Ext-group $\mathrm{Ext}_{\mathrm{us}}^{-1}(\mathfrak{A}, \mathfrak{B})$ instead. One difficulty in working with the strong Ext-group is that it is even more sensitive than KK-theory. For instance, let $u \in \mathfrak{A}$ be a unitary. In contrast to KK-theory where $\mathrm{KK}(\mathrm{Ad}\, u) = \mathrm{KK}(\mathrm{id}_A)$, the automorphism on $\mathrm{Ext}_{\mathrm{us}}^{-1}(\mathfrak{A}, \mathfrak{B})$ induced by $\mathrm{Ad}\, u$ is *not necessarily* the identity map. The same phenomena will never happen for the weak Ext-group $\mathrm{Ext}_{\mathrm{uw}}^{-1}(\mathfrak{A}, \mathfrak{B})$ as it embeds naturally as a subgroup of $\mathrm{KK}^1(\mathfrak{A}, \mathfrak{B})$.

In [11, Theorem 3.9], all full extensions of *non-unital* UCT Kirchberg algebras by stable AF algebras are classified by their six-term exact sequences in K-theory (with order in K_0 of the ideal). We will complete the classification of such extensions obtaining classification in the case where the UCT Kirchberg algebra is unital. This will be divided into two cases: one where the extension algebra is unital and the other where it is non-unital.

In the case of unital extensions, the invariant will be $K_{\mathrm{six}}^{+,\mathrm{u}}$ which is the six-term exact sequence in K-theory together with order and position of the unit in the K_0-groups. The classification is as follows.

THEOREM A. *Let $\mathfrak{e}_i : 0 \to \mathfrak{B}_i \to \mathfrak{E}_i \to \mathfrak{A}_i \to 0$ be unital extensions of C^*-algebras for $i = 1, 2$ such that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are UCT Kirchberg algebras, and $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are stable AF algebras. Then $\mathfrak{E}_1 \cong \mathfrak{E}_2$ if and only if $K_{\mathrm{six}}^{+,\mathrm{u}}(\mathfrak{e}_1) \cong K_{\mathrm{six}}^{+,\mathrm{u}}(\mathfrak{e}_2)$.*

Next we turn our attention to non-unital extensions with unital quotients. A unital extension as considered above will always be full, as the Busby map is unital and the quotient is simple. For non-unital extensions it is in general much harder to determine whether they are full or not. However, when mixing sufficient amounts of finiteness and infiniteness, it turns out that fullness is a very natural criterion, witnessed by the existence of a properly infinite, full projection in the extension algebra (see Theorem 6.5).

In [16], examples were given of non-isomorphic full extensions of the Cuntz algebra $\mathcal{O}_2$ by the stabilised CAR algebra $M_{2^\infty} \otimes \mathbb{K}$, which had isomorphic six-term exact sequences in K-theory with order, scales and units in the K_0-groups. This means that one needs a finer invariant to classify non-unital extensions when the quotient is unital.

For this purpose, we introduce an invariant $\widetilde{K}_{\mathrm{six}}^{+,\Sigma}$ which includes the usual six-term exact sequence of the extension $0 \to \mathfrak{B} \to \mathfrak{E} \xrightarrow{\pi} \mathfrak{A} \to 0$, together with the K-theory of the extension $0 \to \mathfrak{B} \to \pi^{-1}(\mathbb{C}1_{\mathfrak{A}}) \to \mathbb{C} \to 0$. We refer the reader to Section 7 for more details.

THEOREM B. *Let $\mathfrak{e}_i : 0 \to \mathfrak{B}_i \to \mathfrak{E}_i \to \mathfrak{A}_i \to 0$ be full extensions of C^*-algebras for $i = 1, 2$ such that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unital UCT Kirchberg algebras, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are stable AF algebras. Then, $\mathfrak{E}_1 \cong \mathfrak{E}_2$ if and only if $\widetilde{K}_{\mathrm{six}}^{+,\Sigma}(\mathfrak{e}_1) \cong \widetilde{K}_{\mathrm{six}}^{+,\Sigma}(\mathfrak{e}_2)$.*

In Ref. [9], we will compute the range of the invariant $\widetilde{K}_{\mathrm{six}}^{+,\Sigma}$ for graph C^*-algebras with exactly one nontrivial ideal and for which the nontrivial quotient is unital. This will be used to show that an extension of simple graph C^*-algebras is again a graph C^*-algebra, provided there are no K-theoretic obstructions.

[1]A *UCT Kirchberg algebra* is a separable, nuclear, simple, purely infinite C^*-algebra satisfying the UCT in KK-theory.

2. Extensions of C^*-algebras. In this section, we recall some well-known definitions and results about extensions of C^*-algebras. More details can be found in [1, Chapter VII].

For a C^*-algebra $\mathfrak{B}$, we will denote the multiplier algebra by $\mathscr{M}(\mathfrak{B})$, the corona algebra $\mathscr{M}(\mathfrak{B})/\mathfrak{B}$ by $\mathscr{Q}(\mathfrak{B})$, and the canonical $*$-epimorphism from $\mathscr{M}(\mathfrak{B})$ to $\mathscr{Q}(\mathfrak{B})$ by $\pi_{\mathfrak{B}}$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. An *extension of* $\mathfrak{A}$ *by* $\mathfrak{B}$ is a short exact sequence

$$\mathfrak{e} : 0 \to \mathfrak{B} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \mathfrak{A} \to 0$$

of C^*-algebras. Often we just refer to such a short exact sequence above, as an extension of C^*-algebras. At times we identify $\mathfrak{B}$ with its image $\iota(\mathfrak{B})$ in $\mathfrak{E}$, which is a two-sided, closed ideal, and at times we identify $\mathfrak{A}$ with the quotient $\mathfrak{E}/\iota(\mathfrak{B})$.

To any extension of C^*-algebras as above, there are induced $*$-homomorphisms $\sigma : \mathfrak{E} \to \mathscr{M}(\mathfrak{B})$ and $\tau : \mathfrak{A} \to \mathscr{Q}(\mathfrak{B})$, the latter of these called the *Busby map* (or Busby invariant) of $\mathfrak{e}$. We sometimes refer to arbitrary $*$-homomorphisms $\mathfrak{A} \to \mathscr{Q}(\mathfrak{B})$ as Busby maps.

An extension can be recovered up to canonical isomorphism of extensions by its Busby map τ, as the extension

$$0 \to \mathfrak{B} \to \mathfrak{A} \oplus_{\tau,\pi_{\mathfrak{B}}} \mathscr{M}(\mathfrak{B}) \to \mathfrak{A} \to 0$$

where

$$\mathfrak{A} \oplus_{\tau,\pi_{\mathfrak{B}}} \mathscr{M}(\mathfrak{B}) = \{a \oplus m \in \mathfrak{A} \oplus \mathscr{M}(\mathfrak{B}) : \tau(a) = \pi_{\mathfrak{B}}(m)\}$$

is the pull-back of τ and $\pi_{\mathfrak{B}}$.

An extension is *unital* if the extension algebra is unital, or equivalently, if the Busby map is a unital $*$-homomorphism.

A (unital) extension $\mathfrak{e} : 0 \to \mathfrak{B} \to \mathfrak{E} \xrightarrow{\pi} \mathfrak{A} \to 0$ is called *trivial* (or split) if there is a (unital) $*$-homomorphism $\rho : \mathfrak{A} \to \mathfrak{E}$ such that $\pi \circ \rho = \mathrm{id}_{\mathfrak{A}}$.[2] The extension $\mathfrak{e}$ is called *semi-split* if there is a (unital) completely positive map $\eta : \mathfrak{A} \to \mathfrak{E}$ such that $\pi \circ \rho = \mathrm{id}_{\mathfrak{A}}$.

Let $\mathfrak{e}_i : 0 \to \mathfrak{B} \to \mathfrak{E}_i \to \mathfrak{A} \to 0$ be extensions of C^*-algebras with Busby maps τ_i for $i = 1, 2$. We say that $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are *strongly unitarily equivalent*, written $\mathfrak{e}_1 \sim_{\mathrm{s}} \mathfrak{e}_2$, if there exists a unitary $u \in \mathscr{M}(\mathfrak{B})$ such that $\mathrm{Ad}\, \pi_{\mathfrak{B}}(u) \circ \tau_1 = \tau_2$.

By identifying $\mathfrak{E}_i$ with $\mathfrak{A} \oplus_{\tau_i,\pi_{\mathfrak{B}}} \mathscr{M}(\mathfrak{B})$, we obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{B} \arrow[r] \arrow[d, "\cong"', "\mathrm{Ad}\, u"] & \mathfrak{E}_1 \arrow[r] \arrow[d, "(\cong)"', "\mathrm{Ad}(1_{\mathfrak{A}} \oplus u)"] & \mathfrak{A} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{B} \arrow[r] & \mathfrak{E}_2 \arrow[r] & \mathfrak{A} \arrow[r] & 0
\end{tikzcd} \tag{2.1}$$

with exact rows, which shows that $\mathrm{Ad}(1_{\mathfrak{A}} \oplus u) : \mathfrak{E}_1 \xrightarrow{\cong} \mathfrak{E}_2$ is an isomorphism by the five lemma.

Similarly, $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are *weakly unitary equivalent*, written $\mathfrak{e}_1 \sim_{\mathrm{w}} \mathfrak{e}_2$, if there exists a unitary $u \in \mathscr{Q}(\mathfrak{B})$ such that $\mathrm{Ad}\, u \circ \tau_1 = \tau_2$.

[2] Note that a unital extension being trivial is slightly different from an extension – which happens to be unital – being trivial. In fact, the first requires $\rho(1_{\mathfrak{A}}) = 1_{\mathfrak{E}}$ which the other does not, and in general these two notions are different.

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In contrast to strong unitary equivalence, we cannot in general conclude that the extension algebras $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are isomorphic from weak unitary equivalence.

REMARK 2.1 (Cuntz sum). If $\mathfrak{B}$ is a stable C^*-algebra, then there are isometries $s_1, s_2 \in \mathscr{M}(\mathfrak{B})$ such that $s_1 s_1^* + s_2 s_2^* = 1$. Such a pair s_1, s_2 are called *$\mathcal{O}_2$-isometries*.

If $\mathfrak{e}_i : 0 \to \mathfrak{B} \to \mathfrak{E}_i \to \mathfrak{A} \to 0$ are extensions with Busby maps τ_i for $i = 1, 2$, then we let $\mathfrak{e}_1 \oplus_{s_1,s_2} \mathfrak{e}_2$ denote the extension of $\mathfrak{A}$ by $\mathfrak{B}$ with Busby map τ given by

$$\tau(a) = \pi_{\mathfrak{B}}(s_1)\tau_1(a)\pi_{\mathfrak{B}}(s_1)^* + \pi_{\mathfrak{B}}(s_2)\tau_2(a)\pi_{\mathfrak{B}}(s_2)^*$$

for $a \in A$. This construction is independent of the choice of s_1 and s_2 up to strong unitary equivalence, and thus we often write $\mathfrak{e}_1 \oplus \mathfrak{e}_2$, when we only care about the extension up to $\sim_{\mathrm{s}}$.

DEFINITION 2.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable C^*-algebras with $\mathfrak{B}$ stable. We let

- $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B})$ denote the semigroup of extensions of $\mathfrak{A}$ by $\mathfrak{B}$ modulo the relation defined by $[\mathfrak{e}_1] = [\mathfrak{e}_2]$ if and only if there exist trivial extensions $\mathfrak{f}_1, \mathfrak{f}_2$ of $\mathfrak{A}$ by $\mathfrak{B}$ such that

$$\mathfrak{e}_1 \oplus \mathfrak{f}_1 \sim_{\mathrm{w}} \mathfrak{e}_2 \oplus \mathfrak{f}_2,$$

  or equivalently, there exist trivial extensions $\mathfrak{f}_1', \mathfrak{f}_2'$ of $\mathfrak{A}$ by $\mathfrak{B}$ (which can be taken as $\mathfrak{f}_i' = \mathfrak{f}_i \oplus 0$) such that

$$\mathfrak{e}_1 \oplus \mathfrak{f}_1' \sim_{\mathrm{s}} \mathfrak{e}_2 \oplus \mathfrak{f}_2'.$$

Moreover, if $\mathfrak{A}$ is unital then we let

- $\mathrm{Ext}_{\mathrm{us}}(\mathfrak{A}, \mathfrak{B})$ denote the semigroup of *unital* extensions of $\mathfrak{A}$ by $\mathfrak{B}$ modulo the relation defined by $[\mathfrak{e}_1]_{\mathrm{s}} = [\mathfrak{e}_2]_{\mathrm{s}}$ if and only if there exist trivial, *unital* extensions $\mathfrak{f}_1, \mathfrak{f}_2$ of $\mathfrak{A}$ by $\mathfrak{B}$ such that

$$\mathfrak{e}_1 \oplus \mathfrak{f}_1 \sim_{\mathrm{s}} \mathfrak{e}_2 \oplus \mathfrak{f}_2.$$

- $\mathrm{Ext}_{\mathrm{uw}}(\mathfrak{A}, \mathfrak{B})$ denote the semigroup of *unital* extensions of $\mathfrak{A}$ by $\mathfrak{B}$ modulo the relation defined by $[\mathfrak{e}_1]_{\mathrm{w}} = [\mathfrak{e}_2]_{\mathrm{w}}$ if and only if there exist trivial, *unital* extensions $\mathfrak{f}_1, \mathfrak{f}_2$ of $\mathfrak{A}$ by $\mathfrak{B}$ such that

$$\mathfrak{e}_1 \oplus \mathfrak{f}_1 \sim_{\mathrm{w}} \mathfrak{e}_2 \oplus \mathfrak{f}_2.$$

If $\mathfrak{B}$ is not stable, we define $\mathrm{Ext}_{(\mathrm{us}/\mathrm{uw})}(\mathfrak{A}, \mathfrak{B}) := \mathrm{Ext}_{(\mathrm{us}/\mathrm{uw})}(\mathfrak{A}, \mathfrak{B} \otimes \mathbb{K})$.

It is not hard to show that $\mathrm{Ext}_{(\mathrm{us}/\mathrm{uw})}(\mathfrak{A}, \mathfrak{B})$ is an abelian monoid, and that any trivial (unital) extension induces the zero element. Hence, the following makes sense.

DEFINITION 2.3. Let $\mathrm{Ext}^{-1}(\mathfrak{A}, \mathfrak{B})$, $\mathrm{Ext}_{\mathrm{us}}^{-1}(\mathfrak{A}, \mathfrak{B})$ and $\mathrm{Ext}_{\mathrm{uw}}^{-1}(\mathfrak{A}, \mathfrak{B})$ denote the subsemigroups of $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B})$, $\mathrm{Ext}_{\mathrm{us}}(\mathfrak{A}, \mathfrak{B})$ and $\mathrm{Ext}_{\mathrm{uw}}(\mathfrak{A}, \mathfrak{B})$, respectively (whenever these make sense), of elements which have an additive inverse. These subsets are abelian groups.

REMARK 2.4 (Semisplit extensions). Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable C^*-algebras with $\mathfrak{B}$ stable (and $\mathfrak{A}$ unital). As in [1, Section 15.7], it follows that a (unital) extension of $\mathfrak{A}$ by $\mathfrak{B}$ induces an element in $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B})$ (resp. in either $\mathrm{Ext}_{\mathrm{us}}(\mathfrak{A}, \mathfrak{B})$ or $\mathrm{Ext}_{\mathrm{uw}}(\mathfrak{A}, \mathfrak{B})$) which has an additive inverse, if and only if the extension is semisplit.

In particular, if $\mathfrak{A}$ is nuclear it follows from the Choi–Effros Lifting Theorem [5] that

$$\mathrm{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}) = \mathrm{Ext}(\mathfrak{A}, \mathfrak{B}), \quad \mathrm{Ext}_{\mathrm{us}}^{-1}(\mathfrak{A}, \mathfrak{B}) = \mathrm{Ext}_{\mathrm{us}}(\mathfrak{A}, \mathfrak{B}), \quad \mathrm{Ext}_{\mathrm{uw}}^{-1}(\mathfrak{A}, \mathfrak{B}) = \mathrm{Ext}_{\mathrm{uw}}(\mathfrak{A}, \mathfrak{B}).$$

DEFINITION 2.5 (Pull-back and push-out extensions). Let $\mathfrak{e}: 0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$ be an extension of C^*-algebras with Busby map τ, and let $\alpha: \mathfrak{C} \to \mathfrak{A}$ be a $*$-homomorphism. The *pull-back extension* $\mathfrak{e} \cdot \alpha$ is the extension of $\mathfrak{C}$ by $\mathfrak{B}$ with Busby map $\tau \circ \alpha$.

If $\beta: \mathfrak{B} \to \mathfrak{D}$ is a nondegenerate $*$-homomorphism[3] there is an induced unital $*$-homomorphism $\overline{\beta}: \mathscr{Q}(\mathfrak{B}) \to \mathscr{Q}(\mathfrak{D})$.[4] The *push-out extension* $\beta \cdot \mathfrak{e}$ is the extension of $\mathfrak{A}$ by $\mathfrak{D}$ with Busby map $\overline{\beta} \circ \tau$.

If $\eta: \mathscr{Q}(\mathfrak{B}) \to \mathscr{Q}(\mathfrak{D})$ is a $*$-homomorphism, then we let $\eta \cdot \mathfrak{e}$ denote the extension of $\mathfrak{A}$ by $\mathfrak{D}$ with Busby map $\eta \circ \tau$. In particular, with β as above, we have $\beta \cdot \mathfrak{e} = \overline{\beta} \cdot \mathfrak{e}$.

With the notation as above, the push-out and pull-back extensions fit into the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathfrak{e}\cdot\alpha: & 0 \arrow[r] & \mathfrak{B} \arrow[r] \arrow[d, equal] & \mathfrak{E}_\alpha \arrow[r] \arrow[d] & \mathfrak{C} \arrow[r] \arrow[d, "\alpha"] & 0 \\
\mathfrak{e}: & 0 \arrow[r] & \mathfrak{B} \arrow[r] \arrow[d, "\beta"] & \mathfrak{E} \arrow[r] \arrow[d] & \mathfrak{A} \arrow[r] \arrow[d, equal] & 0 \\
\beta\cdot\mathfrak{e}: & 0 \arrow[r] & \mathfrak{D} \arrow[r] & \mathfrak{E}_\beta \arrow[r] & \mathfrak{A} \arrow[r] & 0.
\end{tikzcd}$$

The top two rows form a pull-back diagram and the bottom two rows form a push-out diagram.

REMARK 2.6 (Functoriality). The pull-back/push-out constructions of extensions turn $\mathrm{Ext}_{(\mathrm{us}/\mathrm{uw})}(\mathfrak{A}, \mathfrak{B})$ into a bifunctor with respect to (unital) $*$-homomorphisms in the first variable, and nondegenerate $*$-homomorphisms in the second variable.

A fair warning: while any unital $*$-homomorphism $\eta: \mathscr{Q}(\mathfrak{B}) \to \mathscr{Q}(\mathfrak{D})$ induces a map $\mathfrak{e} \mapsto \eta \cdot \mathfrak{e}$ which preserves $\sim_{\mathrm{w}}$ (and $\sim_{\mathrm{s}}$ if $\mathfrak{B}$ is stable[5]), it does in general not preserve Cuntz sums. This construction will be crucial in Remark 4.10 where we define $\mathfrak{e}_{[u]} = \mathrm{Ad}\, u \cdot \mathfrak{e}_0$ for a unitary $\mathcal{U}(\mathscr{Q}(\mathfrak{B}))$ and a trivial unital extension $\mathfrak{e}_0$.

The following is a celebrated result of Kasparov [18].

THEOREM 2.7 ([18]). *If $\mathfrak{A}$ and $\mathfrak{B}$ are separable C^*-algebras, then* $\mathrm{Ext}^{-1}(\mathfrak{A}, \mathfrak{B})$ *is naturally isomorphic to Kasparov's group $KK^1(\mathfrak{A}, \mathfrak{B})$.*

REMARK 2.8 (Absorbing extensions). Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable C^*-algebras with $\mathfrak{B}$ stable (and $\mathfrak{A}$ unital). A (unital) extension $\mathfrak{e}$ of $\mathfrak{A}$ by $\mathfrak{B}$ is called *absorbing* if $\mathfrak{e} \sim_{\mathrm{s}} \mathfrak{e} \oplus \mathfrak{f}$ for any trivial (unital) extension $\mathfrak{f}$ of $\mathfrak{A}$ by $\mathfrak{B}$.[6]

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[3] A $*$-homomorphism $\beta: \mathfrak{B} \to \mathfrak{D}$ is *nondegenerate* (or proper) if $\overline{\beta(\mathfrak{B})\mathfrak{D}} = \mathfrak{D}$.

