J. Austral. Math. Soc. **23** (Series A) (1977), 402–415.

THE BOOLEAN RING UNIVERSAL OVER A MEET SEMILATTICE

ELLIOTT EVANS

(Received 17 March 1976)

Abstract

The Boolean ring $B[M]$ universal over a meet semilattice M is examined. It is the vector space over the two element field $\mathbf{Z}_2$ with base $M\backslash\{0\}$. The $\mathbf{Z}_2$ linear independence of a meet subsemilattice of a Boolean ring is characterized in order theoretic terms and some ramifications of this on $B[M]$ are considered. The space $\mathcal{F}_p(M)$ of proper filters of M is shown homeomorphic to the Stone space $S(B[M])$ of $B[M]$ if M has no least element, with $\mathcal{F}_p(M) \cup \{M\}$ and $S(B[M])$ homeomorphic otherwise. The congruence lattice $\theta(M)$ of M is compared to the ideal lattice $\mathcal{I}(B[M])$ of $B[M]$ with best results coming if M is a tree with zero when $\theta(M) \cong \mathcal{I}(B[M])$.

1. Introduction

Suppose M is a meet subsemilattice of a Boolean ring B such that if M has a least element then $0_B \in M$. The Boolean ring B is said to be universal over M if the following condition holds: for each Boolean ring R and each meet homomorphism $\psi: M \to R$ mapping M's least element to zero (if M has one), there is exactly one ring homomorphism $\phi: B \to R$ extending ψ. This condition determines B up to M-isomorphism and this B universal over M will be denoted $B[M]$. It is natural to ask how $B[M]$ is built out of M and in general how properties of one of M or $B[M]$ are reflected in the other. It is the purpose of this paper to examine certain of the relationships occuring between M and $B[M]$.

The motivation for this work is twofold: the Mostowski and Tarski (1939) examination of those Boolean rings with a chain basis and the MacNeille (1939) study of Boolean extensions of distributive lattices. Also influential here is the recent paper of Byrd, Mena and Troy (1975) on Boolean rings generated by distributive lattices. The results and techniques of the latter authors are quite similar to those here; the use of these authors' notion of evenly generated ideal resulted in an improvement in the results of Section 3 (making them apply to semilattice trees rather than just those with a zero).

Many of the notions here (universal object, extension and contraction of congruences etc.) can be phrased in the general context of universal algebra, but it is felt here that the majority of the results presented depend on the specific context and so it is in that specific context we remain.

In section 1 we build and characterize $B[M]$ finding that $M\backslash\{0\}$ is a $\mathbf{Z}_2$ vector space base of $B[M]$. (Here $\mathbf{Z}_2$ always denotes the two element field). Hence $B[M]$ is almost the semigroup algebra of M over the field $\mathbf{Z}_2$ (see Clifford and Preston (1961) page 159), and is exactly this if M has no least element. Also the $\mathbf{Z}_2$ linear independence of $M\backslash\{0\}$ in $B[M]$ is characterized in order theoretic terms. Section 2 is mostly a generalization of a result of Mostowski and Tarski (1939) who established a one-one correspondence between the upper ends of a chain C and primes of $B[C]$. We compare the space of filters of M to the Stone space of $B[M]$. In section 3 we use the classical technique of extension and contraction of congruences to compare the congruence lattice $\theta(M)$ of M to the ideal lattice $\mathcal{I}(B[M])$ of $B[M]$. Here we use the evenly generated ideal E_M of $B[M]$ and find that $\theta(M) \cong \mathcal{I}(E_M)$ if and only if M is a semilattice tree, meaning that for each $m \in M$, $(m] = \{y \in M \mid y \leqq m\}$ is a chain. It turns out that $E_M = B[M]$ if M has a least element. So if M is a semilattice tree with a least element then $\theta(M) \cong \mathcal{I}(B[M])$.

Frequently used here are certain properties of Boolean rings which we now list. Let B be a Boolean ring, $x_1, \cdots, x_n$, a, b all elements of B and P a prime ideal of B. Then

(i) $x_1 + \cdots + x_n \leqq x_1 \vee \cdots \vee x_n$,

(ii) if $a \cdot b = 0$ then $a + b = a \vee b$, and

(iii) one of a, b, or $a + b$ is in P.