[4] In fact, β induces a unital $*$-homomorphism $\mathscr{M}(\beta): \mathscr{M}(\mathfrak{B}) \to \mathscr{M}(\mathfrak{D})$ by $\mathscr{M}(\beta)(m)(\beta(b)d) := \beta(mb)d$ for $m \in \mathscr{M}(\mathfrak{B})$, $b \in \mathfrak{B}$ and $d \in \mathfrak{D}$. This $*$-homomorphism descends to a unital $*$-homomorphism $\overline{\beta}: \mathscr{Q}(\mathfrak{B}) \to \mathscr{Q}(\mathfrak{D})$.

[5] In fact, if $\mathfrak{B}$ is stable then the unitary group $\mathcal{U}(\mathscr{M}(\mathfrak{B}))$ is connected, and thus a unitary $u \in \mathscr{Q}(\mathfrak{B})$ lifts to a unitary in $\mathscr{M}(\mathfrak{B})$ exactly when $u \in \mathcal{U}_0(\mathscr{Q}(\mathfrak{B}))$, i.e. the connected component of $1_{\mathscr{Q}}(\mathfrak{B})$ in the unitary group. As $\eta(\mathcal{U}_0(\mathscr{Q}(\mathfrak{B}))) \subseteq \mathcal{U}_0(\mathscr{Q}(\mathfrak{D}))$, and as every unitary in $\mathcal{U}_0(\mathscr{Q}(\mathfrak{D}))$ lifts to a unitary in $\mathscr{M}(\mathfrak{D})$, it easily follows that η preserves strong unitary equivalence classes of extensions.

[6] Just as with triviality, there is a difference between requiring that an extension is absorbing, or that a unital extension is absorbing. Sometimes absorbing unital extensions are said to be unital-absorbing. However, we simply call these absorbing as there is no cause of confusion, since a unital extension can never be absorbing in the general sense (it would have to absorb the extension with zero Busby map, which is impossible).

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By [34] there always exists an absorbing, trivial (unital) extension $\mathfrak{e}_0$ of $\mathfrak{A}$ by $\mathfrak{B}$.[7] In particular, $\mathfrak{e} \oplus \mathfrak{e}_0$ is absorbing for any (unital) extension $\mathfrak{e}$.

In particular, if $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are absorbing extensions of $\mathfrak{A}$ by $\mathfrak{B}$ with $[\mathfrak{e}_1] = [\mathfrak{e}_2]$ in $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B})$, then $\mathfrak{e}_1 \sim_{\mathrm{s}} \mathfrak{e}_2$.

Similarly, if $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are absorbing *unital* extensions of $\mathfrak{A}$ by $\mathfrak{B}$ with $[\mathfrak{e}_1]_{\mathrm{s}} = [\mathfrak{e}_2]_{\mathrm{s}}$ in $\mathrm{Ext}_{\mathrm{us}}(\mathfrak{A}, \mathfrak{B})$ (resp. $[\mathfrak{e}_1]_{\mathrm{w}} = [\mathfrak{e}_2]_{\mathrm{w}}$ in $\mathrm{Ext}_{\mathrm{uw}}(\mathfrak{A}, \mathfrak{B})$), then $\mathfrak{e}_1 \sim_{\mathrm{s}} \mathfrak{e}_2$ (resp. $\mathfrak{e}_1 \sim_{\mathrm{w}} \mathfrak{e}_2$).

REMARK 2.9 (Determining absorption). It seems inconceivable *a priori* that one could ever determine when an extension is absorbing. However, this was done by Elliott and Kucerovsky in [15].

Following [15], an extension $\mathfrak{e} : 0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$ of separable C^*-algebras is called *purely large* if for any $x \in \mathfrak{E} \setminus \mathfrak{B}$, there exists a stable C^*-subalgebra $\mathfrak{D} \subseteq \overline{x^* \mathfrak{B} x}$ such that $\overline{\mathfrak{B}\mathfrak{D}\mathfrak{B}} = \mathfrak{B}$.

By a remarkable result [15, Theorem 6], if $\mathfrak{e} : 0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$ is a unital extension of separable C^*-algebras for which $\mathfrak{A}$ is nuclear and $\mathfrak{B}$ is stable, then $\mathfrak{e}$ is absorbing (in the unital sense) if and only if it is purely large. Similar conditions for when non-unital extensions are absorbing were studied in [16].

A separable C^*-algebra $\mathfrak{B}$ is said to have the *corona factorisation property* if any full projection $p \in \mathscr{M}(\mathfrak{B} \otimes \mathbb{K})$ is equivalent to $1_{\mathscr{M}(\mathfrak{B} \otimes \mathbb{K})}$. Many classes of separable C^*-algebras are known to have the corona factorisation property, e.g. all C^*-algebras with finite nuclear dimension by [28, Corollary 3.5] (building on the work in [24]). In particular, any AF algebra has the corona factorisation property, as these have nuclear dimension zero.

An extension $\mathfrak{e}$ of $\mathfrak{A}$ by $\mathfrak{B}$ with Busby map $\tau : \mathfrak{A} \to \mathscr{Q}(\mathfrak{B})$ is called *full* if for every nonzero $a \in \mathfrak{A}$, $\tau(a)$ generates all of $\mathscr{Q}(\mathfrak{B})$ as a two-sided, closed ideal. As observed by Kucerovsky and Ng in [20], if $\mathfrak{e} : 0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$ is a full extension of separable C^*-algebras, for which $\mathfrak{B}$ is stable and has the corona factorisation property, then $\mathfrak{e}$ is purely large.

3. *K*-theory of unital extensions.

The purpose of this section is to collect some results on the K-theory of extensions of C^*-algebras, with a main focus on what happens to the unit in the K_0-groups under certain operations of unital extensions. While most results in this section are quite elementary and most likely well-known to some experts in the field, we know of no references to these results and have included detailed proofs for completion.

Consider two six-term exact sequences

$$\mathsf{x}^{(i)} : \qquad \begin{tikzcd} H_0^{(i)} \arrow[r] & L_0^{(i)} \arrow[r] & G_0^{(i)} \arrow[d] \\ G_1^{(i)} \arrow[u] & L_1^{(i)} \arrow[l] & H_1^{(i)} \arrow[l] \end{tikzcd}$$

for $i = 1, 2$. A homomorphism $(\psi_*, \rho_*, \phi_*) : \mathsf{x}^{(1)} \to \mathsf{x}^{(2)}$ of six-term exact sequences consists of homomorphisms

$$\phi_* : G_*^{(1)} \to G_*^{(2)}, \qquad \psi_* : H_*^{(1)} \to H_*^{(2)}, \qquad \rho_* : L_*^{(1)} \to L_*^{(2)}$$

making the obvious diagram commute.

[7] This requires that $\mathfrak{A}$ and $\mathfrak{B}$ are separable. Although the definition of absorption makes sense without separability, we stick to this case.

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We may also consider six-term exact sequences with certain distinguished elements, which in our case will always be elements in $x_i \in L_0^{(i)}$ and $y_i \in G_0^{(i)}$ for $i = 1, 2$, and will correspond to the classes of the units in our K_0-groups. If this is the case, we only consider homomorphisms such that $\rho_0(x_1) = x_2$ and $\phi_0(y_1) = y_2$.

If $G_*^{(1)} = G_*^{(2)} =: G_*$ and $H_*^{(1)} = H_*^{(2)} =: H_*$ then we say that $\mathsf{x}^{(1)}$ and $\mathsf{x}^{(2)}$ are *congruent*, written $\mathsf{x}^{(1)} \equiv \mathsf{x}^{(2)}$, if there exists a homomorphism of the form $(\mathrm{id}_{H_*}, \rho_*, \mathrm{id}_{G_*}) : \mathsf{x}^{(1)} \to \mathsf{x}^{(2)}$. Note that by the five lemma, this forces ρ_* to be an isomorphism, but in general many different ρ_* can implement a congruence.

If any of the groups in the six-term exact sequences contain distinguished elements, we require that our homomorphisms preserve these elements. In particular, when considering congruence with $x_i \in L_0^{(i)}$ and $y_i \in G_0^{(i)} = G_0$ being our distinguished elements, we only consider the case $y_1 = y_2$.

Definition 3.1. For an extension $\mathfrak{e} : 0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$ of (unital) C^*-algebras, we let $K_{\mathrm{six}}(\mathfrak{e})$ (resp. $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e})$) denote the six-term exact sequence in K-theory (resp. with distinguished elements $[1_{\mathfrak{E}}] \in K_0(\mathfrak{E})$ and $[1_{\mathfrak{A}}] \in K_0(\mathfrak{A})$).

Note that two extensions $\mathfrak{e}$ and $\mathfrak{f}$ can *only* have congruent six-term exact sequences, if the two ideals are *equal* and the two quotients are *equal* (isomorphisms are not enough for the definition to make sense). So both extensions *have to* be extensions of $\mathfrak{A}$ by $\mathfrak{B}$ for the definition of congruence to make sense.

The following two lemmas are well-known, but we fill in the proofs for completeness.

Lemma 3.2. *Let $\mathfrak{e}_1$ and $\mathfrak{e}_2$ be unital extensions of $\mathfrak{A}$ by $\mathfrak{B}$ which are strongly unitarily equivalent. Then $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_1) \equiv K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_2)$.*

Proof. If $u \in \mathscr{M}(\mathfrak{B})$ implements the strong unitary equivalence, then applying K-theory to the diagram (2.1) and using that $K_*(\mathrm{Ad}\, u) = \mathrm{id}_{K_*(\mathfrak{B})} : K_*(\mathfrak{B}) \to K_*(\mathfrak{B})$, one obtains a congruence $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_1) \equiv K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_2)$. □

Lemma 3.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ unital and $\mathfrak{B}$ stable. Let $\mathfrak{e} : 0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$ be a unital extension, and let $\mathfrak{e}_0$ be a trivial unital extension of $\mathfrak{A}$ by $\mathfrak{B}$. Then $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e})$ and $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e} \oplus \mathfrak{e}_0)$ are congruent.*

Proof. Let $s_1, s_2 \in \mathscr{M}(\mathfrak{B})$ be $\mathcal{O}_2$-isometries so that $\mathfrak{e} \oplus \mathfrak{e}_0 = \mathfrak{e} \oplus_{s_1,s_2} \mathfrak{e}_0$. Let $\pi : \mathfrak{E} \to \mathfrak{A}$ be the quotient map, $\sigma : \mathfrak{E} \to \mathscr{M}(\mathfrak{B})$ be the canonical unital $*$-homomorphism, and $\phi : \mathfrak{A} \to \mathscr{M}(\mathfrak{B})$ be a unital $*$-homomorphism which lifts τ_0.

The extension algebra $\mathfrak{F}$ of $\mathfrak{e} \oplus_{s_1,s_2} \mathfrak{e}_0$ is by definition

$$\mathfrak{F} = \{a \oplus m \in \mathfrak{A} \oplus \mathscr{M}(\mathfrak{B}) : \pi_{\mathfrak{B}}(s_1)\tau(a)\pi_{\mathfrak{B}}(s_1)^* + \pi_{\mathfrak{B}}(s_2)\tau_0(a)\pi_{\mathfrak{B}}(s_2)^* = \pi_{\mathfrak{B}}(m)\}.$$

Define the unital $*$-homomorphism $\Psi : \mathfrak{E} \to \mathfrak{F}$ by

$$\Psi(y) = \pi(y) \oplus \big(s_1\sigma(y)s_1^* + s_2\phi(\pi(y))s_2^*\big).$$

This is clearly well-defined and induces a unital $*$-homomorphism of extensions by

$$\begin{array}{rc}
\mathfrak{e} : & 0 \longrightarrow \mathfrak{B} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{A} \longrightarrow 0 \\
 & \qquad\ \ \downarrow{\scriptstyle s_1(-)s_1^*} \quad\ \ \downarrow{\scriptstyle \Psi} \qquad \| \qquad\qquad \\
\mathfrak{e} \oplus_{s_1,s_2} \mathfrak{e}_0 : & 0 \longrightarrow \mathfrak{B} \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{A} \longrightarrow 0.
\end{array}$$

As $(s_1(-)s_1^*)_* = \mathrm{id}_{K_*(\mathfrak{B})} : K_*(\mathfrak{B}) \to K_*(\mathfrak{B})$, applying K-theory to the above diagram induces a congruence $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}) \equiv K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e} \oplus_{s_1,s_2} \mathfrak{e}_0)$. □

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Corollary 3.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable C^*-algebras with $\mathfrak{A}$ unital and B stable. Suppose that $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are unital extensions of $\mathfrak{A}$ by $\mathfrak{B}$ for which $[\mathfrak{e}_1]_{\mathrm{s}} = [\mathfrak{e}_2]_{\mathrm{s}}$ in* $\mathrm{Ext}_{\mathrm{us}}(\mathfrak{A}, \mathfrak{B})$*. Then $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_1) \equiv K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_2)$.*

Proof. By definition of $\mathrm{Ext}_{\mathrm{us}}$, there are trivial, unital extensions $\mathfrak{f}_1, \mathfrak{f}_2$, such that $\mathfrak{e}_1 \oplus \mathfrak{f}_1$ and $\mathfrak{e}_2 \oplus \mathfrak{f}_2$ are strongly unitarily equivalent. Hence, the result follows from Lemmas 3.2 and 3.3. □

Lemma 3.5. *Let $\mathfrak{e}: 0 \to \mathfrak{B} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \mathfrak{A} \to 0$ be a unital extension C^*-algebras with boundary map $\delta_*: K_*(\mathfrak{A}) \to K_{1-*}(\mathfrak{B})$ in K-theory, let $u \in \mathcal{Q}(\mathfrak{B})$ be a unitary, and let $\chi_1: K_1(\mathcal{Q}(\mathfrak{B})) \to K_0(\mathfrak{B})$ denote the index map in K-theory. Then $K_{\mathrm{six}}^{\mathrm{u}}(\mathrm{Ad}\, u \cdot \mathfrak{e})$ (see Definition 2.5) is congruent to*

$$\begin{tikzcd}
K_0(\mathfrak{B}) \arrow[r, "\iota_0"] & (K_0(\mathfrak{E}), [1_{\mathfrak{E}}] + \iota_0(\chi_1([u]))) \arrow[r, "\pi_0"] & (K_0(\mathfrak{A}), [1_{\mathfrak{A}}]) \arrow[d, "\delta_0"] \\
K_1(\mathfrak{A}) \arrow[u, "\delta_1"] & K_1(\mathfrak{E}) \arrow[l, "\pi_1"'] & K_1(\mathfrak{B}). \arrow[l, "\iota_1"']
\end{tikzcd}$$

Proof. Let $a \in \mathcal{M}(\mathfrak{B})$ be a lift of u with $\|a\| = 1$, and define

$$v := \begin{pmatrix} a & 0 \\ (1-a^*a)^{1/2} & 0 \end{pmatrix} \in M_2(\mathcal{M}(\mathfrak{B})), \qquad v_c := \begin{pmatrix} a \\ (1-a^*a)^{1/2} \end{pmatrix} \in M_{2,1}(\mathcal{M}(\mathfrak{B})).$$

Then v is a partial isometry for which $v^*v = 1_{\mathcal{M}(\mathfrak{B})} \oplus 0$. It is well-known (see e.g. [30, Section 9.2] that

$$\chi_1([u]) = [1_{M_2(\tilde{\mathfrak{B}})} - vv^*] - [0 \oplus 1_{\tilde{\mathfrak{B}}}] \in K_0(\mathfrak{B}). \tag{3.1}$$

Let τ denote the Busby map of $\mathfrak{e}$, and identify $\mathfrak{E}$ with the pull-back $\mathfrak{A} \oplus_{\tau, \pi_{\mathfrak{B}}} \mathcal{M}(\mathfrak{B})$. Define

$$\mathfrak{E}_2 := \{a \oplus y \in \mathfrak{A} \oplus M_2(\mathcal{M}(\mathfrak{B})) : (\mathrm{Ad}\, u \circ \tau(a)) \oplus 0 = M_2(\pi_{\mathfrak{B}})(y) \in M_2(\mathcal{Q}(\mathfrak{B}))\},$$

i.e. $\mathfrak{E}_2$ is the pull-back $\mathfrak{A} \oplus_{(\mathrm{Ad}\, u \circ \tau) \oplus 0, M_2(\pi_{\mathfrak{B}})} M_2(\mathcal{M}(\mathfrak{B}))$. We obtain an embedding

$$\mathrm{Ad}(1 \oplus v_c): \mathfrak{A} \oplus_{\tau, \pi_{\mathfrak{B}}} \mathcal{M}(\mathfrak{B}) \to \mathfrak{E}_2.$$

Similarly, identify the extension algebra $\mathfrak{E}_u$ of $\mathrm{Ad}\, u \cdot \mathfrak{e}$ with the pull-back $\mathfrak{A} \oplus_{\mathrm{Ad}\, u \circ \tau, \pi_{\mathfrak{B}}} \mathcal{M}(\mathfrak{B})$. The embedding $\mathcal{M}(\mathfrak{B}) \to M_2(\mathcal{M}(\mathfrak{B}))$ into the $(1, 1)$-corner induces an embedding

$$\mathrm{id}_{\mathfrak{A}} \oplus j: \mathfrak{A} \oplus_{\mathrm{Ad}\, u \circ \tau, \pi_{\mathfrak{B}}} \mathcal{M}(\mathfrak{B}) \to \mathfrak{E}_2.$$

We get the following diagram where all rows are short exact sequences and all maps are $*$-homomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{B} \arrow[r, "\iota"] \arrow[d, "\mathrm{Ad}\, v_c"] & \mathfrak{E} \arrow[r, "\pi"] \arrow[d, "\mathrm{Ad}(1 \oplus v_c)"] & \mathfrak{A} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_2(\mathfrak{B}) \arrow[r, "\iota^{(2)}"] & \mathfrak{E}_2 \arrow[r] & \mathfrak{A} \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{B} \arrow[r] \arrow[u, "j"] & \mathfrak{E}_u \arrow[r] \arrow[u, "\mathrm{id}_{\mathfrak{A}} \oplus j"] & \mathfrak{A} \arrow[r] \arrow[u, equal] & 0.
\end{tikzcd}$$

Note that $(\operatorname{Ad} v_c)_*, j_* \colon K_*(\mathfrak{B}) \to K_*(M_2(\mathfrak{B}))$ are the same map, namely the canonical isomorphism. In particular, by considering the induced maps of six-term exact sequences, the five lemma implies that $(\mathrm{id}_{\mathfrak{A}} \oplus j)_* \colon K_*(\mathfrak{E}_u) \to K_*(\mathfrak{E}_2)$ and $\operatorname{Ad}(1 \oplus v_c)_* \colon K_*(\mathfrak{E}) \to K_*(\mathfrak{E}_2)$ are isomorphisms. As $(\operatorname{Ad} v_c)_* = j_*$, it follows that

$$\operatorname{Ad}(1 \oplus v_c)_*^{-1} \circ (\mathrm{id}_{\mathfrak{A}} \oplus j)_* \colon K_*(\mathfrak{E}_u) \to K_*(\mathfrak{E})$$

induces a congruence $K_{\mathrm{six}}(\operatorname{Ad} u \cdot \mathfrak{e}) \equiv K_{\mathrm{six}}(\mathfrak{e})$ which does not necessarily preserve the class of the unit since $\operatorname{Ad}(1 \oplus v_c)$ and $\mathrm{id}_{\mathfrak{A}} \oplus j$ are not unital maps. Thus, it remains to prove that

$$\operatorname{Ad}(1 \oplus v_c)_0^{-1}((\mathrm{id}_{\mathfrak{A}} \oplus j)_0([1_{\mathfrak{E}_u}])) = [1_{\mathfrak{E}}] + \iota_0(\chi_1([u])),$$

or alternatively, that

$$(\operatorname{Ad}(1 \oplus v_c))_0([1_{\mathfrak{E}}] + \iota_0(\chi_1([u]))) = (\mathrm{id}_{\mathfrak{A}} \oplus j)_0([1_{\mathfrak{E}_u}]) = [1_{\mathfrak{A}} \oplus (1_{\mathscr{M}(\mathfrak{B})} \oplus 0)] \in K_0(\mathfrak{E}_2). \tag{3.2}$$

Note that the unitisation

$$\widetilde{\mathfrak{E}}_2 = \mathfrak{E}_2 + \mathbb{C}(0_{\mathfrak{A}} \oplus (0 \oplus 1_{\mathscr{M}(\mathfrak{B})})) \subseteq \mathfrak{A} \oplus M_2(\mathscr{M}(\mathfrak{B})).$$

As $(\operatorname{Ad} v_c)_0 = j_0 \colon K_0(\mathfrak{B}) \xrightarrow{\cong} K_0(M_2(\mathfrak{B}))$ is the canonical isomorphism, it follows from (3.1) (using that $1_{\mathfrak{A}} \oplus vv^* \in \mathfrak{E}_2$) that

$$\begin{aligned} \operatorname{Ad}(1 \oplus v_c)_0 \circ \iota_0(\chi_1([u])) &= \iota_0^{(2)} \circ j_0(\chi_1([u])) \\ &= [1_{\widetilde{\mathfrak{E}}_2} - (1_{\mathfrak{A}} \oplus vv^*)] - [0_{\mathfrak{A}} \oplus (0 \oplus 1_{\mathscr{M}(\mathfrak{B})})] \\ &= [1_{\mathfrak{A}} \oplus (1_{\mathscr{M}(\mathfrak{B})} \oplus 0)] - [1_{\mathfrak{A}} \oplus vv^*] \in K_0(\mathfrak{E}_2). \end{aligned} \tag{3.3}$$

Clearly

$$\operatorname{Ad}(1 \oplus v_c)_0([1_{\mathfrak{E}}]) = [1_{\mathfrak{A}} \oplus v_c v_c^*] = [1_{\mathfrak{A}} \oplus vv^*] \in K_0(\mathfrak{E}_2),$$

and combining this with (3.3) yields (3.2). □

Recall that if L_1, L_2, and G are abelian groups and $\phi_i \colon L_i \to G$ are homomorphisms, then

$$L_1 \oplus_{\phi_1, \phi_2} L_2 = \{x_1 \oplus x_2 \in L_1 \oplus L_2 : \phi_1(x_1) = \phi_2(x_2)\}$$

is the pull-back. When there is no doubt of what the maps ϕ_i are, we simply write $L_1 \oplus_G L_2$ instead of $L_1 \oplus_{\phi_1, \phi_2} L_2$.