2. The Boolean rings universal over a meet semilattice

Let B be a Boolean ring, $M \subseteq B$. We call M an admissible subsemilattice of B if and only if M is a meet subsemilattice of B which has a least element only in case 0 is in M. Each meet semilattice is an admissible subsemilattice of some Boolean ring, a fact that can be seen by the following construction.

PROPOSITION 2.1. *Suppose M is a meet semilattice. Then there is a Boolean ring B so that* (i) *M is an admissible subsemilattice of B and* (ii) *$M\backslash\{0\}$ is a $\mathbf{Z}_2$ vector space base for B.*

PROOF. This is almost the same as constructing the semigroup algebra $\mathbf{Z}_2[M]$ of the semigroup M over the field $\mathbf{Z}_2$ (see Clifford and Preston (1961), page 159). Let B be the $\mathbf{Z}_2$ vector space whose base is the set of nonzero

https://doi.org/10.1017/S1446788700019558 Published online by Cambridge University Press

elements of M (i.e. if M has a least element throw it out, otherwise leave M alone). If M has a least element identify it with the zero of B. We can view $M \subseteq B$. The meet operation on M extends to an associative, bilinear multiplication on B under which B becomes a Boolean ring. The described properties of B then follow easily. Note that if M has no least element then B is actually $\mathbf{Z}_2[M]$.

For any Boolean ring B and any meet homomorphism $\phi: M \to B$, call ϕ admissible if whenever M has a least element m_0 then $\phi(m_0)$ is the zero of the ring B. The ring constructed in 2.1 is significant because of the next result.

PROPOSITION 2.2. *Let M be a meet semilattice, B a Boolean ring, $\phi: M \to B$ admissible. The following statements are equivalent*:

(i) *ϕ is the universal admissible map into a Boolean ring (i.e. if R is a Boolean ring and $\psi: M \to R$ is admissible then there is exactly one ring homomorphism $\sigma: B \to R$ so that $\sigma \circ \phi = \psi$).*

(ii) *$(\phi(m) \mid m \in M$, m not least in $M)$ is a basis for B as a $\mathbf{Z}_2$ vector space.*

(iii) *$(\phi(m) \mid m \in M$, m not least in $M)$ is $\mathbf{Z}_2$ linearly independent in B and ring generates B.*

PROOF. Since $\phi[M]$ is a subsemilattice of B and B is a Boolean ring, (ii) and (iii) are certainly equivalent. We show (i) equivalent to (ii). Assume (ii) holds. Note that because of (ii), ϕ is necessarily one-one. Hence we can view $M \subseteq B$, B a vector space with basis $M \backslash \{0\}$. (Actually ϕ is an order embedding). Let R be any Boolean ring and $\psi: M \to R$ admissible. Now there is exactly one $\mathbf{Z}_2$ linear map $\sigma: B \to R$ so that $\sigma \circ \phi = \psi$. But σ actually preserves the multiplication in B of elements of $\phi[M]$; that is, if $m, n \in M$, $\sigma(\phi m \cdot \phi n) = \sigma(\phi(mn)) = \psi(mn) = \psi m \psi n = \sigma(\phi m) \cdot \sigma(\phi n)$. So σ preserves the multiplication on a generating subset of B. Hence σ preserves multiplication and so is a ring homomorphism so that $\sigma \circ \phi = \psi$. Its uniqueness is clear. Finally by 2.1 and the standard techniques we get (i) implies (ii).

NOTE. We will call the Boolean ring described in 2.2 the Boolean ring universal over M and write $B[M]$ to denote it. The conditions of 2.2 imply that the map $\phi: M \to B[M]$ is an admissible order embedding. The universal Boolean ring over M can then be characterized as a Boolean ring B wherein M is an admissible subsemilattice whose nonzero element form a $\mathbf{Z}_2$ vector space basis for B. Every Boolean ring is a $\mathbf{Z}_2$ vector space and so has a $\mathbf{Z}_2$ vector space basis; our interest is in those with a multiplicatively closed basis. Our goal now is to translate $\mathbf{Z}_2$ linear independence of an admissible subsemilattive into some order theoretic statement.

PROPOSITION 2.3. *Let M be an admissible subsemilattice of the Boolean* *ring B. Then $M\backslash\{0\}$ is $\mathbf{Z}_2$ linearly independent in B if and only if M satisfies the following condition in B:*

$$\text{LI} \qquad \text{if } m_1, \cdots, m_k,\ m \in M \text{ and if each } m_i < m \text{ then } \bigvee_{i=1}^{k} m_i < m.$$

Actually the "if" part does not require the admissibility of M in B.