REMARK 3.6. Recall that if $\mathsf{x}_i : 0 \to H \xrightarrow{\iota^{(i)}} L_i \xrightarrow{\pi^{(i)}} G \to 0$ are extensions of abelian groups for $i = 1, 2$, then their *Baer sum* $\mathsf{x}_1 \oplus \mathsf{x}_2$ is the extension given by

$$0 \to H \xrightarrow{\iota^{(1)}} \frac{L_1 \oplus_G L_2}{\{(\iota^{(1)}(x), -\iota^{(2)}(x)) : x \in H\}} \xrightarrow{\pi^{(1)}} G \to 0.$$

Addition in the group $\operatorname{Ext}(G, H)$ is given by the Baer sum.

The following proposition is an explicit formula for computing $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_1 \oplus \mathfrak{e}_2)$ using a similar construction as the Baer sum, when we know that the boundary maps for one of $\mathfrak{e}_1$ or $\mathfrak{e}_2$ vanishes.

PROPOSITION 3.7. *Let* $\mathfrak{e}_i : 0 \to \mathfrak{B} \xrightarrow{\iota^{(i)}} \mathfrak{E}_i \xrightarrow{\pi^{(i)}} \mathfrak{A} \to 0$ *be unital extensions of* C^**-algebras for* $i = 1, 2$ *such that* $\mathfrak{B}$ *is stable. Let* $\delta_*^{(i)} \colon K_*(\mathfrak{A}) \to K_{1-*}(\mathfrak{B})$ *denote the boundary map of* $\mathfrak{e}_i$ *in K-theory for* $i = 1, 2$. *If* $\delta_*^{(2)} = 0$, *then* $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_1 \oplus \mathfrak{e}_2)$ *is congruent to*

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$$\begin{tikzcd}
K_0(\mathfrak{B}) \arrow[r, "\iota_0^{(1)}"] & \left(\frac{K_0(\mathfrak{E}_1)\oplus_{K_0(\mathfrak{A})}K_0(\mathfrak{E}_2)}{\{(\iota_0^{(1)}(x),-\iota_0^{(2)}(x)):x\in K_0(\mathfrak{B})\}}, [1_{\mathfrak{E}_1}]\oplus[1_{\mathfrak{E}_2}]\right) \arrow[r, "\pi_0^{(1)}"] & (K_0(\mathfrak{A}), [1_{\mathfrak{A}}]) \arrow[d, "\delta_0^{(1)}"] \\
K_1(\mathfrak{A}) \arrow[u, "\delta_1^{(1)}"] & \frac{K_1(\mathfrak{E}_1)\oplus_{K_1(\mathfrak{A})}K_1(\mathfrak{E}_2)}{\{(\iota_1^{(1)}(y),-\iota_1^{(2)}(y)):y\in K_1(\mathfrak{B})\}} \arrow[l, "\pi_1^{(1)}"'] & K_1(\mathfrak{B}). \arrow[l, "\iota_1^{(1)}"']
\end{tikzcd}$$

The same result also holds in the not necessarily unital case by removing all units from the statement.

Proof. For the not necessarily unital case, one simply ignores any mentioning of units in the argument below.

We fix $\mathcal{O}_2$-isometries $s_1, s_2 \in \mathscr{M}(\mathfrak{B})$, and identify $\mathfrak{e}_1 \oplus \mathfrak{e}_2$ with $\mathfrak{e}_1 \oplus_{s_1,s_2} \mathfrak{e}_2$, which we denote as $0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$. Construct the pull-back diagram

$$\begin{tikzcd}
 & \mathfrak{B} \arrow[r, equal] \arrow[d, tail] & \mathfrak{B} \arrow[d, tail, "\iota^{(1)}"] \\
\mathfrak{B} \arrow[r, tail] \arrow[d, equal] & \mathfrak{E}_0 \arrow[r, two heads] \arrow[d] & \mathfrak{E}_1 \arrow[d, two heads, "\pi^{(1)}"] \\
\mathfrak{B} \arrow[r, tail, "\iota^{(2)}"] & \mathfrak{E}_2 \arrow[r, two heads, "\pi^{(2)}"] & \mathfrak{A}.
\end{tikzcd} \tag{3.4}$$

Applying K-theory to this diagram, and using that $\delta_*^{(2)} = 0$, one gets the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & K_0(\mathfrak{B}) \arrow[r, equal] \arrow[d] & K_0(\mathfrak{B}) \arrow[d, "\iota_0^{(1)}"] & \\
K_1(\mathfrak{E}_1) \arrow[r, "0"] \arrow[d, "\pi_1^{(1)}"] & K_0(\mathfrak{B}) \arrow[r] \arrow[d, equal] & K_0(\mathfrak{E}_0) \arrow[r] \arrow[d] & K_0(\mathfrak{E}_1) \arrow[r, "0"] \arrow[d, "\pi_0^{(1)}"] & K_1(\mathfrak{B}) \arrow[d, equal] \\
K_1(\mathfrak{A}) \arrow[r, "0"] & K_0(\mathfrak{B}) \arrow[r, "\iota_0^{(2)}"] & K_0(\mathfrak{E}_2) \arrow[r, "\pi_0^{(2)}"] & K_0(\mathfrak{A}) \arrow[r, "0"] & K_1(\mathfrak{B}).
\end{tikzcd}$$

Hence, $K_0(\mathfrak{E}_0) \cong K_0(\mathfrak{E}_1) \oplus_{K_0(\mathfrak{A})} K_0(\mathfrak{E}_2)$ canonically, and this isomorphism takes $[1_{\mathfrak{E}_0}] \in K_0(\mathfrak{E}_0)$ to the element $[1_{\mathfrak{E}_1}] \oplus [1_{\mathfrak{E}_2}] \in K_0(\mathfrak{E}_1) \oplus_{K_0(\mathfrak{A})} K_0(\mathfrak{E}_2)$.

The pull-back diagram (3.4) induces a short exact sequence $\mathfrak{e}_0 : 0 \to \mathfrak{B} \oplus \mathfrak{B} \to \mathfrak{E}_0 \to \mathfrak{A} \to 0$, where $\mathfrak{B} \oplus 0$ is the "top $\mathfrak{B}$" and $0 \oplus \mathfrak{B}$ is the "left $\mathfrak{B}$" in (3.4). Let $\Phi \colon \mathfrak{B} \oplus \mathfrak{B} \to \mathfrak{B}$ be the Cuntz sum map $\Phi(b_1 \oplus b_2) = s_1 b_1 s_1^* + s_2 b_2 s_2^*$. We obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{B} \oplus \mathfrak{B} \arrow[r] \arrow[d, "\Phi"] & \mathfrak{E}_0 \arrow[r] \arrow[d] & \mathfrak{A} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{B} \arrow[r] & \mathfrak{E} \arrow[r] & \mathfrak{A} \arrow[r] & 0,
\end{tikzcd} \tag{3.5}$$

for which the $*$-homomorphism $\mathfrak{E}_0 \to \mathfrak{E}$ is unital. Applying K-theory to this diagram, and using the canonical identification $K_0(\mathfrak{E}_0) \cong K_0(\mathfrak{E}_1) \oplus_{K_0(\mathfrak{A})} K_0(\mathfrak{E}_2)$ as well as the fact that $\delta_*^{(2)} = 0$, one obtains the following commutative diagram with exact rows:

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$$\begin{tikzcd}
K_1(\mathfrak{A}) \arrow[r, "\delta_1^{(1)}\times 0"] \arrow[d, equal] & K_0(\mathfrak{B})\oplus K_0(\mathfrak{B}) \arrow[r, "(\iota_0^{(1)},\iota_0^{(2)})"] \arrow[d, "\mathrm{Sum}"] & K_0(\mathfrak{E}_1)\oplus_{K_0(\mathfrak{A})} K_0(\mathfrak{E}_2) \arrow[r, "\pi_0^{(1)}"] \arrow[d] & K_0(\mathfrak{A}) \arrow[r, "\delta_0^{(1)}\times 0"] \arrow[d, equal] & K_1(\mathfrak{B})^2 \arrow[d, "\mathrm{Sum}"] \\
K_1(\mathfrak{A}) \arrow[r, "\delta_1^{(1)}"] & K_0(\mathfrak{B}) \arrow[r] & K_0(\mathfrak{E}) \arrow[r] & K_0(\mathfrak{A}) \arrow[r, "\delta_0^{(1)}"] & K_1(\mathfrak{B}).
\end{tikzcd}$$

A diagram chase shows that $K_0(\mathfrak{E}_1)\oplus_{K_0(\mathfrak{A})} K_0(\mathfrak{E}_2)\to K_0(\mathfrak{E})$ is surjective, with kernel $\{(\iota_0^{(1)}(x), -\iota_0^{(2)}(x)) : x\in K_0(\mathfrak{B})\}$. As the map $\mathfrak{E}_0\to\mathfrak{E}$ was unital, $[1_{\mathfrak{E}_1}]\oplus[1_{\mathfrak{E}_2}]$ is mapped to $[1_{\mathfrak{E}}]$. Hence, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
K_1(\mathfrak{A}) \arrow[r, "\delta_1^{(1)}\times 0"] \arrow[d, equal] & \frac{K_0(\mathfrak{B})\oplus K_0(\mathfrak{B})}{\ker \mathrm{Sum}} \arrow[r, "(\iota_0^{(1)},\iota_0^{(2)})"] \arrow[d, "\mathrm{Sum}", "\cong"'] & \frac{K_0(\mathfrak{E}_1)\oplus_{K_0(\mathfrak{A})} K_0(\mathfrak{E}_2)}{\{(\iota_0^{(1)}(x),-\iota_0^{(2)}(x)):x\in K_0(\mathfrak{B})\}} \arrow[r, "\pi_0^{(1)}"] \arrow[d, "\cong"'] & K_0(\mathfrak{A}) \arrow[r, "\delta_0^{(1)}\times 0"] \arrow[d, equal] & \frac{K_1(\mathfrak{B})^2}{\ker \mathrm{Sum}} \arrow[d, "\mathrm{Sum}", "\cong"'] \\
K_1(\mathfrak{A}) \arrow[r, "\delta_1^{(1)}"] & K_0(\mathfrak{B}) \arrow[r] & K_0(\mathfrak{E}) \arrow[r] & K_0(\mathfrak{A}) \arrow[r, "\delta_0^{(1)}"] & K_1(\mathfrak{B}).
\end{tikzcd}$$

The element $[1_{\mathfrak{E}}]$ exactly corresponds to $[1_{\mathfrak{E}_1}]\oplus[1_{\mathfrak{E}_2}]$ via the above isomorphism. By identifying $K_0(\mathfrak{B})$ with $\frac{K_0(\mathfrak{B})\oplus K_0(\mathfrak{B})}{\ker \mathrm{Sum}}$ via the map $x\mapsto(x, 0)$, one obtains part of the desired congruence. Running the same argument as above where one interchange K_0 and K_1, one obtains the rest of the congruence. $\square$

4. A universal coefficient theorem. Recall that a separable C^*-algebra $\mathfrak{A}$ satisfies the UCT (in KK-theory) if and only if there is a short exact sequence

$$0\to \mathrm{Ext}(K_*(\mathfrak{A}), K_*(\mathfrak{B}))\to \mathrm{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}) \xrightarrow{\gamma_{\mathfrak{A},\mathfrak{B}}} \mathrm{Hom}(K_*(\mathfrak{A}), K_{1-*}(\mathfrak{B}))\to 0 \tag{4.1}$$

for every separable C^*-algebra $\mathfrak{B}$. Here we made the canonical identification $\mathrm{KK}^1(\mathfrak{A}, \mathfrak{B})\cong \mathrm{Ext}^{-1}(\mathfrak{A}, \mathfrak{B})$ (see Theorem 2.7). In this section, we prove UCTs for the unital Ext-groups $\mathrm{Ext}^{-1}_{\mathrm{us}}$ and $\mathrm{Ext}^{-1}_{\mathrm{uw}}$. Such UCTs were stated in [**32**] without a proof, and were proved in [**36**] under the assumption that $\mathfrak{B}$ has an approximate identity of projections.[8] We give a complete proof without this additional assumption and prove that the UCT's are natural in both variables. Naturality is crucial for our applications and was not established in [**36**].

DEFINITION 4.1. Given abelian groups K, H and an element $h\in H$, we can form the pointed Ext-group of (H, h) by K by considering pointed extensions

$$0\to K\to (G, g)\xrightarrow{\phi}(H, h)\to 0$$

for which $\phi(g)=h$. The set $\mathrm{Ext}((H, h), K)$ of congruence classes of such extensions is an abelian group as in the classical case with $\mathrm{Ext}(H, K)$ (see Remark 3.6).

REMARK 4.2. There is a homomorphism $K\to \mathrm{Ext}((H, h), K)$ given by

$$k\mapsto [K\rightarrowtail (K\oplus H, k\oplus h)\twoheadrightarrow (H, h)].$$

[8] While this is not stated explicitly in [**36**, Theorems 4.8 and 4.9], it can be deduced from the proof that $\mathfrak{B}$ is assumed to have an approximate identity of projections.

The kernel of this map is $\{\psi(h) : \psi \in \mathrm{Hom}(H, K)\}$. It easily follows that there is a short exact sequence

$$0 \to K/\{\psi(h) : \psi \in \mathrm{Hom}(H, K)\} \to \mathrm{Ext}((H, h), K) \to \mathrm{Ext}(H, K) \to 0.$$

NOTATION 4.3. For abelian groups H and K, and $h \in H$, we let $\mathrm{Hom}((H, h), K)$ denote the subgroup of $\mathrm{Hom}(H, K)$ consisting of homomorphisms δ for which $\delta(h) = 0$.

NOTATION 4.4. We write $\mathrm{Ext}((K_*(\mathfrak{A}), [1_{\mathfrak{A}}]), K_*(\mathfrak{B}))$ for the group

$$\mathrm{Ext}((K_0(\mathfrak{A}), [1_{\mathfrak{A}}]), K_0(\mathfrak{B})) \oplus \mathrm{Ext}(K_1(\mathfrak{A}), K_1(\mathfrak{B}))$$

and $\mathrm{Hom}((K_*(\mathfrak{A}), [1_{\mathfrak{A}}]), K_{*+1}(\mathfrak{B}))$ for the group

$$\mathrm{Hom}((K_0(\mathfrak{A}), [1_{\mathfrak{A}}]), K_1(\mathfrak{B})) \oplus \mathrm{Hom}(K_1(\mathfrak{A}), K_0(\mathfrak{B})).$$

REMARK 4.5. It is easily seen that there is a homomorphism

$$\tilde{\gamma}_{\mathfrak{A},\mathfrak{B}} \colon \mathrm{Ext}^{-1}_{\mathrm{us}}(\mathfrak{A}, \mathfrak{B}) \to \mathrm{Hom}((K_*(\mathfrak{A}), [1_{\mathfrak{A}}]), K_{*+1}(\mathfrak{B})),$$

given by mapping $[\mathfrak{e}]_{\mathrm{s}}$ to its boundary map in K-theory.

Similarly, there is a map

$$\widetilde{\kappa}_{\mathfrak{A},\mathfrak{B}} \colon \ker \widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}} \to \mathrm{Ext}((K_*(\mathfrak{A}), [1_{\mathfrak{A}}]), K_*(\mathfrak{B}))$$

given by mapping $[\mathfrak{e}]_{\mathrm{s}}$ to its induced six-term exact sequence in K-theory with position of the unit. This is well defined since the boundary maps vanish, but *a priori* it is not obviously a homomorphism (it is a homomorphism by Corollary 4.6).

The following is an immediate consequence of Proposition 3.7 and the definition of the sum in the pointed Ext-group.

COROLLARY 4.6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable C^*-algebras for which $\mathfrak{A}$ is unital. Then the map*

$$\widetilde{\kappa}_{\mathfrak{A},\mathfrak{B}} \colon \ker \widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}} \to \mathrm{Ext}((K_*(\mathfrak{A}), [1_{\mathfrak{A}}]), K_*(\mathfrak{B}))$$

defined in Remark 4.5 is a homomorphism.

We introduce the following nonstandard notation to ease what follows.

NOTATION 4.7. Let $\mathfrak{A}$ be a unital separable C^*-algebra and $\mathfrak{B}$ be a separable C^*-algebra. We define

$$\Gamma_{\mathfrak{A},\mathfrak{B}} := \{\psi([1_{\mathfrak{A}}]) : \psi \in \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{B}))\}.$$

REMARK 4.8. If $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras with $\mathfrak{A}$ unital, then

$$0 \to K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \to \mathrm{Ext}((K_*(\mathfrak{A}), [1_{\mathfrak{A}}]), K_*(\mathfrak{B})) \to \mathrm{Ext}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to 0$$

is a short exact sequence by Remark 4.2.

For a unital C^*-algebra $\mathfrak{D}$, we let $\mathcal{U}(\mathfrak{D})$ denote its unitary group, and let $\mathcal{U}_0(\mathfrak{D})$ denote the connected component of $1_{\mathfrak{D}}$ in $\mathcal{U}(\mathfrak{D})$. Recall that a unital C^*-algebra $\mathfrak{D}$ is K_1-*surjective* (resp. K_1-*injective*) if the canonical homomorphism $\mathcal{U}(\mathfrak{D})/\mathcal{U}_0(\mathfrak{D}) \to K_1(\mathfrak{D})$ is surjective (resp. injective), and K_1-*bijective* if it is both K_1-surjective and K_1-injective.

While the following result is well-known to experts, we know of no reference and thus include a proof.

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PROPOSITION 4.9. *If $\mathfrak{B}$ is a stable C^*-algebra then the corona algebra $\mathcal{Q}(\mathfrak{B})$ is K_1-bijective.*

Proof. Stability of $\mathfrak{B}$ implies that $\mathcal{Q}(\mathfrak{B})$ is properly infinite and thus K_1-surjective by [6]. For K_1-injectivity, let $u \in \mathcal{U}(\mathcal{Q}(\mathfrak{B}))$ be such that $[u] = 0$ in $K_1(\mathcal{Q}(\mathfrak{B}))$. By [23, Corollary 2.5], the connected stable rank of $\mathfrak{B}$ is at most 2. Consequently, the general stable rank[9] of $\mathfrak{B}$ is at most 2. By [22, Theorem 2] (which relies on results in [27]), it follows that u lifts to $\widetilde{u} \in \mathcal{U}(\mathcal{M}(\mathfrak{B}))$. By [7], one has $\mathcal{U}(\mathcal{M}(\mathfrak{B})) = \mathcal{U}_0(\mathcal{M}(\mathfrak{B}))$, and thus $u \in \mathcal{U}_0(\mathcal{Q}(\mathfrak{B}))$. Hence, $\mathcal{Q}(\mathfrak{B})$ is K_1-injective. □

REMARK 4.10. Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable C^*-algebras for which $\mathfrak{A}$ is unital and $\mathfrak{B}$ is stable. For every $x \in K_0(\mathfrak{B}) \cong K_1(\mathcal{Q}(\mathfrak{B}))$, there is an induced semisplit, unital extension $\mathfrak{e}_x$ of $\mathfrak{A}$ by $\mathfrak{B}$ (uniquely determined up to strong unitary equivalence) given as follows: Let $\tau_0 : \mathfrak{A} \to \mathcal{Q}(\mathfrak{B})$ be the Busby map of a trivial, absorbing unital extension [34], and let $u \in \mathcal{U}(\mathcal{Q}(\mathfrak{B}))$ be a unitary being mapped to x under the natural isomorphism $K_1(\mathcal{Q}(\mathfrak{B})) \xrightarrow{\cong} K_0(\mathfrak{B})$. Then, $\mathfrak{e}_x$ is the extension with Busby map $\operatorname{Ad} u \circ \tau_0$.

As τ_0 is uniquely determined up to strong unitary equivalence, and since $K_1(\mathcal{Q}(\mathfrak{B})) = \mathcal{U}(\mathcal{Q}(\mathfrak{B}))/\mathcal{U}_0(\mathcal{Q}(\mathfrak{B}))$ by Proposition 4.9, it easily follows that $\mathfrak{e}_x$ is unique up to strong unitary equivalence.

The following elementary lemma will be used frequently.

LEMMA 4.11. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable C^*-algebras for which $\mathfrak{A}$ is unital and $\mathfrak{B}$ is stable. Let $\mathfrak{e}$ be a unital extension of $\mathfrak{A}$ by $\mathfrak{B}$, and let $u \in \mathcal{U}(\mathcal{Q}(\mathfrak{B}))$. Then*

$$[\operatorname{Ad} u \cdot \mathfrak{e}]_s = [\mathfrak{e}]_s + [\mathfrak{e}_{[u]_1}]_s \in \operatorname{Ext}_{us}(A, B).$$

In particular, the map

$$K_0(\mathfrak{B}) \to \operatorname{Ext}_{us}^{-1}(\mathfrak{A}, \mathfrak{B}), \qquad x \mapsto [\mathfrak{e}_x]_s$$

is a group homomorphism.

Proof. Let $s_1, s_2 \in \mathcal{M}(\mathfrak{B})$ be $\mathcal{O}_2$-isometries, and let $\oplus$ denote the Cuntz sum induced by this choice of isometries. Then, we have

$$\operatorname{Ad}(u \oplus u^*) \circ (\tau_{\mathfrak{e}} \oplus \tau_{\mathfrak{e}_{[u]_1}}) = \operatorname{Ad}(u \oplus u^*) \circ (\tau_{\mathfrak{e}} \oplus (\operatorname{Ad} u \circ \tau_0)) = (\operatorname{Ad} u \circ \tau_{\mathfrak{e}}) \oplus \tau_0 \qquad (4.2)$$

where τ_0 is an absorbing, trivial unital extension. As $u \oplus u^*$ lifts to a unitary in $\mathcal{M}(\mathfrak{B})$, the result follows. □

The following is an immediate consequence of Lemma 3.5 applied to the case where $\mathfrak{e}$ is a trivial unital extension.

COROLLARY 4.12. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable C^*-algebras for which $\mathfrak{A}$ is unital and $\mathfrak{B}$ is stable, and let $x \in K_0(\mathfrak{B})$. Then $\mathfrak{e}_x$ induces the element*

$$[0 \to K_0(\mathfrak{B}) \to (K_0(\mathfrak{B}) \oplus K_0(\mathfrak{A}), x \oplus [1_{\mathfrak{A}}]) \to (K_0(\mathfrak{A}), [1_{\mathfrak{A}}]) \to 0]$$

in $\operatorname{Ext}((K_0(\mathfrak{A}), [1_{\mathfrak{A}}]), K_0(\mathfrak{B}))$.

Recall that $\gamma_{\mathfrak{A},\mathfrak{B}} : \operatorname{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}) \to \operatorname{Hom}(K_*(\mathfrak{A}), K_{1-*}(\mathfrak{B}))$ denotes the canonical homomorphism.

[9] Not to be confused with the topological stable rank, which in modern terms is usually just referred to as stable rank.

LEMMA 4.13. *Let $\mathfrak{A}$ be a separable, unital C^*-algebra satisfying the UCT, and let $\mathfrak{B}$ be a separable, stable C^*-algebra. Then there is an exact sequence*

$$0 \to K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \to \mathrm{Ext}_{\mathrm{us}}^{-1}(\mathfrak{A}, \mathfrak{B}) \to \mathrm{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}).$$

Moreover, the map $\mathrm{Ext}_{\mathrm{uw}}^{-1}(\mathfrak{A}, \mathfrak{B}) \to \mathrm{Ext}^{-1}(\mathfrak{A}, \mathfrak{B})$ *is an isomorphism onto*

$$\gamma_{\mathfrak{A},\mathfrak{B}}^{-1}(\mathit{Hom}((K_*(\mathfrak{A}), [1_{\mathfrak{A}}]), K_{*+1}(\mathfrak{B}))) \quad (\subseteq \mathrm{Ext}^{-1}(\mathfrak{A}, \mathfrak{B})).$$

Proof. By a result of Skandalis [33, Remarque 2.8] (see also [32] or [21] for a proof), there is an exact sequence of the form

$$\begin{tikzcd} K_0(\mathfrak{B}) \arrow[r] & \mathrm{Ext}_{\mathrm{us}}^{-1}(\mathfrak{A}, \mathfrak{B}) \arrow[r] & \mathrm{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}) \arrow[d, "\iota_1^*"] \\ \mathrm{KK}(\mathfrak{A}, \mathfrak{B}) \arrow[u, "\iota_0^*"] & & K_1(\mathfrak{B}) \end{tikzcd}$$

where ι_i^* is induced from the unital $*$-homomorphism $\iota\colon \mathbb{C} \to \mathfrak{A}$. It is easily seen that $\iota_0^*\colon \mathrm{KK}(\mathfrak{A}, \mathfrak{B}) \to K_0(\mathfrak{B})$ factors as

$$\mathrm{KK}(\mathfrak{A}, \mathfrak{B}) \xrightarrow{\gamma_0} \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{B})) \xrightarrow{\mathrm{ev}_{[1_{\mathfrak{A}}]}} K_0(\mathfrak{B})$$

where $\mathrm{ev}_{[1_{\mathfrak{A}}]}$ is evaluation at $[1_{\mathfrak{A}}]$. Similarly, $\iota_1^*\colon\ \mathrm{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}) \to K_1(\mathfrak{B})$ factors as

$$\mathrm{Ext}^{-1}(\mathfrak{A}, \mathfrak{B}) \xrightarrow{\gamma_0} \mathrm{Hom}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \xrightarrow{\mathrm{ev}_{[1_{\mathfrak{A}}]}} K_1(\mathfrak{B}).$$

Since $\mathfrak{A}$ satisfies the UCT, γ_0 is surjective and thus $\mathrm{im}(\iota_0^*) = \Gamma_{\mathfrak{A},\mathfrak{B}}$. Hence, the exact sequence collapses to an exact sequence

$$0 \to K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \to \mathrm{Ext}_{\mathrm{us}}^{-1}(\mathfrak{A}, \mathfrak{B}) \to \mathrm{Ext}^{-1}(\mathfrak{A}, \mathfrak{B})$$

where the image of $\mathrm{Ext}_{\mathrm{us}}^{-1}(\mathfrak{A}, \mathfrak{B}) \to \mathrm{Ext}^{-1}(\mathfrak{A}, \mathfrak{B})$ is $\ker \iota_1^*$. By using the above exact sequence, it easily follows that $\ker \iota_1^* = \gamma_{\mathfrak{A},\mathfrak{B}}^{-1}(\mathrm{Hom}((K_*(\mathfrak{A}), [1_{\mathfrak{A}}]), K_{*+1}(\mathfrak{B})))$, so we obtain a short exact sequence

$$0 \to K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \to \mathrm{Ext}_{\mathrm{us}}^{-1}(\mathfrak{A}, \mathfrak{B}) \to \gamma_{\mathfrak{A},\mathfrak{B}}^{-1}(\mathrm{Hom}((K_*(\mathfrak{A}), [1_{\mathfrak{A}}]), K_{*+1}(\mathfrak{B}))) \to 0.$$

Using Lemma 4.11 it follows that the quotient $\mathrm{Ext}_{\mathrm{us}}^{-1}(\mathfrak{A}, \mathfrak{B})/(K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}})$ is canonically isomorphic to $\mathrm{Ext}_{\mathrm{uw}}^{-1}(\mathfrak{A}, \mathfrak{B})$. Combined with the above short exact sequence, it follows that $\mathrm{Ext}_{\mathrm{uw}}^{-1}(\mathfrak{A}, \mathfrak{B}) \to \mathrm{Ext}^{-1}(\mathfrak{A}, \mathfrak{B})$ is injective and its image is

$$\gamma_{\mathfrak{A},\mathfrak{B}}^{-1}(\mathrm{Hom}((K_*(\mathfrak{A}), [1_{\mathfrak{A}}]), K_{*+1}(\mathfrak{B})))$$

as desired. □

We can now assemble the pieces provided by the previous results in this section and obtain the following UCT. This is a minor improvement on the UCT sequences proved by Wei [36, Theorems 4.8 and 4.9], in which the C^*-algebra $\mathfrak{B}$ was required to have an approximate identity of projections. Also, Wei does not prove that the UCTs for the unital Ext-groups are natural, which will be important in our applications.

THEOREM 4.14. *Let $\mathfrak{A}$ be a unital, separable C^*-algebra satisfying the UCT, and let $\mathfrak{B}$ be a separable C^*-algebra. There is a commutative diagram*

$$\begin{tikzcd}
K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \arrow[r, equal] \arrow[d, tail] & K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \arrow[d, tail] & \\
\mathrm{Ext}((K_*(\mathfrak{A}),[1_{\mathfrak{A}}]),K_*(\mathfrak{B})) \arrow[r, tail] \arrow[d, two heads] & \mathrm{Ext}^{-1}_{\mathrm{us}}(\mathfrak{A},\mathfrak{B}) \arrow[r, two heads, "\widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}}"] \arrow[d, two heads] & Hom((K_*(\mathfrak{A}),[1_{\mathfrak{A}}]),K_{*+1}(\mathfrak{B})) \arrow[d, equal] \\
\mathrm{Ext}(K_*(\mathfrak{A}),K_*(\mathfrak{B})) \arrow[r, tail] & \mathrm{Ext}^{-1}_{\mathrm{uw}}(\mathfrak{A},\mathfrak{B}) \arrow[r, two heads, "\gamma_{\mathfrak{A},\mathfrak{B}}"] & Hom((K_*(\mathfrak{A}),[1_{\mathfrak{A}}]),K_{*+1}(\mathfrak{B})).
\end{tikzcd} \tag{4.3}$$

for which all rows and columns are short exact sequences. This diagram is natural with respect to unital $*$-homomorphisms in the first variable, and with respect to nondegenerate $*$-homomorphisms in the second variable.

Proof. By replacing $\mathfrak{B}$ with $\mathfrak{B}\otimes\mathbb{K}$, we may assume that $\mathfrak{B}$ is stable.

By Lemma 4.13 and the UCT for Ext^{-1} (see (4.1)), we obtain a short exact sequence

$$0\to \mathrm{Ext}(K_*(\mathfrak{A}),K_*(\mathfrak{B}))\to \mathrm{Ext}^{-1}_{\mathrm{uw}}(\mathfrak{A},\mathfrak{B}) \xrightarrow{\gamma_{\mathfrak{A},\mathfrak{B}}} \mathrm{Hom}((K_*(\mathfrak{A}),[1_{\mathfrak{A}}]),K_{*+1}(\mathfrak{B}))\to 0. \tag{4.4}$$

The map $\mathrm{Ext}(K_*(\mathfrak{A}),K_*(\mathfrak{B}))\to \ker\gamma_{\mathfrak{A},\mathfrak{B}}$ above, which is an isomorphism by exactness, is exactly the inverse of the isomorphism

$$\kappa_{\mathfrak{A},\mathfrak{B}}\colon\ \ker\gamma_{\mathfrak{A},\mathfrak{B}} \xrightarrow{\cong} \mathrm{Ext}(K_*(\mathfrak{A}),K_*(\mathfrak{B}))$$

given by applying K-theory to a given extension (which induce short exact sequences by vanishing of the boundary maps). That $\kappa_{\mathfrak{A},\mathfrak{B}}$ is an isomorphism follows from the UCT. The homomorphism

$$\widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}}\colon\ \mathrm{Ext}^{-1}_{\mathrm{us}}(\mathfrak{A},\mathfrak{B})\to \mathrm{Hom}((K_*(\mathfrak{A}),[1_{\mathfrak{A}}]),K_{*+1}(\mathfrak{B}))$$

is the composition of the surjective homomorphisms $\mathrm{Ext}^{-1}_{\mathrm{us}}\to\mathrm{Ext}^{-1}_{\mathrm{uw}}$ and $\gamma_{\mathfrak{A},\mathfrak{B}}$ from (4.4), so $\widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}}$ is surjective. Hence, we obtain the following commutative diagram:

$$\begin{tikzcd}
K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \arrow[r, equal] \arrow[d, tail] & K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \arrow[d, tail] & \\
\ker\widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}} \arrow[r, tail] \arrow[d, two heads] & \mathrm{Ext}^{-1}_{\mathrm{us}}(\mathfrak{A},\mathfrak{B}) \arrow[r, two heads, "\widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}}"] \arrow[d, two heads] & \mathrm{Hom}((K_*(\mathfrak{A}),[1_{\mathfrak{A}}]),K_{1-*}(\mathfrak{B})) \arrow[d, equal] \\
\ker\gamma_{\mathfrak{A},\mathfrak{B}} \arrow[r, tail] & \mathrm{Ext}^{-1}_{\mathrm{uw}}(\mathfrak{A},\mathfrak{B}) \arrow[r, two heads, "\gamma_{\mathfrak{A},\mathfrak{B}}"] & \mathrm{Hom}((K_*(\mathfrak{A}),[1_{\mathfrak{A}}]),K_{1-*}(\mathfrak{B}))
\end{tikzcd} \tag{4.5}$$

for which the rows and columns are short exact sequences. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \arrow[r] \arrow[d, equal] & \ker\widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}} \arrow[r] \arrow[d, "\widetilde{\kappa}_{\mathfrak{A},\mathfrak{B}}", "(\cong)"'] & \ker\gamma_{\mathfrak{A},\mathfrak{B}} \arrow[r] \arrow[d, "\kappa_{\mathfrak{A},\mathfrak{B}}", "\cong"'] & 0 \\
0 \arrow[r] & K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \arrow[r] & \mathrm{Ext}((K_*(\mathfrak{A}),[1_{\mathfrak{A}}]),K_*(\mathfrak{B})) \arrow[r] & \mathrm{Ext}(K_*(\mathfrak{A}),K_*(\mathfrak{B})) \arrow[r] & 0
\end{tikzcd} \tag{4.6}$$

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which has exact rows. The map $\widetilde{\kappa}_{\mathfrak{A},\mathfrak{B}}$ is a homomorphism by Corollary 4.6, and clearly the right square above commutes. The left square above commutes by Remark 4.2 and Corollary 4.12. Hence, $\widetilde{\kappa}_{\mathfrak{A},\mathfrak{B}}$ is an isomorphism by the five lemma. By gluing together the diagrams (4.5) and (4.6) in the obvious way, we obtain the desired diagram (4.3).

It remains to be shown that diagram (4.3) is natural in both variables. For verifying this let $\mathfrak{C}$ be separable, unital C^*-algebra satisfying the UCT, let $\phi\colon \mathfrak{C}\to\mathfrak{A}$ be a unital $*$-homomorphism, let $\mathfrak{D}$ be a separable, stable C^*-algebra, and let $\psi\colon \mathfrak{B}\to\mathfrak{D}$ be a nondegenerate $*$-homomorphism. We first check that diagram (4.5) is natural, and then (4.6).

It is well-known that $\mathrm{Ext}^{-1}_{\mathrm{us}}(\mathfrak{A},\mathfrak{B})\to\mathrm{Ext}^{-1}_{\mathrm{uw}}(\mathfrak{A},\mathfrak{B})$ is natural, and by naturality of six-term exact sequences the maps $\widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}}$ and $\gamma_{\mathfrak{A},\mathfrak{B}}$ are natural.

Again by naturality of six-term exact sequences, it follows that

$$\phi^*(\ker\widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}})\subseteq\ker\widetilde{\gamma}_{\mathfrak{C},\mathfrak{B}},\qquad\text{and}\qquad\psi_*(\ker\widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}})\subseteq\ker\widetilde{\gamma}_{\mathfrak{A},\mathfrak{D}}.$$

Hence, the inclusion $\ker\widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}}\hookrightarrow\mathrm{Ext}^{-1}_{\mathrm{us}}(\mathfrak{A},\mathfrak{B})$ is natural in both variables. Similarly, the inclusion $\ker\gamma_{\mathfrak{A},\mathfrak{B}}\hookrightarrow\mathrm{Ext}^{-1}_{\mathrm{uw}}(\mathfrak{A},\mathfrak{B})$ and the map $\ker\widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}}\to\ker\gamma_{\mathfrak{A},\mathfrak{B}}$ are natural in both variables. This implies that diagram (4.5) is natural. Hence, it remains to check that the diagram (4.6) is natural.

It is straightforward to verify that the maps in the lower row of (4.6) are natural (this is purely algebraic, and of course uses that $\phi_0([1_{\mathfrak{C}}])=[1_{\mathfrak{A}}]$). We saw above that $\ker\widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}}\to\ker\gamma_{\mathfrak{A},\mathfrak{B}}$ is natural.

We will show that $\widetilde{\kappa}_{\mathfrak{A},\mathfrak{B}}$ is natural in the first variable. Let $\mathfrak{e}:0\to\mathfrak{B}\to\mathfrak{E}\to\mathfrak{A}\to 0$ be a unital extension inducing an element in $\ker\widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}}$, i.e. $\mathfrak{e}$ has vanishing boundary maps in K-theory. Construct the pull-back diagram

$$\begin{tikzcd}
\mathfrak{e}\cdot\phi: & 0 \arrow[r] & \mathfrak{B} \arrow[r] \arrow[d, equal] & \mathfrak{E}_\phi \arrow[r] \arrow[d] & \mathfrak{C} \arrow[r] \arrow[d, "\phi"] & 0 \\
\mathfrak{e}: & 0 \arrow[r] & \mathfrak{B} \arrow[r] & \mathfrak{E} \arrow[r] & \mathfrak{A} \arrow[r] & 0.
\end{tikzcd}\tag{4.7}$$

As ϕ is a unital map, $\mathfrak{E}_\phi$ is unital and the map $\mathfrak{E}_\phi\to\mathfrak{E}$ is unital. As $\phi^*([\mathfrak{e}]_{\mathrm{s}})=[\mathfrak{e}\cdot\phi]_{\mathrm{s}}$, we should check that

$$\widetilde{\kappa}_{\mathfrak{C},\mathfrak{B}}([\mathfrak{e}\cdot\phi]_{\mathrm{s}})=(\phi_*)^*(\widetilde{\kappa}_{\mathfrak{A},\mathfrak{B}}([\mathfrak{e}]_{\mathrm{s}})).\tag{4.8}$$

Applying K-theory to the pull-back diagram (4.7), and using that both $\mathfrak{e}$ and $\mathfrak{e}\cdot\phi$ have vanishing boundary maps, we obtain the diagram

$$\begin{tikzcd}
\widetilde{\kappa}_{\mathfrak{C},\mathfrak{B}}([\mathfrak{e}\cdot\phi]): & 0 \arrow[r] & K_*(\mathfrak{B}) \arrow[r] \arrow[d, equal] & (K_*(\mathfrak{E}_\phi),[1_{\mathfrak{E}_\phi}]) \arrow[r] \arrow[d] & (K_*(\mathfrak{C}),[1_{\mathfrak{C}}]) \arrow[r] \arrow[d, "\phi_*"] & 0 \\
\widetilde{\kappa}_{\mathfrak{A},\mathfrak{B}}([\mathfrak{e}]): & 0 \arrow[r] & K_*(\mathfrak{B}) \arrow[r] & (K_*(\mathfrak{E}),[1_{\mathfrak{E}}]) \arrow[r] & (K_*(\mathfrak{A}),[1_{\mathfrak{A}}]) \arrow[r] & 0.
\end{tikzcd}$$

Since this is a pull-back diagram, it follows that (4.8) holds. Hence, $\widetilde{\kappa}_{\mathfrak{A},\mathfrak{B}}$ is natural in the first variable. That $\widetilde{\kappa}_{\mathfrak{A},\mathfrak{B}}$ is natural in the second variable, and that $\kappa_{\mathfrak{A},\mathfrak{B}}$ is natural in both variables, is checked in a similar fashion.