PROOF. For the "if" part suppose M is just a meet subsemilattice of B satisfying LI. Assume by way of contradiction, that $M\backslash\{0\}$ is not linearly independent. Then there is a $\mathbf{Z}_2$ linear combination:

$$\sum_{\substack{m \in M \\ m \neq 0}} \lambda_m \cdot m = 0 \quad \text{where} \quad \lambda_m \in \mathbf{Z}_2, \quad a.a.\lambda_m = 0,$$

where not all λ_m's are zero. Note that there are no repetitions of elements of M in this sum. Let $m_1, \cdots, m_k$ be the nonzero elements of M whose coefficients in the above sum are not zero. Since our field of scalars is $\mathbf{Z}_2$ the above linear combination now reads:

$$m_1 + \cdots + m_k = 0$$

with $m_i \neq m_j$ if $i \neq j$. Without loss of generality assume $k > 1$. Relabel the elements, if necessary, so that m_1 is maximal in the list $m_1, \cdots, m_k$. Multiplying the last equation through by m_1 and solving for m_1 we get

$$m_1 = m_1 \cdot m_2 + \cdots + m_1 \cdot m_k.$$

Each of $m_1m_2, m_1m_3, \cdots, m_1m_k$ is in M. If each of $m_1m_2, \cdots, m_1m_k$ is less than m_1 then condition LI gives:

$$m_1 = m_1m_2 + \cdots + m_1m_k \leqq \bigvee_{j=1}^{k} m_1m_j < m_1$$

a contradiction. Hence for some $j > 1$, $m_1m_j = m_1$ implying $m_1 \leqq m_j$. But m_1 is maximal among $m_1, \cdots, m_k$ so $m_1 = m_j$. But this is a contradiction. Thus condition LI forces $M\backslash\{0\}$ to be $\mathbf{Z}_2$ linearly independent in B.

Assume now M is an admissible subsemilattice of B. We show the "only if" part in the case M has a least element. The reader should make the appropriate adjustments in the other case. In $P(M) = \{D \mid D \subseteq M\}$ consider the interval $R = [\{0\}, M] = \{D \subseteq M \mid 0 \in D\}$. (Note: it is the admissibility of M that puts $0 \in M$.) The interval R is a Boolean lattice whose join and meet are set theoretic union and intersection respectively. Consider the map $\rho: M \to R$

given by $\rho(m)=(m]=\{y\in M\mid y\leqq m\}$. This ρ is an admissible map into a Boolean ring. Assume now that $M\backslash\{0\}$ is $\mathbf{Z}_2$ linearly independent in B. The subring of B generated by M, $\langle M\rangle_B$, is by 1.2 isomorphic to $B[M]$. Hence with $\rho: M\to R$ admissible, there is exactly one ring homomorphism $\psi:\langle M\rangle_B\to R$ extending ρ. Notice that for elements of $\langle M\rangle_B$ the join in $\langle M\rangle_B$ is the same as their join in B. If x, $y\in M_B$ then $\psi(x\vee y)=\psi(x+y+x\cdot y)=\psi x+\psi y+\psi x\cdot\psi y=\psi x\vee\psi y$ (latter is join in R) $=\psi x\cup\psi y$. Then for $x_1,\cdots,x_n$ in $\langle M\rangle_B$ we get

$$\bigvee_{i=1}^{n} x_i \quad\text{in}\quad \langle M\rangle_B \quad\text{and}\quad \psi\left(\bigvee_{i=1}^{n} x_i\right)=\bigcup_{i=1}^{n}\psi(x_i).$$

We now show M satisfies LI in B. Suppose not. Then there are elements $m_1,\cdots,m_k$, m in M with each $m_i<m$ but with their join in B, $\bigvee_{i=1}^{n} m_i=m$. But applying ψ to this last equality gives $\bigcup_{i=1}^{n}\psi m_i=\psi m$. Because ψ extends ρ this last equation is actually $\bigcup_{i=1}^{n}(m_i]=(m]$. This in turn forces $m\leqq m_i$ for some i which is a contradiction.