It remains to check that $K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}}\to\ker\widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}}$ is natural in both variables. For this purpose, fix a unitary in $u\in\mathcal{Q}(\mathfrak{B})$ inducing an arbitrary element in $K_0(\mathfrak{B})$. Let $\mathfrak{e}_{\mathfrak{A},\mathfrak{B}}$ and

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$\mathfrak{e}_{\mathfrak{C},\mathfrak{B}}$ be absorbing, unital extensions of $\mathfrak{A}$ by $\mathfrak{B}$ and of $\mathfrak{C}$ by $\mathfrak{B}$, respectively. By definition, we have

$$[u]+\Gamma_{\mathfrak{A},\mathfrak{B}} \mapsto [\operatorname{Ad} u \cdot \mathfrak{e}_{\mathfrak{A},\mathfrak{B}}]_{\mathrm{s}} \in \ker \widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}}, \qquad [u]+\Gamma_{\mathfrak{C},\mathfrak{B}} \mapsto [\operatorname{Ad} u \cdot \mathfrak{e}_{\mathfrak{C},\mathfrak{B}}]_{\mathrm{s}} \in \ker \widetilde{\gamma}_{\mathfrak{C},\mathfrak{B}}.$$

In order to check that $K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \to \ker \widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}}$ is natural in the first variable, we should therefore verify that

$$\phi^*([\operatorname{Ad} u \cdot \mathfrak{e}_{\mathfrak{A},\mathfrak{B}}]_{\mathrm{s}}) = [\operatorname{Ad} u \cdot \mathfrak{e}_{\mathfrak{C},\mathfrak{B}}]_{\mathrm{s}}.$$

This follows easily from Lemma 4.11 since

$$\phi^*([\operatorname{Ad} u \cdot \mathfrak{e}_{\mathfrak{A},\mathfrak{B}}]_{\mathrm{s}}) = [\operatorname{Ad} u \cdot (\mathfrak{e}_{\mathfrak{A},\mathfrak{B}} \cdot \phi)]_{\mathrm{s}} = [\mathfrak{e}_{\mathfrak{A},\mathfrak{B}} \cdot \phi]_{\mathrm{s}} + [\operatorname{Ad} u \cdot \mathfrak{e}_{\mathfrak{C},\mathfrak{B}}]_{\mathrm{s}} = [\operatorname{Ad} u \cdot \mathfrak{e}_{\mathfrak{C},\mathfrak{B}}]_{\mathrm{s}},$$

where we used that $\mathfrak{e}_{\mathfrak{A},\mathfrak{B}} \cdot \phi$ is trivial so that $[\mathfrak{e}_{\mathfrak{A},\mathfrak{B}} \cdot \phi]_{\mathrm{s}} = 0$. Hence, $K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \to \ker \widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}}$ is natural in the first variable. For the second variable, let $\overline{\psi}\colon \mathscr{Q}(\mathfrak{B}) \to \mathscr{Q}(\mathfrak{D})$ be the induced $*$-homomorphism, and let $\mathfrak{e}_{\mathfrak{A},\mathfrak{D}}$ be an absorbing, unital extension of $\mathfrak{A}$ by $\mathfrak{D}$. Note that $\psi_*([u]) = [\overline{\psi}(u)]$. As above, we get

$$\begin{aligned}
\psi_*([\operatorname{Ad} u \cdot \mathfrak{e}_{\mathfrak{A},\mathfrak{B}}]_{\mathrm{s}}) &= [(\overline{\psi} \circ \operatorname{Ad} u) \cdot \mathfrak{e}_{\mathfrak{A},\mathfrak{B}}]_{\mathrm{s}} \\
&= [\operatorname{Ad} \overline{\psi}(u) \cdot (\overline{\psi} \cdot \mathfrak{e}_{\mathfrak{A},\mathfrak{B}})]_{\mathrm{s}} \\
&\overset{\text{Lem. 4.11}}{=} [\overline{\psi} \cdot \mathfrak{e}_{\mathfrak{A},\mathfrak{B}}]_{\mathrm{s}} + [\operatorname{Ad} \overline{\psi}(u) \cdot \mathfrak{e}_{\mathfrak{A},\mathfrak{D}}]_{\mathrm{s}} \\
&= [\operatorname{Ad} \overline{\psi}(u) \cdot \mathfrak{e}_{\mathfrak{A},\mathfrak{D}}]_{\mathrm{s}}.
\end{aligned}$$

As $[\operatorname{Ad} \overline{\psi}(u) \cdot \mathfrak{e}_{\mathfrak{A},\mathfrak{D}}]_{\mathrm{s}}$ is the image of $\psi_*([u]) + \Gamma_{\mathfrak{A},\mathfrak{D}}$ via the map $K_0(\mathfrak{B})/\Gamma_{\mathfrak{A},\mathfrak{B}} \to \ker \widetilde{\gamma}_{\mathfrak{A},\mathfrak{B}}$, it follows that this map is natural in the second variable, thus finishing the proof. □

5. Classification of unital extensions. In this section, we will apply our UCT to obtain classification results for certain unital extensions of C^*-algebras via their six-term exact sequence in K-theory.

The main idea is the following: suppose $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are absorbing, semisplit unital extensions of $\mathfrak{A}$ by $\mathfrak{B}$, and suppose that $[\mathfrak{e}_1]_{\mathrm{w}} = [\mathfrak{e}_2]_{\mathrm{w}} \in \operatorname{Ext}^{-1}_{\mathrm{uw}}(\mathfrak{A}, \mathfrak{B})$. By Theorem 4.14, there is an element $x \in K_0(\mathfrak{B})$ such that $[\mathfrak{e}_1]_{\mathrm{s}} = [\mathfrak{e}_2 \oplus \mathfrak{e}_x]_{\mathrm{s}}$, and in particular $\mathfrak{e}_1 \cong \mathfrak{e}_2 \oplus \mathfrak{e}_x$ by absorption. So the goal will be to prove, under certain conditions, that $\mathfrak{e}_2 \oplus \mathfrak{e}_x \cong \mathfrak{e}_2$.

As a technical devise, we introduce the following notation.

NOTATION 5.1. If $\delta_* \in \operatorname{Hom}((K_*(\mathfrak{A}), [1_{\mathfrak{A}}]), K_{*+1}(\mathfrak{B}))$, then we define

$$\Gamma^{\delta_*}_{\mathfrak{A},\mathfrak{B}} := q_{\delta_1}^{-1}(\{\phi([1_{\mathfrak{A}}]_0) : \phi \in \operatorname{Hom}(\ker \delta_0, \operatorname{coker}\delta_1)\})$$

where $q_{\delta_1}\colon K_0(\mathfrak{B}) \to \operatorname{coker}\delta_1$ is the canonical epimorphism.

Note that we always have $\Gamma_{\mathfrak{A},\mathfrak{B}} = \Gamma^0_{\mathfrak{A},\mathfrak{B}} \subseteq \Gamma^{\delta_*}_{\mathfrak{A},\mathfrak{B}}$ (see Notation 4.7). The following is essentially [**36**, Theorem 3.5], but without assuming that $\mathfrak{B}$ has an approximate identity of projections.

LEMMA 5.2. *Let* $\mathfrak{e}\colon 0 \to \mathfrak{B} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \mathfrak{A} \to 0$ *be a unital extension of separable C^*-algebras with $\mathfrak{B}$ stable, let $\delta_*\colon K_*(\mathfrak{A}) \to K_{1-*}(\mathfrak{B})$ denote the induced boundary map in K-theory, and let $x \in K_0(\mathfrak{B})$. Then $K^{\mathrm{u}}_{\mathrm{six}}(\mathfrak{e}) \equiv K^{\mathrm{u}}_{\mathrm{six}}(\mathfrak{e} \oplus \mathfrak{e}_x)$ if and only if $x \in \Gamma^{\delta_*}_{\mathfrak{A},\mathfrak{B}}$.*

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Proof. By Lemmas 4.11, 3.2, 3.3 and 3.5, $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e} \oplus \mathfrak{e}_x)$ is congruent to

$$\begin{tikzcd} K_0(\mathfrak{B}) \arrow[r, "\iota_0"] & (K_0(\mathfrak{E}), [1_{\mathfrak{E}}] + \iota_0(x)) \arrow[r, "\pi_0"] & (K_0(\mathfrak{A}), [1_{\mathfrak{A}}]) \arrow[d, "\delta_0"] \\ K_1(\mathfrak{A}) \arrow[u, "\delta_1"] & K_1(\mathfrak{E}) \arrow[l, "\pi_1"'] & K_1(\mathfrak{B}). \arrow[l, "\iota_1"'] \end{tikzcd} \tag{5.1}$$

If $x \in \Gamma^{\delta_*}_{\mathfrak{A},\mathfrak{B}}$, then there is a homomorphism $\phi\colon \ker\delta_0 \to \operatorname{coker}\delta_1$ such that $q_{\delta_1}(x) = \phi([1_{\mathfrak{A}}])$. Define $\eta_0 = \mathrm{id}_{K_0(\mathfrak{E})} + \bar{\iota}_0 \circ \phi \circ \pi_0 \colon K_0(\mathfrak{E}) \to K_0(\mathfrak{E})$, where $\bar{\iota}_0\colon \operatorname{coker}\delta_1 \to K_0(\mathfrak{E})$ is the injective homomorphism induced by ι_0. Letting $\eta_1 = \mathrm{id}_{K_1(\mathfrak{E})}$ it easily follows that $\eta_*\colon K_*(\mathfrak{E}) \to K_*(\mathfrak{E})$ induces a congruence between $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e})$ and the sequence (5.1).

Now suppose that $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e})$ is congruent to $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e} \oplus \mathfrak{e}_x)$ which in turn is congruent to the sequence (5.1). There is a homomorphism $\eta_*\colon K_*(\mathfrak{E}) \to K_*(\mathfrak{E})$ such that $\eta_0([1_{\mathfrak{E}}]) = [1_{\mathfrak{E}}] + \iota_0(x)$ and the following diagram with exact rows:

$$\begin{tikzcd} K_1(\mathfrak{A}) \arrow[r, "\delta_1"] \arrow[d, equal] & K_0(\mathfrak{B}) \arrow[r, "\iota_0"] \arrow[d, equal] & K_0(\mathfrak{E}) \arrow[r, "\pi_0"] \arrow[d, "\eta_0"] & K_0(\mathfrak{A}) \arrow[r, "\delta_0"] \arrow[d, equal] & K_1(\mathfrak{B}) \arrow[d, equal] \\ K_1(\mathfrak{A}) \arrow[r, "\delta_1"] & K_0(\mathfrak{B}) \arrow[r, "\iota_0"] & K_0(\mathfrak{E}) \arrow[r, "\pi_0"] & K_0(\mathfrak{A}) \arrow[r, "\delta_0"] & K_1(\mathfrak{B}) \end{tikzcd}$$

commutes. By a standard diagram chase, there is a homomorphism $\phi \in \operatorname{Hom}(\ker\delta_0, \operatorname{coker}\delta_1)$ such that $\eta_0 = \mathrm{id}_{K_0(\mathfrak{E})} + \bar{\iota}_0 \circ \phi \circ \pi_0$, where $\bar{\iota}_0\colon \operatorname{coker}\delta_1 \to K_0(\mathfrak{E})$ is the map induced by ι_0. Hence,

$$[1_{\mathfrak{E}}] + \iota_0(x) = \eta_0([1_{\mathfrak{E}}]) = [1_{\mathfrak{E}}] + \bar{\iota}_0 \circ \phi([1_{\mathfrak{A}}]).$$

Letting $q_{\delta_1}\colon K_0(\mathfrak{B}) \to \operatorname{coker}\delta_1$ denote the quotient map, we get $\bar{\iota}_0(q_{\delta_1}(x)) = \iota_0(x) = \bar{\iota}_0 \circ \phi([1_{\mathfrak{A}}])$ which implies $q_{\delta_1}(x) = \phi([1_{\mathfrak{A}}])$ since $\bar{\iota}_0$ is injective. Thus, $x \in \Gamma^{\delta_*}_{\mathfrak{A},\mathfrak{B}}$. □

PROPOSITION 5.3. *Let $\mathfrak{A}$ be a separable C^*-algebra satisfying the UCT, and let $\alpha \in Aut(\mathfrak{A})$ be an isomorphism such that $K_*(\alpha) = K_*(id_{\mathfrak{A}})$. Then the induced Pimsner–Voiculescu sequence collapses to a short exact sequence*

$$0 \to K_{1-*}(\mathfrak{A}) \to K_{1-*}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \to K_*(\mathfrak{A}) \to 0, \tag{5.2}$$

and the induced element in $\operatorname{Ext}(K_*(\mathfrak{A}), K_{1-*}(\mathfrak{A}))$ *is mapped to*

$$KK(\alpha) - KK(id_{\mathfrak{A}}) \in KK(\mathfrak{A}, \mathfrak{A})$$

via the map $\operatorname{Ext}(K_*(\mathfrak{A}), K_{1-*}(\mathfrak{A})) \to KK(\mathfrak{A}, \mathfrak{A})$ *from the UCT.*

Proof. That the Pimsner–Voiculescu sequence collapses to a short exact sequence is obvious.

Let $\mathfrak{M} := \{f \in C([0,1], \mathfrak{A}) : \alpha(f(0)) = f(1)\}$ be the mapping torus of α and $\mathrm{id}_{\mathfrak{A}}$. It is well-known that the extension

$$0 \to C_0((0,1), \mathfrak{A}) \to \mathfrak{M} \to \mathfrak{A} \to 0$$

induces a short exact sequence

$$0 \to K_{1-*}(\mathfrak{A}) \to K_*(\mathfrak{M}) \to K_*(\mathfrak{A}) \to 0$$

which represents the element in $\operatorname{Ext}(K_*(\mathfrak{A}), K_{1-*}(\mathfrak{A}))$ induced by $\mathrm{KK}(\alpha) - \mathrm{KK}(\mathrm{id}_{\mathfrak{A}})$. By [**1**, Section 10.4] it follows that this extension is congruent to (5.2). □

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The following lemma is an immediate consequence of the Elliott–Kucerovsky absorption theorem.

LEMMA 5.4. *Let $\mathfrak{A}$ and $\mathfrak{C}$ be separable, unital, nuclear C^*-algebras, with a unital embedding $\iota\colon \mathfrak{A} \to \mathfrak{C}$, and let $\mathfrak{B}$ be a separable stable C^*-algebra. If $\mathfrak{e}$ is an absorbing, unital extension of $\mathfrak{C}$ by $\mathfrak{B}$, then $\mathfrak{e} \cdot \iota$ is an absorbing, unital extension of $\mathfrak{A}$ by $\mathfrak{B}$.*

Proof. It follows immediately from the definition of pure largeness that $\mathfrak{e} \cdot \iota$ is also purely large, so the result follows from [15, Theorem 6]. □

In the following, we consider

$$\Gamma^{(0,\delta_1)}_{\mathfrak{A},\mathfrak{B}} = q_{\delta_1}^{-1}(\{\psi([1_{\mathfrak{A}}]) : \psi \in \mathrm{Hom}(K_0(\mathfrak{A}), \mathrm{coker}\delta_1)\}),$$

which is a special case of Notation 5.1. Clearly

$$\Gamma^{(0,\delta_1)}_{\mathfrak{A},\mathfrak{B}} \subseteq \Gamma^{\delta_*}_{\mathfrak{A},\mathfrak{B}} \subseteq K_0(\mathfrak{B}).$$

The following lemma is the main technical tool to obtain our classification of unital extensions. While the conditions on $\mathfrak{A}$ in the following lemma might look slightly technical, we emphasise that any unital UCT Kirchberg algebra has these properties; K_1-surjectivity follows from [6] and the condition on automorphisms follows from the Kirchberg–Phillips theorem [19, 26].

LEMMA 5.5. *Let $\mathfrak{e} : 0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$ be a unital extension of separable C^*-algebras with boundary map $\delta_*\colon K_*(\mathfrak{A}) \to K_{1-*}(\mathfrak{B})$ in K-theory. Suppose that $\mathfrak{B}$ is stable, and that $\mathfrak{A}$ is nuclear, K_1-surjective, satisfies the UCT, and that for any $\mathsf{y} \in KK(\mathfrak{A}, \mathfrak{A})$ for which $K_*(\mathsf{y}) = K_*(id_{\mathfrak{A}})$, there is an automorphism $\alpha \in Aut(\mathfrak{A})$ such that $KK(\alpha) = \mathsf{y}$. Then for any $x \in \Gamma^{(0,\delta_1)}_{\mathfrak{A},\mathfrak{B}}$ there is an automorphism $\beta \in Aut(\mathfrak{A})$ for which $K_*(\beta) = id_{K_*(\mathfrak{A})}$, and*

$$[\mathfrak{e} \cdot \beta]_s = [\mathfrak{e}]_s + [\mathfrak{e}_x]_s \in \mathrm{Ext}_{us}(\mathfrak{A}, \mathfrak{B}).$$

Proof. Let $\mathfrak{e}_0$ be an absorbing, trivial, unital extension $\mathfrak{e}_0$. Since

$$[(\mathfrak{e} \oplus \mathfrak{e}_0) \cdot \beta]_s = [\mathfrak{e} \cdot \beta]_s + [\mathfrak{e}_0 \cdot \beta]_s = [\mathfrak{e} \cdot \beta]_s$$

for any automorphism $\beta \in \mathrm{Aut}(\mathfrak{A})$, it follows that we may replace $\mathfrak{e}$ with $\mathfrak{e} \oplus \mathfrak{e}_0$ without loss of generality, and thus assume that $\mathfrak{e}$ is absorbing.

As $x \in \Gamma^{(0,\delta_1)}_{\mathfrak{A},\mathfrak{B}}$ we may find a homomorphism $\psi\colon K_0(\mathfrak{A}) \to K_0(\mathfrak{B})/\mathrm{im}\,\delta_1$, such that $\psi([1_{\mathfrak{A}}]) = x + \mathrm{im}\,\delta_1$. Let $0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} K_0(\mathfrak{A}) \to 0$ be a free resolution.[10] As F_0 and F_1 are free, we may construct the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r, "f_1"] \arrow[d, "\psi_1"'] & F_0 \arrow[r, "f_0"] \arrow[d, "\psi_0"'] & K_0(\mathfrak{A}) \arrow[r] \arrow[d, "\psi"'] & 0 \\
 & K_1(\mathfrak{A}) \arrow[r, "\delta_1"] & K_0(\mathfrak{B}) \arrow[r] & K_0(\mathfrak{B})/\mathrm{im}\,\delta_1 \arrow[r] & 0.
\end{tikzcd}$$

[10] I.e. a short exact sequence with both F_0 and F_1 free abelian groups.

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Letting G denote the push-out of ψ_1 and f_1, we get the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[r, "f_1"] \arrow[d, "\psi_1"'] & F_0 \arrow[r, "f_0"] \arrow[d, "\tilde{\psi}_0"'] & K_0(\mathfrak{A}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_1(\mathfrak{A}) \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d, dotted, "\phi"'] & K_0(\mathfrak{A}) \arrow[r] \arrow[d, "\psi"'] & 0 \\ & K_1(\mathfrak{A}) \arrow[r, "\delta_1"] & K_0(\mathfrak{B}) \arrow[r] & K_0(\mathfrak{B})/\operatorname{im}\delta_1 \arrow[r] & 0. \end{tikzcd} \tag{5.3}$$

with exact rows. The homomorphism $\phi\colon G \to K_0(\mathfrak{B})$ making the diagram commute, exists by the universal property of push-outs. Let $\mathsf{x}_* \in \operatorname{Ext}(K_*(\mathfrak{A}), K_{1-*}(\mathfrak{A})) \subseteq KK(\mathfrak{A}, \mathfrak{A})$ be such that

$$\mathsf{x}_0 = \left[0 \to K_1(\mathfrak{A}) \to G \to K_0(\mathfrak{A}) \to 0\right] \in \operatorname{Ext}(K_0(\mathfrak{A}), K_1(\mathfrak{A})),$$

and x_1 is the trivial extension. As $K_*(\mathsf{x}_*)$ is the zero map, it follows from our hypothesis on $\mathfrak{A}$ that there is an automorphism $\alpha \in \operatorname{Aut}(\mathfrak{A})$ such that $KK(\alpha) = KK(\mathrm{id}_A) + \mathsf{x}_*$.

Applying Proposition 5.3, the Pimsner–Voiculescu sequence for the C^*-dynamical system $(\mathfrak{A}, \alpha, \mathbb{Z})$ collapses to a short exact sequence

$$0 \to K_{1-*}(\mathfrak{A}) \xrightarrow{\iota_{1-*}} K_{1-*}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \to K_*(\mathfrak{A}) \to 0,$$

which exactly induces the element $\mathsf{x}_* \in \operatorname{Ext}(K_*(\mathfrak{A}), K_{1-*}(\mathfrak{A}))$. Here, $\iota\colon \mathfrak{A} \to \mathfrak{A} \rtimes_\alpha \mathbb{Z}$ is the inclusion map. In particular, we may assume that $K_0(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) = K_0(\mathfrak{A}) \oplus K_1(\mathfrak{A})$, and $K_1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) = G$, and thus we have a homomorphism

$$(\delta_0 \oplus 0, \phi)\colon K_*(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \to K_{1-*}(\mathfrak{B}).$$

As $\iota_*\colon K_*(\mathfrak{A}) \to K_*(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$ is injective, it induces a surjection

$$\iota^*\colon \operatorname{Ext}(K_*(\mathfrak{A} \rtimes_\alpha \mathbb{Z}), K_*(\mathfrak{B})) \twoheadrightarrow \operatorname{Ext}(K_*(\mathfrak{A}), K_*(\mathfrak{B})).$$

As $\mathfrak{A}$ satisfies the UCT, so does $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ by [31]. Thus, by Theorem 4.14, we get the following commutative diagram:

$$\begin{tikzcd} \operatorname{Ext}(K_*(\mathfrak{A} \rtimes_\alpha \mathbb{Z}), K_*(\mathfrak{B})) \arrow[r, tail] \arrow[d, "\iota^*"] & \operatorname{Ext}_{\mathrm{uw}}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}, \mathfrak{B}) \arrow[r, two heads] \arrow[d, "\iota^*"] & \operatorname{Hom}((K_*(\mathfrak{A} \rtimes_\alpha \mathbb{Z}), [1]), K_{1-*}(\mathfrak{B})) \arrow[d, "\iota^*"] \\ \operatorname{Ext}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \arrow[r, tail] & \operatorname{Ext}_{\mathrm{uw}}(\mathfrak{A}, \mathfrak{B}) \arrow[r, two heads] & \operatorname{Hom}((K_*(\mathfrak{A}), [1]), K_{1-*}(\mathfrak{B})) \end{tikzcd}$$

for which the rows are short exact sequences. We may pick $[\mathfrak{f}']_{\mathrm{w}} \in \operatorname{Ext}_{\mathrm{uw}}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}, \mathfrak{B})$ which lifts the homomorphism $(\delta_0 \oplus 0, \phi)$. Recall that we identified $G = K_1(\mathfrak{A} \rtimes_\alpha \mathbb{Z})$, so by (5.3), we have

$$\iota^*(\delta_0 \oplus 0, \phi) = ((\delta_0 \oplus 0) \circ \iota_0, \phi \circ \iota_1) = (\delta_0, \delta_1) = \delta_*.$$

Thus, $\iota^*([\mathfrak{f}']_{\mathrm{w}})$ and $[\mathfrak{e}]_{\mathrm{w}}$ induce the same element in Hom. Thus, by doing a diagram chase in the above diagram (using surjectivity of the left vertical map), there is an element $[\mathfrak{f}'']_{\mathrm{w}} \in \operatorname{Ext}_{\mathrm{uw}}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}, \mathfrak{B})$ vanishing in Hom, such that $\iota^*([\mathfrak{f}']_{\mathrm{w}} + [\mathfrak{f}'']_{\mathrm{w}}) = [\mathfrak{e}]_{\mathrm{w}}$. Let $\mathfrak{f}$ be an absorbing unital extension of $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ by $\mathfrak{B}$ such that $[\mathfrak{f}]_{\mathrm{w}} = [\mathfrak{f}']_{\mathrm{w}} + [\mathfrak{f}'']_{\mathrm{w}}$. Then $[\mathfrak{f} \cdot \iota]_{\mathrm{w}} = [\mathfrak{e}]_{\mathrm{w}}$.

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Let $\tau\colon \mathfrak{A} \to \mathcal{Q}(\mathfrak{B})$ be the Busby map of $\mathfrak{e}$, and $\eta\colon \mathfrak{A} \rtimes_\alpha \mathbb{Z} \to \mathcal{Q}(\mathfrak{B})$ be the Busby map of $\mathfrak{f}$. In particular, $\eta \circ \iota$ is the Busby map of $\mathfrak{f} \cdot \iota$. Recall from the beginning of the proof that we assumed that $\mathfrak{e}$ was absorbing, and by Lemma 5.4, $\mathfrak{f} \cdot \iota$ is also absorbing. Thus, as $[\mathfrak{f} \cdot \iota]_w = [\mathfrak{e}]_w$, there is a unitary $u \in \mathcal{Q}(\mathfrak{B})$ such that

$$\operatorname{Ad} u \circ \tau = \eta \circ \iota.$$

Let $w \in \mathfrak{A} \rtimes_\alpha \mathbb{Z}$ denote the canonical unitary, so that $\operatorname{Ad} w \circ \iota = \iota \circ \alpha$. Then

$$\begin{aligned}
\tau \circ \alpha &= \operatorname{Ad} u^* \circ \eta \circ \iota \circ \alpha \\
&= \operatorname{Ad} u^* \circ \eta \circ \operatorname{Ad} w \circ \iota \\
&= \operatorname{Ad} u^* \circ \operatorname{Ad} \eta(w) \circ \eta \circ \iota \\
&= \operatorname{Ad} u^* \circ \operatorname{Ad} \eta(w) \circ \operatorname{Ad} u \circ \tau \\
&= \operatorname{Ad}(u^* \eta(w) u) \circ \tau.
\end{aligned}$$

Hence, it follows from Lemma 4.11 that

$$[\mathfrak{e} \cdot \alpha]_{\mathrm{s}} = [\mathfrak{e}]_{\mathrm{s}} + [\mathfrak{e}_{[u^*\eta(w)u]}]_{\mathrm{s}} = [\mathfrak{e}]_{\mathrm{s}} + [\mathfrak{e}_{[\eta(w)]}]_{\mathrm{s}} \in \operatorname{Ext}_{\mathrm{us}}(\mathfrak{A}, \mathfrak{B}).$$

Recall that $[\mathfrak{f}]_{\mathrm{w}} = [\mathfrak{f}']_{\mathrm{w}} + [\mathfrak{f}'']_{\mathrm{w}}$, where $[\mathfrak{f}'']_{\mathrm{w}}$ vanishes in Hom, and $[\mathfrak{f}']_{\mathrm{w}}$ induces the homomorphism $(\delta_0 \oplus 0, \phi)\colon K_*(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \to K_{1-*}(\mathfrak{B})$. Thus, $[\mathfrak{f}]_{\mathrm{w}}$ also induces the homomorphism $(\delta_0 \oplus 0, \phi)$, so in particular

$$K_1(\eta) = \phi\colon K_1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \to K_1(\mathcal{Q}(\mathfrak{B})) = K_0(\mathfrak{B}).$$

It follows that

$$[\mathfrak{e} \cdot \alpha]_{\mathrm{s}} = [\mathfrak{e}]_{\mathrm{s}} + [\mathfrak{e}_{\phi([w])}]_{\mathrm{s}} \in \operatorname{Ext}_{\mathrm{us}}(\mathfrak{A}, \mathfrak{B}). \tag{5.4}$$

By commutativity of the lower right square in (5.3), the two compositions

$$K_1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \xrightarrow{\phi} K_0(\mathfrak{B}) \to K_0(\mathfrak{B})/\operatorname{im} \delta_1, \qquad K_1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \to K_0(\mathfrak{A}) \xrightarrow{\psi} K_0(\mathfrak{B})/\operatorname{im} \delta_1,$$

are the same. It is well known that $[w]$ is mapped to $[1_{\mathfrak{A}}]$ via the map $K_1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \to K_0(\mathfrak{A})$.[11] Thus, we have

$$\phi([w]) + \operatorname{im} \delta_1 = \psi([1_{\mathfrak{A}}]) = x + \operatorname{im} \delta_1,$$

where $x \in \Gamma^{(0,\delta_1)}_{\mathfrak{A},\mathfrak{B}}$ is our given element from the statement of the lemma. As $\mathfrak{A}$ is K_1-surjective, we may find a unitary $v \in \mathfrak{A}$ such that

$$\phi([w]) + \delta_1([v]) = x. \tag{5.5}$$

Let $\beta = \operatorname{Ad} v \circ \alpha$ be the induced automorphism on $\mathfrak{A}$. By construction $K_*(\alpha) = \mathrm{id}_{K_*(\mathfrak{A})}$ and thus $K_*(\beta) = \mathrm{id}_{K_*(\mathfrak{A})}$. By Lemma 4.11, it follows that

$$\begin{aligned}
[\mathfrak{e} \cdot \beta]_{\mathrm{s}} &= [\mathfrak{e} \cdot \alpha]_{\mathrm{s}} + [\mathfrak{e}_{\delta_1([v]_1)}]_{\mathrm{s}} \\
&\overset{(5.4)}{=} [\mathfrak{e}]_{\mathrm{s}} + [\mathfrak{e}_{\phi([w]_1)}]_{\mathrm{s}} + [\mathfrak{e}_{\delta_1([v]_1)}]_{\mathrm{s}} \\
&= [\mathfrak{e}]_{\mathrm{s}} + [\mathfrak{e}_{\phi([w]_1)+\delta_1([v]_1)}]_{\mathrm{s}} \\
&\overset{(5.5)}{=} [\mathfrak{e}]_{\mathrm{s}} + [\mathfrak{e}_x]_{\mathrm{s}}
\end{aligned}$$

as desired. □

[11] The proof of this is identical to the proof showing that the map $K_1(C(\mathbb{T})) \to K_0(\mathbb{K})$ induced by the usual Toeplitz extension sends the class of the canonical unitary in $C(\mathbb{T})$ to $[e_{11}]_0$.

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Proposition 5.6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable C^*-algebras, with $\mathfrak{A}$ unital, nuclear and satisfying the UCT, and $\mathfrak{B}$ stable. Suppose that $\mathfrak{A}$ is K_1-surjective and that for any $\mathsf{y} \in KK(\mathfrak{A}, \mathfrak{A})$ for which $K_*(\mathsf{y}) = K_*(id_{\mathfrak{A}})$, there is an automorphism $\alpha \in Aut(\mathfrak{A})$ such that $KK(\alpha) = \mathsf{y}$. Let $\mathfrak{e}_1$ and $\mathfrak{e}_2$ be unital extensions of $\mathfrak{A}$ by $\mathfrak{B}$ and suppose that*

(*a*) $[\mathfrak{e}_1]_{\mathrm{w}} = [\mathfrak{e}_2]_{\mathrm{w}}$ *in* $\mathrm{Ext}_{\mathrm{uw}}(\mathfrak{A}, \mathfrak{B})$*,*
(*b*) $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_1) \equiv K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_2)$*,*
(*c*) *the exponential maps $\delta_0 \colon K_0(\mathfrak{A}) \to K_{1-*}(\mathfrak{B})$ induced by $\mathfrak{e}_1$ and $\mathfrak{e}_2$ vanish.*

Then there is an automorphism $\beta \in Aut(\mathfrak{A})$ with $K_*(\beta) = id_{K_*(\mathfrak{A})}$ such that $[\mathfrak{e}_1 \cdot \beta] = [\mathfrak{e}_2]$ in $\mathrm{Ext}_{\mathrm{us}}(\mathfrak{A}, \mathfrak{B})$.

Proof. Let $\delta_* \colon K_*(\mathfrak{A}) \to K_{1-*}(\mathfrak{B})$ be the connecting maps in the six-term exact sequences of $\mathfrak{e}_1$ and $\mathfrak{e}_2$, which agree since $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_1) \equiv K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_2)$. As $[\mathfrak{e}_1]_{\mathrm{w}} = [\mathfrak{e}_2]_{\mathrm{w}}$, it follows from Theorem 4.14 that there is an $x \in K_0(\mathfrak{B})$ such that $[\mathfrak{e}_1 \oplus \mathfrak{e}_x]_{\mathrm{s}} = [\mathfrak{e}_2]_{\mathrm{s}}$ in $\mathrm{Ext}_{\mathrm{us}}(\mathfrak{A}, \mathfrak{B})$. As

$$K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_1 \oplus \mathfrak{e}_x) \overset{\text{Cor. 3.4}}{\equiv} K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_2) \equiv K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_1),$$

it follows from Lemma 5.2 that $x \in \Gamma_{\mathfrak{A},\mathfrak{B}}^{\delta_*}$. As $\delta_0 = 0$, it clearly holds that $\Gamma_{\mathfrak{A},\mathfrak{B}}^{\delta_*} = \Gamma_{\mathfrak{A},\mathfrak{B}}^{(0,\delta_1)}$ and thus Lemma 5.5 provides an automorphism $\beta \in \mathrm{Aut}(\mathfrak{A})$ such that

$$[\mathfrak{e}_1 \cdot \beta]_{\mathrm{s}} = [\mathfrak{e}_1]_{\mathrm{s}} + [\mathfrak{e}_x]_{\mathrm{s}} = [\mathfrak{e}_2]_{\mathrm{s}} \in \mathrm{Ext}_{\mathrm{us}}(\mathfrak{A}, \mathfrak{B})$$

as wanted. □

Remark 5.7. The only thing Condition (*c*) was used for in Proposition 5.6 was so that $\Gamma_{\mathfrak{A},\mathfrak{B}}^{(0,\delta_1)} = \Gamma_{\mathfrak{A},\mathfrak{B}}^{\delta_*}$. Hence, one may replace Condition (*c*) with this more general condition in order to obtain the conclusion of Proposition 5.6.

In particular, Condition (*c*) in Proposition 5.6 may be replaced by any of the following statements, as these all imply that $\Gamma_{\mathfrak{A},\mathfrak{B}}^{(0,\delta_1)} = \Gamma_{\mathfrak{A},\mathfrak{B}}^{\delta_*}$. Proving that (*c*1)–(*c*6) imply $\Gamma_{\mathfrak{A},\mathfrak{B}}^{(0,\delta_1)} = \Gamma_{\mathfrak{A},\mathfrak{B}}^{\delta_*}$ is left to the reader.

(*c*1) The class of the unit $[1_{\mathfrak{A}}]$ vanishes in $K_0(\mathfrak{A})$.
(*c*2) The exponential map δ_0 is injective.
(*c*3) The index map δ_1 is surjective.
(*c*4) $K_0(\mathfrak{A}) \cong \mathbb{Z} \oplus G$, such that $[1_{\mathfrak{A}}] = (1, g)$ for some $g \in G$.
(*c*5) $\ker \delta_0$ is a direct summand in $K_0(\mathfrak{A})$.
(*c*6) $K_0(\mathfrak{E})$ is divisible.

Proposition 5.8. *Let $\mathfrak{e}_i \colon 0 \to \mathfrak{B} \to \mathfrak{E}_i \to \mathfrak{A} \to 0$ be unital extensions of C^*-algebras for $i = 1, 2$ such that $\mathfrak{A}$ is a unital UCT Kirchberg algebra, and $\mathfrak{B}$ is a stable AF algebra. If $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_1) \equiv K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_2)$ then there is an automorphism $\alpha \in Aut(\mathfrak{A})$ such that $\mathfrak{e}_1$ and $\mathfrak{e}_2 \cdot \alpha$ are strongly unitarily equivalent.*

In particular, if $K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_1) \equiv K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_2)$ then $\mathfrak{E}_1 \cong \mathfrak{E}_2$.

Proof. We identify $\mathrm{Ext}(\mathfrak{A}, \mathfrak{B}) \cong \mathrm{KK}^1(\mathfrak{A}, \mathfrak{B})$ in the usual way (see Theorem 2.7). By [**11**, Theorem 2.3] (which is based on [**29**, Theorem 3.2]), there exist $\mathsf{x} \in \mathrm{KK}(\mathfrak{A}, \mathfrak{A})$ and $\mathsf{y} \in \mathrm{KK}(\mathfrak{B}, \mathfrak{B})$ such that $K_*(\mathsf{x}) = K_*(\mathrm{id}_{\mathfrak{A}})$, $K_*(\mathsf{y}) = K_*(\mathrm{id}_{\mathfrak{B}})$, $[\mathfrak{e}_1] \times \mathsf{y} = \mathsf{x} \times [\mathfrak{e}_2]$ in $\mathrm{KK}^1(\mathfrak{A}, \mathfrak{B})$. Since $\mathfrak{B}$ is an AF algebra, we have that $\mathsf{y} = \mathrm{KK}(\mathrm{id}_{\mathfrak{B}})$. Thus, $[\mathfrak{e}_1] = \mathsf{x} \times [\mathfrak{e}_2]$. Since $\mathfrak{A}$ is a UCT Kirchberg algebra, by the Kirchberg–Phillips theorem [19], [26] there exists an isomorphism $\alpha^{(1)}\colon \mathfrak{A} \to \mathfrak{A}$ such that $\mathsf{x} = \mathrm{KK}(\alpha^{(1)})$. By [29, Proposition 1.1], we get

$$[\mathfrak{e}_2 \cdot \alpha^{(1)}] = \mathsf{x} \times [\mathfrak{e}_2] = [\mathfrak{e}_1] \in \mathrm{KK}^1(\mathfrak{A}, \mathfrak{B}) \cong \mathrm{Ext}(\mathfrak{A}, \mathfrak{B}).$$

By Lemma 4.13 it follows that $[\mathfrak{e}_1]_{\mathrm{w}} = [\mathfrak{e}_2 \cdot \alpha^{(1)}]_{\mathrm{w}}$ in $\mathrm{Ext}_{\mathrm{uw}}(\mathfrak{A}, \mathfrak{B})$. Now, as $K_*(\alpha^{(1)}) = \mathrm{id}_{K_*(\mathfrak{A})}$, it follows that

$$K^{\mathrm{u}}_{\mathrm{six}}(\mathfrak{e}_2 \cdot \alpha^{(1)}) \equiv K^{\mathrm{u}}_{\mathrm{six}}(\mathfrak{e}_2) \equiv K^{\mathrm{u}}_{\mathrm{six}}(\mathfrak{e}_1).$$

By [6] $\mathfrak{A}$ is K_1-surjective, and by the Kirchberg–Phillips theorem (cited above) $\mathfrak{A}$ satisfies the condition in Proposition 5.6 about automorphisms. Hence, this proposition produces an automorphism $\alpha^{(2)} \in \mathrm{Aut}(\mathfrak{A})$ with $K_*(\alpha^{(2)}) = \mathrm{id}_{K_*(\mathfrak{A})}$ such that

$$[\mathfrak{e}_1]_{\mathrm{s}} = [\mathfrak{e}_2 \cdot \alpha]_{\mathrm{s}} \in \mathrm{Ext}_{\mathrm{us}}(\mathfrak{A}, \mathfrak{B})$$

where $\alpha = \alpha^{(1)} \circ \alpha^{(2)}$.

As $\mathfrak{A}$ is simple, unital and nuclear, $\mathfrak{B}$ is stable with the corona factorisation property, and the extensions $\mathfrak{e}_1$ and $\mathfrak{e}_2 \cdot \alpha$ are unital, it follows that $\mathfrak{e}_1$ and $\mathfrak{e}_2 \cdot \alpha$ are full and thus absorbing. Hence, $\mathfrak{e}_1$ and $\mathfrak{e}_2 \cdot \alpha$ are strongly unitarily equivalent.

The "in particular" part follows since the extension algebra of $\mathfrak{e}_2$ is isomorphic to the extension algebra of $\mathfrak{e}_2 \cdot \alpha$ and since strong unitary equivalence implies isomorphism of the extension algebras. $\square$

By $K^{+,\mathrm{u}}_{\mathrm{six}}(\mathfrak{e})$ we mean the six-term exact sequence in K-theory with order in all K_0-groups. The following is the main classification result of this section and is Theorem A.

THEOREM 5.9. *Let $\mathfrak{e}_i : 0 \to \mathfrak{B}_i \to \mathfrak{E}_i \to \mathfrak{A}_i \to 0$ be unital extensions of C^*-algebras for $i = 1, 2$, such that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unital UCT Kirchberg algebras and $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are stable AF algebras. Then $\mathfrak{E}_1 \cong \mathfrak{E}_2$ if and only if $K^{+,\mathrm{u}}_{\mathrm{six}}(\mathfrak{e}_1) \cong K^{+,\mathrm{u}}_{\mathrm{six}}(\mathfrak{e}_2)$.*

Proof. Suppose $\mathfrak{E}_1 \cong \mathfrak{E}_2$. As the extension $\mathfrak{e}_i$ is unital, and as $\mathfrak{A}_i$ is simple, it follows that the extension $\mathfrak{e}_i$ is full. As $\mathfrak{B}_i$ is stable, it therefore follows that $\mathfrak{B}_i$ is the unique maximal ideal in $\mathfrak{E}_i$ for $i = 1, 2$.[12] It follows that the extensions $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are isomorphic, and thus $K^{+,\mathrm{u}}_{\mathrm{six}}(\mathfrak{e}_1) \cong K^{+,\mathrm{u}}_{\mathrm{six}}(\mathfrak{e}_2)$.

Now suppose that there is an isomorphism $K^{+,\mathrm{u}}_{\mathrm{six}}(\mathfrak{e}_1) \xrightarrow{\cong} K^{+,\mathrm{u}}_{\mathrm{six}}(\mathfrak{e}_2)$ induced by

$$\phi_*\colon K_*(\mathfrak{A}_1) \xrightarrow{\cong} K_*(\mathfrak{A}_2), \qquad \psi_*\colon K^+_*(\mathfrak{B}_1) \xrightarrow{\cong} K^+_*(\mathfrak{B}_2), \qquad \rho_*\colon K_*(\mathfrak{E}_1) \xrightarrow{\cong} K_*(\mathfrak{E}_2).$$

By the Kirchberg–Phillips theorem [19], [26] we find an isomorphism $\alpha\colon \mathfrak{A}_1 \xrightarrow{\cong} \mathfrak{A}_2$ such that $K_*(\alpha) = \phi_*$. Similarly, by Elliott's classification of AF algebras [13], we find an isomorphism $\beta\colon \mathfrak{B}_1 \xrightarrow{\cong} \mathfrak{B}_2$ such that $K_*(\beta) = \psi_*$. We obtain the following commutative diagram:

[12] Clearly $\mathfrak{B}_1$ is a maximal ideal as the corresponding quotient is simple. If $\mathfrak{J} \subseteq \mathfrak{E}_1$ is a two-sided, closed ideal such that $\mathfrak{J} \not\subseteq \mathfrak{B}_1$, then there is an element $x \in \mathfrak{J} \setminus \mathfrak{B}_1$ inducing a nonzero element in $\mathfrak{A}_1$. As the extension is full and $\mathfrak{B}_1$ is stable, it follows that x induces a full element in $\mathscr{M}(\mathfrak{B}_1)$. Hence, $\overline{\mathfrak{B}_1 x \mathfrak{B}_1} = \mathfrak{B}_1$ so $\mathfrak{B}_1 \subsetneq \mathfrak{J}$ and thus $\mathfrak{J} = \mathfrak{E}_1$ by maximality of $\mathfrak{B}_1$. The same argument works for $\mathfrak{E}_2$.

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$$\begin{tikzcd}
\mathfrak{e}_1 : & 0 \arrow[r] & \mathfrak{B}_1 \arrow[r] \arrow[d, "\beta", "\cong"'] & \mathfrak{E}_1 \arrow[r] \arrow[d, "\eta^{(1)}", "\cong"'] & \mathfrak{A}_1 \arrow[r] \arrow[d, equal] & 0 \\
\beta \cdot \mathfrak{e}_1 : & 0 \arrow[r] & \mathfrak{B}_2 \arrow[r] \arrow[d, equal] & \mathfrak{E}_1' \arrow[r] & \mathfrak{A}_1 \arrow[r] \arrow[d, equal] & 0 \\
\mathfrak{e}_2 \cdot \alpha : & 0 \arrow[r] & \mathfrak{B}_2 \arrow[r] \arrow[d, equal] & \mathfrak{E}_2' \arrow[r] \arrow[d, "\eta^{(2)}", "\cong"'] & \mathfrak{A}_1 \arrow[r] \arrow[d, "\alpha", "\cong"'] & 0 \\
\mathfrak{e}_2 : & 0 \arrow[r] & \mathfrak{B}_2 \arrow[r] & \mathfrak{E}_2 \arrow[r] & \mathfrak{A}_2 \arrow[r] & 0
\end{tikzcd} \tag{5.6}$$

which has exact rows. It is easy to see that the map

$$K_*(\eta^{(2)})^{-1} \circ \rho_* \circ K_*(\eta^{(1)})^{-1} \colon K_*(\mathfrak{E}_1') \to K_*(\mathfrak{E}_2')$$

induces a congruence $K_{\mathrm{six}}^{\mathrm{u}}(\beta \cdot \mathfrak{e}_1) \equiv K_{\mathrm{six}}^{\mathrm{u}}(\mathfrak{e}_2 \cdot \alpha)$. By Proposition 5.8, it follows that $\mathfrak{E}_1' \cong \mathfrak{E}_2'$, so it follows that $\mathfrak{E}_1 \cong \mathfrak{E}_2$. □

6. Determining when extensions are full. In this section, we characterise when certain extensions are full with a stable ideal. We show that when the ideal is sufficiently finite (e.g. an AF algebra) and the quotient is sufficiently infinite (e.g. a Kirchberg algebra), then this is characterised by the existence of a properly infinite, full projection in the extension algebra.

LEMMA 6.1. *Let $\mathfrak{B}$ be a σ-unital C^*-algebra with stable rank one. Then $\mathfrak{B}$ is stable if and only if there exists a projection $p \in \mathscr{M}(\mathfrak{B})$ which is properly infinite, and which is strictly full, i.e. $\overline{\mathfrak{B}p\mathfrak{B}} = \mathfrak{B}$.*

In particular, if $p \in \mathscr{M}(\mathfrak{B})$ is a strictly full, properly infinite projection, then $p\mathfrak{B}p$ is stable.

Proof. If $\mathfrak{B}$ is stable, then $1_{\mathscr{M}(\mathfrak{B})} \in \mathscr{M}(\mathfrak{B})$ is a strictly full, properly infinite projection.

Conversely, suppose $p \in \mathscr{M}(\mathfrak{B})$ is a strictly full, properly infinite projection. Let $p_1, p_2, \ldots \in \mathscr{M}(\mathfrak{B})$ be a sequence of pairwise orthogonal projections in $\mathscr{M}(\mathfrak{B})$, such that $p_i \leq p$ and $p \sim p_i$ for all $i \in \mathbb{N}$. Then the hereditary C^*-subalgebra $\mathfrak{B}_0$ of $\mathfrak{B}$ generated by $p_1, p_2, \ldots$ is isomorphic to $p\mathfrak{B}p \otimes \mathbb{K}$. As p is strictly full it follows that $\mathfrak{B}_0 \subseteq \mathfrak{B}$ is a stable, full, hereditary C^*-subalgebra. It is an easy consequence of [25, Lemma 4.6] that $\mathfrak{B}$ is stable (as any strictly positive element in $\mathfrak{B}_0$ induces a full, properly infinite element in the scale of the Cuntz semigroup of $\mathfrak{B}$).

"In particular" is immediate since $\mathscr{M}(p\mathfrak{B}p) \cong p\mathscr{M}(\mathfrak{B})p$ canonically, and since $p\mathfrak{B}p$ is σ-unital with stable rank one. □

The following is essentially [2, Proposition 2.7].

LEMMA 6.2. *Let $\mathfrak{A}$, $\mathfrak{C}$ and $\mathfrak{D}$ be C^*-algebras and suppose that $\phi \colon \mathfrak{A} \to \mathfrak{D}$ and $\pi : \mathfrak{C} \to \mathfrak{D}$ are $*$-homomorphisms for which π is surjective. Suppose that $p \in \mathfrak{A}$ and $q \in \mathfrak{C}$ are projections such that $\phi(p) = \pi(q)$ and $\phi(p)\mathfrak{D}\phi(p)$ is K_1-injective. If both p and q are properly infinite, then $p \oplus q$ is properly infinite in the pull-back $\mathfrak{A} \oplus_{\phi,\pi} \mathfrak{C}$.*

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Proof. By replacing $\mathfrak{A}$, $\mathfrak{C}$ and $\mathfrak{D}$ with $p\mathfrak{A}p$, $q\mathfrak{C}q$ and $\phi(p)\mathfrak{D}\phi(p)$, we may assume that $\mathfrak{A}$, $\mathfrak{C}$ and $\mathfrak{D}$ are unital and properly infinite, that ϕ and π are unital maps, and that $\mathfrak{D}$ is K_1-injective. Under these assumptions, we should show that $\mathfrak{A} \oplus_{\phi,\pi} \mathfrak{C}$ is properly infinite.

The result now follows from [**2**, Proposition 2.7]. In fact, although said result assumes that both maps are surjective (corresponding in our case to ϕ and π), they only use that one map is surjective. We fill in the proof for completion.

Let $s_1, s_2, s_3 \in \mathfrak{A}$ and $t_1, t_2, t_3 \in \mathfrak{C}$ be isometries with mutually orthogonal range projections. Let

$$v := \sum_{j=1}^{2} \phi(s_j)\pi(t_j)^* \in \mathfrak{D}$$

which is a partial isometry satisfying $\phi(s_j) = v\pi(t_j)$ for $j = 1, 2$. Note that

$$1_{\mathfrak{D}} \sim \phi(s_3 s_3^*) \leq 1_{\mathfrak{D}} - vv^*, \qquad 1_{\mathfrak{D}} \sim \pi(t_3 t_3^*) \leq 1_{\mathfrak{D}} - v^*v.$$

It follows that $1_{\mathfrak{D}} - vv^*$ and $1_{\mathfrak{D}} - v^*v$ are properly infinite and full in $\mathfrak{D}$. By [**2**, Lemma 2.4(*i*)] there is a unitary $u \in \mathfrak{D}$ with $[u] = 0 \in K_1(\mathfrak{D})$ such that $v = uv^*v$. As $\mathfrak{D}$ is K_1-injective, it follows that u is homotopic to 1, and thus lifts to a unitary $\widetilde{u} \in \mathfrak{C}$.

Clearly $\widetilde{u}t_1, \widetilde{u}t_2 \in \mathfrak{C}$ are isometries with orthogonal range projections, and

$$\pi(\widetilde{u}t_j) = u\pi(t_j) = v\pi(t_j) = \phi(s_j)$$

so $s_j \oplus \widetilde{u}t_j \in \mathfrak{A} \oplus_{\phi,\pi} \mathfrak{C}$ for $j = 1, 2$ are isometries with orthogonal range projections. Hence, $\mathfrak{A} \oplus_{\phi,\pi} \mathfrak{C}$ is properly infinite. □

By the above lemma we deduce the following property about proper infiniteness of projections in purely large extensions (see Remark 2.9).

PROPOSITION 6.3. *Let* $0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$ *be a purely large extension of separable* C^**-algebras such that* $\mathfrak{B}$ *is stable, and suppose that* $p \in \mathfrak{E} \setminus \mathfrak{B}$ *is a projection. Then* p *is properly infinite if and only if* $p + \mathfrak{B} \in \mathfrak{A}$ *is properly infinite.*

Proof. "Only if" is trivial. To prove "if", assume that the image of p in $\mathfrak{A}$ is properly infinite. Let $\tau : \mathfrak{A} \to \mathscr{Q}(\mathfrak{B})$ be the Busby map of the extension. We may identify $\mathfrak{E}$ with the pull-back $\mathfrak{A} \oplus_{\tau,\pi_{\mathfrak{B}}} \mathscr{M}(\mathfrak{B})$. Let $q \in \mathscr{M}(\mathfrak{B})$ be the projection induced by p. As purely large extensions are full,[13] it follows that q is full in $\mathscr{M}(\mathfrak{B})$. As our given extension is purely large, it easily follows that the extension

$$0 \to \mathfrak{B} \to \mathfrak{B} + \mathbb{C}q \to \mathbb{C} \to 0$$

is purely large. By [**16**, Proposition 2.7] it follows that q is a properly infinite, full projection in $\mathscr{M}(\mathfrak{B})$. Hence, $q\mathfrak{B}q \cong \mathfrak{B}$ is stable, and thus $\pi_{\mathfrak{B}}(q)\mathscr{Q}(\mathfrak{B})\pi_{\mathfrak{B}}(q) \cong \mathscr{Q}(\mathfrak{B})$ is K_1-injective by Proposition 4.9. By Lemma 6.2, p is properly infinite. □

PROPOSITION 6.4. *Let* $\mathfrak{e} : 0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$ *be an extension of separable* C^**-algebras for which* $\mathfrak{A}$ *is simple and* $\mathfrak{B}$ *has stable rank one and the corona factorisation property. Suppose that there is a projection* $p \in \mathfrak{E} \setminus \mathfrak{B}$ *such that* $p + \mathfrak{B} \in \mathfrak{A}$ *is properly infinite. Then* $\mathfrak{B}$ *is stable and* $\mathfrak{e}$ *is full if and only if* p *is full and properly infinite in* $\mathfrak{E}$.

[13] It is easy to see that an extension $\mathfrak{e}$ is full if and only if the Cuntz sum $\mathfrak{e} \oplus 0$ is full. If $\mathfrak{e}$ is purely large, then $\mathfrak{e} \oplus 0$ is nuclearly absorbing by [**16**, Corollary 2.4]. As $\mathfrak{e} \oplus 0$ absorbs any full, trivial, weakly nuclear extension (which always exist), it follows that $\mathfrak{e} \oplus 0$ – and thus also $\mathfrak{e}$ – is full.

Proof. "Only if" follows from Proposition 6.3 as $\mathfrak{e}$ is purely large by the corona factorisation property. For "if" suppose that p is full and properly infinite. Then $\mathfrak{B} = \overline{\mathfrak{B}p\mathfrak{B}}$ by fullness of p. By Lemma 6.1, it follows that $\mathfrak{B}$ is stable and $\mathfrak{B} \cong p\mathfrak{B}p$. Hence, by [3, Theorem 4.23], p induces a full projection in $\mathscr{M}(\mathfrak{B})$. As $\mathfrak{A}$ is simple, and as $p + \mathfrak{B} \in \mathfrak{A}$ is mapped to a full projection in $\mathscr{Q}(\mathfrak{B})$ via the Busby map, it follows that the extension $\mathfrak{e}$ is full. □

The following can be used to characterise when the extensions we wish to classify are full.

THEOREM 6.5. *Let $\mathfrak{e} : 0 \to \mathfrak{B} \to \mathfrak{E} \to \mathfrak{A} \to 0$ be an extension of C^*-algebras such that $\mathfrak{A}$ is a Kirchberg algebra and $\mathfrak{B}$ is an AF algebra. The following are equivalent.*

(*i*) *$\mathfrak{B}$ is stable and the extension $\mathfrak{e}$ is full,*
(*ii*) *$\mathfrak{E}$ contains a full, properly infinite projection,*
(*iii*) *any projection $p \in \mathfrak{E} \setminus \mathfrak{B}$ is full and properly infinite (in $\mathfrak{E}$).*

Proof. (*i*) ⇒ (*iii*): Suppose that $p \in \mathfrak{E} \setminus \mathfrak{B}$ is a projection. Fullness of $\mathfrak{e}$ and simplicity of $\mathfrak{A}$ imply that p is full. As $\mathfrak{B}$ has the corona factorisation property by virtue of being an AF algebra, it follows from Proposition 6.3 that p is properly infinite.

(*iii*) ⇒ (*ii*): Let $q \in \mathfrak{A}$ be a nonzero projection. By [4, Proposition 3.15], q lifts to a projection $p \in \mathfrak{E} \setminus \mathfrak{B}$, which is properly infinite and full by assumption.

(*ii*) ⇒ (*i*): Follows from Proposition 6.4 □

7. Classification of non-unital extensions. In [16, Section 4], an example was given of two non-unital, full extensions $\mathfrak{e}_i : 0 \to \mathfrak{B}_i \to \mathfrak{E}_i \to \mathfrak{A}_i \to 0$ such that $\mathfrak{A}_i \cong \mathcal{O}_2$, $\mathfrak{B}_i \cong M_{2^\infty} \otimes \mathbb{K}$, $K_{\mathrm{six}}(\mathfrak{e}_1) \cong K_{\mathrm{six}}(\mathfrak{e}_2)$ (with order, scale and units preserved), but for which $\mathfrak{E}_1 \ncong \mathfrak{E}_2$. In this section, we will describe how to obtain classification of such (and more general) extensions. Note that our invariant needs to carry more information than the six-term exact sequence alone.

The following lemma indicates the main trick that will be used to get classification of non-unital extensions with unital quotients. It implies that if one can arrange that the corresponding Busby maps have the same unit, and that the units in the quotients lift to projections, then the classification problem can be reduced to the unital case.

LEMMA 7.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ unital, and let $\tau_i \colon \mathfrak{A} \to \mathscr{Q}(\mathfrak{B})$ be (not necessarily unital) Busby maps for $i = 1, 2$. Suppose that $\tau_1(1_{\mathfrak{A}}) = \tau_2(1_{\mathfrak{A}})$, and that this projection lifts to a projection $p \in \mathscr{M}(\mathfrak{B})$. If the unital extensions*

$$0 \to p\mathfrak{B}p \to (1_{\mathfrak{A}} \oplus p)(\mathfrak{A} \oplus_{\tau_i, \pi_{\mathfrak{B}}} \mathscr{M}(\mathfrak{B}))(1_{\mathfrak{A}} \oplus p) \to \mathfrak{A} \to 0 \tag{7.1}$$

for $i = 1, 2$ are strongly unitarily equivalent, then so are the extensions induced by τ_1 and τ_2.

Proof. The Busby maps $\widetilde{\tau}_i$ of the extensions (7.1) are just the corestrictions of the Busby maps τ_i to $\tau_i(1_{\mathfrak{A}})\mathscr{Q}(B)\tau_i(1_{\mathfrak{A}}) \cong \mathscr{Q}(p\mathfrak{B}p)$ (the canonical isomorphism). By assumption there is a unitary $\widetilde{u} \in \mathscr{M}(p\mathfrak{B}p)$ such that $\operatorname{Ad} \pi_{p\mathfrak{B}p}(\widetilde{u}) \circ \widetilde{\tau}_1 = \widetilde{\tau}_2$. Using the canonical identification $\mathscr{M}(p\mathfrak{B}p) \cong p\mathscr{M}(\mathfrak{B})p$, let $u = \widetilde{u} + (1_{\mathscr{M}(B)} - p)$. Then u is a unitary in $\mathscr{M}(\mathfrak{B})$ satisfying $\operatorname{Ad} \pi_{\mathfrak{B}}(u) \circ \tau_1 = \tau_2$. □

The next goal will be to arrange that $\tau_1(1_{\mathfrak{A}}) = \tau_2(1_{\mathfrak{A}}) \in \mathscr{Q}(\mathfrak{B})$ by twisting one extension by an automorphism on $\mathfrak{B}$. For this purpose, we introduce the following notation.

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NOTATION 7.2. Let $\mathfrak{e}: 0 \to \mathfrak{B} \to \mathfrak{E} \xrightarrow{\pi} \mathfrak{A} \to 0$ be an extension of C^*-algebras where $\mathfrak{A}$ is unital, but $\mathfrak{E}$ is not necessarily unital. Let

$$\mathfrak{D}_{\mathfrak{e}} := \pi^{-1}(\mathbb{C}1_{\mathfrak{A}}) \subseteq \mathfrak{E}.$$

In the case where $\mathfrak{E}$ is unital, then $\mathfrak{D} = \widetilde{\mathfrak{B}}$ is the (forced) unitisation of $\mathfrak{B}$.

LEMMA 7.3. *Let $\mathfrak{e}_i : 0 \to \mathfrak{B}_i \to \mathfrak{E}_i \to \mathfrak{A}_i \to 0$ be extensions of C^*-algebras for $i = 1, 2$ with Busby maps $\tau_i : \mathfrak{A}_i \to \mathcal{Q}(\mathfrak{B}_i)$. Suppose that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unital, that $\beta : \mathfrak{B}_1 \xrightarrow{\cong} \mathfrak{B}_2$ is an isomorphism, and let $\overline{\beta} : \mathcal{Q}(\mathfrak{B}_1) \xrightarrow{\cong} \mathcal{Q}(\mathfrak{B}_2)$ be the induced isomorphism of corona algebras. Then $\overline{\beta} \circ \tau_1(1_{\mathfrak{A}_1}) = \tau_2(1_{\mathfrak{A}_2})$ if and only if there is a $*$-homomorphism $\mu : \mathfrak{D}_{\mathfrak{e}_1} \to \mathfrak{D}_{\mathfrak{e}_2}$ (necessarily unique and necessarily an isomorphism) making the diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{B}_1 \arrow[r] \arrow[d, "\beta"] & \mathfrak{D}_{\mathfrak{e}_1} \arrow[r] \arrow[d, "\mu"] & \mathbb{C} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{B}_2 \arrow[r] & \mathfrak{D}_{\mathfrak{e}_2} \arrow[r] & \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

commute.

Proof. The Busby maps of the extensions in the above diagram are $\mathbb{C} \ni \lambda \mapsto \lambda\tau_i(1_{\mathfrak{A}_i})$ so the result follows immediately from [**10**, Theorem 2.2]. $\square$

When considering the ordered K-theory $K_*^+(\mathfrak{A}) = (K_0^+(\mathfrak{A}), K_1(\mathfrak{A}))$ for *unital* C^*-algebras $\mathfrak{A}$, we will often add the class of the unit to the invariant

$$K_*^{+,u}(\mathfrak{A}) := (K_0^+(\mathfrak{A}), [1_{\mathfrak{A}}]_0, K_1(\mathfrak{A})).$$

Alternatively, we may consider the unital embedding $j : \mathbb{C} \hookrightarrow \mathfrak{A}$. This gives an induced diagram

$$j_* : K_*^+(\mathbb{C}) \to K_*^+(\mathfrak{A}). \tag{7.2}$$

This diagram contains exactly the same information as $K_*^{+,u}(\mathfrak{A})$, thus motivating the following construction.

Suppose $\mathfrak{e} : 0 \to \mathfrak{B} \xrightarrow{\iota} \mathfrak{E} \xrightarrow{\pi} \mathfrak{A} \to 0$ is an extension of C^*-algebras for which $\mathfrak{A}$ is unital, but where $\mathfrak{E}$ is not necessarily unital. We assume for convenience that $1_{\mathfrak{A}}$ lifts to a projection in $\mathfrak{E}$.

Again, we have a unital embedding $j : \mathbb{C} \hookrightarrow \mathfrak{A}$, and we obtain the following pull-back diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{B} \arrow[r] \arrow[d, equal] & \mathfrak{D}_{\mathfrak{e}} \arrow[r] \arrow[d, hook] & \mathbb{C} \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & \mathfrak{B} \arrow[r, "\iota"] & \mathfrak{E} \arrow[r, "\pi"] & \mathfrak{A} \arrow[r] & 0,
\end{tikzcd}$$

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where $\mathfrak{D}_{\mathfrak{e}}$ is as in Notation 7.2. Our invariant will be to apply K-theory with order and scale to this diagram, thus obtaining the following commutative diagram:

$$\begin{tikzcd}
K_0^{+,\Sigma}(\mathfrak{B}) \arrow[r] \arrow[d, equal] & K_0^{+,\Sigma}(\mathfrak{D}_{\mathfrak{e}}) \arrow[r] \arrow[d] & K_0^{+,\Sigma}(\mathbb{C}) \arrow[d, "j_0"] \\
K_0^{+,\Sigma}(\mathfrak{B}) \arrow[r, "\iota_0"] & K_0^{+,\Sigma}(\mathfrak{E}) \arrow[r, "\pi_0"] & K_0^{+,\Sigma}(\mathfrak{A}) \arrow[d, "\delta_0"] \\
K_1(\mathfrak{A}) \arrow[u, "\delta_1"] & K_1(\mathfrak{E}) \arrow[l, "\pi_1"'] & K_1(\mathfrak{B}). \arrow[l, "\iota_1"']
\end{tikzcd}$$

We denoted this diagram by $\widetilde{K}_{\text{six}}^{+,\Sigma}(\mathfrak{e})$. Homomorphisms between such diagrams are defined in the obvious way.

Suppose that $\mathfrak{e}_i\colon 0 \to \mathfrak{B}_i \to \mathfrak{E}_i \to \mathfrak{A}_i \to 0$ are extensions of C^*-algebras for $i = 1, 2$ with $\mathfrak{A}_i$ unital. Suppose that there is a commutative diagram

$$\begin{tikzcd}
\mathfrak{e}_1 : & 0 \arrow[r] & \mathfrak{B}_1 \arrow[r] \arrow[d, "\beta"] & \mathfrak{E}_1 \arrow[r] \arrow[d, "\eta"] & \mathfrak{A}_1 \arrow[r] \arrow[d, "\alpha"] & 0 \\
\mathfrak{e}_2 : & 0 \arrow[r] & \mathfrak{B}_2 \arrow[r] & \mathfrak{E}_2 \arrow[r] & \mathfrak{A}_2 \arrow[r] & 0
\end{tikzcd}$$

where all maps are $*$-homomorphisms, and α is unital. Then $\eta(\mathfrak{D}_{\mathfrak{e}_1}) \subseteq \mathfrak{D}_{\mathfrak{e}_2}$ and thus it easily follows that (β, η, α) induces a homomorphism $\widetilde{K}_{\text{six}}^{+,\Sigma}(\mathfrak{e}_1) \to \widetilde{K}_{\text{six}}^{+,\Sigma}(\mathfrak{e}_2)$.

In the cases we will be considering below, we assume that $\mathfrak{A}$ is a unital UCT Kirchberg algebra, $\mathfrak{B}$ is a stable AF algebra, and $\mathfrak{E}$ contains a full, properly infinite projection. Hence, the order and scale can be ignored in $K_0(\mathfrak{E})$ and $K_0(\mathfrak{A})$, and the scale of $K_0(\mathfrak{B})$ can be ignored when considering $\widetilde{K}_{\text{six}}^{+,\Sigma}(\mathfrak{e})$.

We obtain our final classification result which is exactly Theorem B.

Theorem 7.4. *Let $\mathfrak{e}_i : 0 \to \mathfrak{B}_i \to \mathfrak{E}_i \to \mathfrak{A}_i \to 0$ be full extensions of C^*-algebras for $i = 1, 2$, such that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are* unital *UCT Kirchberg algebras, and $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are stable AF algebras. Then $\mathfrak{E}_1 \cong \mathfrak{E}_2$ if and only if $\widetilde{K}_{\text{six}}^{+,\Sigma}(\mathfrak{e}_1) \cong \widetilde{K}_{\text{six}}^{+,\Sigma}(\mathfrak{e}_2)$.*

Proof. Suppose $\mathfrak{E}_1 \cong \mathfrak{E}_2$. As the extension $\mathfrak{e}_i$ is full, as $\mathfrak{A}_i$ is simple and $\mathfrak{B}_i$ is stable, it follows that $\mathfrak{B}_i$ is the unique maximal ideal in $\mathfrak{E}_i$ for $i = 1, 2$ (see Footnote 12). It follows that the extensions $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are isomorphic, and thus $\widetilde{K}_{\text{six}}^{+,\Sigma}(\mathfrak{e}_1) \cong \widetilde{K}_{\text{six}}^{+,\Sigma}(\mathfrak{e}_2)$.

For the converse, suppose that $\widetilde{K}_{\text{six}}^{+,\Sigma}(\mathfrak{e}_1) \cong \widetilde{K}_{\text{six}}^{+,\Sigma}(\mathfrak{e}_2)$, and let

$$\begin{aligned}
\phi_*\colon K_*^{+,\Sigma}(\mathfrak{A}_1) &\xrightarrow{\cong} K_*^{+,\Sigma}(\mathfrak{A}_2), & \psi_*\colon K_*^{+,\Sigma}(\mathfrak{B}_1) &\xrightarrow{\cong} K_*^{+,\Sigma}(\mathfrak{B}_2), \\
\rho_*\colon K_*^{+,\Sigma}(\mathfrak{E}_1) &\xrightarrow{\cong} K_*^{+,\Sigma}(\mathfrak{E}_2), & \theta_0\colon K_0^{+,\Sigma}(\mathfrak{D}_{\mathfrak{e}_1}) &\xrightarrow{\cong} K_0^{+,\Sigma}(\mathfrak{D}_{\mathfrak{e}_2})
\end{aligned}$$

be a collection of isomorphisms inducing the isomorphism on $\widetilde{K}_{\text{six}}^{+,\Sigma}$. We first show that we may assume that $\mathfrak{A} = \mathfrak{A}_1 = \mathfrak{A}_2$, $\mathfrak{B} = \mathfrak{B}_1 = \mathfrak{B}_2$, $\phi_* = \mathrm{id}_{K_*(\mathfrak{A})}$, $\psi_* = \mathrm{id}_{K_*(\mathfrak{B})}$, that $\tau_1(1_{\mathfrak{A}}) = \tau_2(1_{\mathfrak{A}})$, where τ_i is the Busby map of $\mathfrak{e}_i$ for $i = 1, 2$, and that $\theta_0 = K_0(\mu)$ where $\mu\colon \mathfrak{D}_{\mathfrak{e}_1} \to \mathfrak{D}_{\mathfrak{e}_2}$ is the isomorphism provided by Lemma 7.3.

By the Kirchberg–Phillips theorem [**19**], [**26**], we may pick an isomorphism $\alpha\colon \mathfrak{A}_1 \xrightarrow{\cong} \mathfrak{A}_2$ such that $K_*(\alpha) = \phi_*$.

As $\mathfrak{D}_{\mathfrak{e}_i}$ is an extension of two AF algebras, it is itself an AF algebra by [**8**, Chapter 9]. Hence, by Elliott's classification of AF algebras [**13**] we may pick an isomorphism

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$\mu\colon \mathfrak{D}_{\mathfrak{e}_1} \xrightarrow{\cong} \mathfrak{D}_{\mathfrak{e}_2}$ such that $K_0(\mu)=\theta_0$. In particular, μ restricts to an isomorphism $\beta\colon \mathfrak{B}_1 \xrightarrow{\cong} \mathfrak{B}_2$ satisfying $K_0(\beta)=\psi_0$.

Forming the push-out extension $\beta\cdot\mathfrak{e}_1$ and the pull-back extension $\mathfrak{e}_2\cdot\alpha$, we obtain a diagram identical to (5.6). By Lemma 7.3, we get

$$\overline{\beta}\circ\tau_1(1_{\mathfrak{A}_1})=\tau_2(1_{\mathfrak{A}_2})=\tau_2\circ\alpha(1_{\mathfrak{A}_1}).$$

Let

$$\mu^{(1)}\colon \mathfrak{D}_{\mathfrak{e}_1} \xrightarrow{\cong} \mathfrak{D}_{\beta\cdot\mathfrak{e}_1}, \qquad \mu^{(2)}\colon \mathfrak{D}_{\mathfrak{e}_2\cdot\alpha} \xrightarrow{\cong} \mathfrak{D}_{\mathfrak{e}_2}$$

be the induced isomorphisms, i.e. the restriction–corestriction of $\eta^{(1)}$ and $\eta^{(2)}$, respectively.

Now, it follows from (5.6) (by inverting the isomorphisms) that we obtain induced isomorphisms $\widetilde{K}^{+,\Sigma}_{\mathrm{six}}(\beta\cdot\mathfrak{e}_1) \xrightarrow{\cong} \widetilde{K}^{+,\Sigma}_{\mathrm{six}}(\mathfrak{e}_1)$ and $\widetilde{K}^{+,\Sigma}_{\mathrm{six}}(\mathfrak{e}_2) \xrightarrow{\cong} \widetilde{K}^{+,\Sigma}_{\mathrm{six}}(\mathfrak{e}_2\cdot\alpha)$. By composing these isomorphisms with the already given isomorphism $\widetilde{K}^{+,\Sigma}_{\mathrm{six}}(\mathfrak{e}_1) \xrightarrow{\cong} \widetilde{K}^{+,\Sigma}_{\mathrm{six}}(\mathfrak{e}_2)$, it follows that the compositions

$$\begin{aligned} K_*(\alpha)^{-1}\circ\phi_*\circ K_*(\mathrm{id}_{\mathfrak{A}_1})^{-1}&=\mathrm{id}_{K_*(\mathfrak{A}_1)}, & K_*(\mathrm{id}_{\mathfrak{B}_2})^{-1}\circ\psi_*\circ K_*(\beta)^{-1}&=\mathrm{id}_{K_*(\mathfrak{B}_2)}\\ K_*(\eta^{(2)})^{-1}\circ\rho_*\circ K_*(\eta^{(1)})^{-1}&, & K_0(\mu^{(2)})^{-1}\circ\theta_0\circ K_0(\mu^{(1)})^{-1}& \end{aligned}$$

give rise to an isomorphism $\widetilde{K}^{+,\Sigma}_{\mathrm{six}}(\beta\cdot\mathfrak{e}_1) \xrightarrow{\cong} \widetilde{K}^{+,\Sigma}_{\mathrm{six}}(\mathfrak{e}_2\cdot\alpha)$. Moreover, observe that $\mu^{(0)}:=(\mu^{(2)})^{-1}\circ\mu\circ(\mu^{(1)})^{-1}$ is the unique (by Lemma 7.3) $*$-homomorphism making the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{B}_2 \arrow[r] \arrow[d, equal] & \mathfrak{D}_{\beta\cdot\mathfrak{e}_1} \arrow[r] \arrow[d, "\mu^{(0)}"] & \mathbb{C} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathfrak{B}_2 \arrow[r] & \mathfrak{D}_{\mathfrak{e}_2\cdot\alpha} \arrow[r] & \mathbb{C} \arrow[r] & 0 \end{tikzcd}$$

commute, and that $K_0(\mu^{(0)})=K_0(\mu^{(2)})^{-1}\circ\theta_0\circ K_0(\mu^{(1)})^{-1}$.

Therefore, without loss of generality, we may assume that $\mathfrak{A}=\mathfrak{A}_1=\mathfrak{A}_2$, $\mathfrak{B}=\mathfrak{B}_1=\mathfrak{B}_2$, $\phi_*=\mathrm{id}_{K_*(\mathfrak{A})}$, $\psi_*=\mathrm{id}_{K_*(\mathfrak{B})}$ that $\tau_1(1_{\mathfrak{A}})=\tau_2(1_{\mathfrak{A}})$, and that that $\theta_0=K_0(\mu)$ where $\mu\colon \mathfrak{D}_{\mathfrak{e}_1}\to\mathfrak{D}_{\mathfrak{e}_2}$ is the map provided by Lemma 7.3 (with $\beta=\mathrm{id}_{\mathfrak{B}}$).

As $\mathfrak{B}$ has real rank zero and $K_1(\mathfrak{B})=0$, the projection $\tau_1(1_{\mathfrak{A}})\in\mathscr{Q}(\mathfrak{B})$ lifts to a projection $p\in\mathscr{M}(\mathfrak{B})$ by [4, Corollary 3.16]. In particular, by identifying $\mathfrak{E}_i$ with $\mathfrak{A}\oplus_{\tau_i,\pi_{\mathfrak{B}}}\mathscr{M}(\mathfrak{B})$ in the canonical way, $1_{\mathfrak{A}}\oplus p$ defines a projection both in $\mathfrak{E}_1$ and in $\mathfrak{E}_2$ since $\tau_1(1_{\mathfrak{A}})=\tau_2(1_{\mathfrak{A}})$. Note that when identifying $\mathfrak{D}_{\mathfrak{e}_1}$ and $\mathfrak{D}_{\mathfrak{e}_2}$ in a canonical way with a subalgebra of $\mathfrak{A}\oplus\mathscr{M}(\mathfrak{B})$, then μ is simply the identity map. Hence, $\mu(1_{\mathfrak{A}}\oplus p)=1_{\mathfrak{A}}\oplus p$. In particular, by commutativity of the diagram

$$\begin{tikzcd} K_0(\mathfrak{D}_{\mathfrak{e}_1}) \arrow[r] \arrow[d, "\theta_0=K_0(\mu)"] & K_0(\mathfrak{E}_1) \arrow[d, "\rho_0"] \\ K_0(\mathfrak{D}_{\mathfrak{e}_2}) \arrow[r] & K_0(\mathfrak{E}_2), \end{tikzcd}$$

which is part of $\widetilde{K}^{+,\Sigma}_{\mathrm{six}}(\mathfrak{e}_1)\to\widetilde{K}^{+,\Sigma}_{\mathrm{six}}(\mathfrak{e}_2)$, it follows that $\rho_0([1_{\mathfrak{A}}\oplus p])=[1_{\mathfrak{A}}\oplus p]$.

By Theorem 6.5 it follows that $1_{\mathfrak{A}}\oplus p$ is a full, properly infinite projection in both $\mathfrak{E}_1$ and $\mathfrak{E}_2$. Moreover, $p\mathfrak{B}p$ is a full and stable corner in $\mathfrak{B}$ by Lemma 6.1. Let

$$\iota\colon p\mathfrak{B}p\hookrightarrow\mathfrak{B}, \qquad \iota_i\colon (1_{\mathfrak{A}}\oplus p)\mathfrak{E}_i(1_{\mathfrak{A}}\oplus p)\hookrightarrow\mathfrak{E}_i$$

for $i=1,2$ denote the inclusions, which are all inclusions of full, hereditary, C^*-subalgebras in separable C^*-algebras and thus induce isomorphisms in K-theory. Since $\rho_0([1_{\mathfrak{A}} \oplus p]) = [1_{\mathfrak{A}} \oplus p]$ it follows that the map

$$K_*(\iota_2)^{-1} \circ \rho_* \circ K_*(\iota_1)\colon K_*((1_{\mathfrak{A}} \oplus p)\mathfrak{E}_1(1_{\mathfrak{A}} \oplus p)) \to K_*((1_{\mathfrak{A}} \oplus p)\mathfrak{E}_2(1_{\mathfrak{A}} \oplus p))$$

induces a congruence $K_{\mathrm{six}}^{\mathrm{u}}(p\mathfrak{e}_1 p) \equiv K_{\mathrm{six}}^{\mathrm{u}}(p\mathfrak{e}_2 p)$, where $p\mathfrak{e}_i p$ denotes the unital extension

$$0 \to p\mathfrak{B}p \to (1_{\mathfrak{A}} \oplus p)\mathfrak{E}_1(1_{\mathfrak{A}} \oplus p) \to \mathfrak{A} \to 0$$

for $i=1,2$.

Thus, by Proposition 5.8 there is an automorphism $\alpha \in \mathrm{Aut}(\mathfrak{A})$ such that $p\mathfrak{e}_1 p$ and $p\mathfrak{e}_2 p \cdot \alpha = p(\mathfrak{e}_2 \cdot \alpha)p$ are strongly unitarily equivalent. By Lemma 7.1 it follows that $\mathfrak{e}_1$ and $\mathfrak{e}_2 \cdot \alpha$ are strongly unitarily equivalent. As the extension algebra of $\mathfrak{e}_2 \cdot \alpha$ is isomorphic to $\mathfrak{E}_2$, it follows that $\mathfrak{E}_1 \cong \mathfrak{E}_2$ as desired. □

REMARK 7.5. In a future paper [9], we compute the range of the invariant $\widetilde{K}_{\mathrm{six}}^{+,\Sigma}$ for graph C^*-algebras with a unique, nontrivial ideal. This will be used to show that an extension of two simple graph C^*-algebras is again a graph C^*-algebra, provided that there are no K-theoretic obstructions.

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