Comment. At this point we mention a simple consequence of independence which will be used often. Suppose $M\subseteq B$, B a Boolean ring. Suppose $M\backslash\{0\}$ is $\mathbf{Z}_2$ independent. Assume $m,m_1,\cdots,m_k\in M$, $m\neq 0$ and $m=m_1+\cdots+m_k$. Then $m=m_i$ for some $i\in\{1,\cdots,k\}$.

We mention some consequences of 2.3. First, proper joins that exist in M are lost in the transition to $B[M]$. To clarify: suppose m_1, m_2 are incomparable elements of M and suppose $m=\sup_M\{m_1,m_2\}$. Then because of LI it is clear that $m\neq\sup_{B[M]}\{m_1,m_2\}$. Thus $B[M]$ is a purely semilattice theoretic object.

For another application recall that a meet semilattice M is called a semilattice tree if for each $m\in M$ the set $(m]=\{y\in M\mid y\leqq m\}$ is a chain. We then have

Corollary 2.4. *Let T be a semilattice tree. The nonzero elements of T are $\mathbf{Z}_2$ linearly independent in any Boolean ring wherein T is a meet subsemilattice. Thus $B[T]$ is characterized by the conditions: it is a Boolean ring in which T is admissible and which T ring generates.*

As a final application of 2.3 we can now detect when $B[M]$ will have an identity.

Proposition 2.5. *For a meet semilattice M, the Boolean ring $B[M]$ has a largest element (is a Boolean lattice) if and only if M has finitely many maximal elements $u_1,\cdots,u_k$ so that each element $m\in M$ is dominated by at least one of these* (i.e. $M=\bigcup_{i=1}^{k}(u_i]$).

PROOF. Suppose first that $B[M]$ has a 1. This is uniquely expressible as a $\mathbf{Z}_2$ linear combination of $M\backslash\{0\}$. Let $m_1, \cdots, m_t$ be the elements of $M\backslash\{0\}$ whose coefficients in this linear combination are nonzero. So $1 = m_1 + \cdots + m_t$ with $m_i \neq m_j$ if $i \neq j$. Relabel $m_1, \cdots, m_t$ if necessary so that $m_1, m_2, \cdots, m_k$ are the elements of $\{m_1, \cdots, m_t\}$ which are maximal in this set. (So $k \leqq t$.) Then each m_j $(1 \leqq j \leqq t)$ is less than or equal to some m_i for $i \leqq k$. Let x be any element of M. Then $x = x \cdot 1 = x \cdot m_1 + x \cdot m_2 + \cdots + x \cdot m_t$. Since $M\backslash\{0\}$ is $\mathbf{Z}_2$ independent in $B[M]$, M satisfies LI in $B[M]$. If each of $x \cdot m_1, \cdots, x \cdot m_t$ were less than x then

$$x = x \cdot m_1 + \cdots + x \cdot m_t \leqq \bigvee_{i=1}^{t} x \cdot m_i < x,$$

a contradiction. So for some $i \in \{1, \cdots, t\}$, $x = x \cdot m_i$; so that $x \leqq m_i$. Therefore each element of M is dominated by one of $m_1, \cdots, m_t$ and so by one of $m_1, \cdots, m_k$. Hence $M = \bigcup_{i=1}^{k} (m_i]$ and it follows that $\{m_i, \cdots, m_k\}$ is the set of maximal elements of M.

To prove the converse suppose $M = \bigcup_{i=1}^{k}(u_i]$ where $u_1, \cdots, u_k$ are the maximal elements of M. We claim that their join in $B[M]$, $u_1 \vee \cdots \vee u_k$, is the maximum element of $B[M]$. Take any $m \in M$. Then $m \leqq u_i$ for some i hence $m \leqq u_1 \vee \cdots \vee u_k$ so that $m \cdot (u_1 \vee \cdots \vee u_k) = m$. So multiplication by $u_1 \vee \cdots \vee u_k$ of any element of M leaves the latter element fixed. Since $M\backslash\{0\}$ is a base for $B[M]$ it follows that for any $x \in B[M]$, $x \cdot (u_1 \vee \cdots \vee u_k) = x$. So our claim about $u_1 \vee \cdots \vee u_k$ is proven, and $B[M]$ has a 1.

3. Filters of M, primes of $B[M]$

Unless otherwise indicated B denotes $B[M]$. For any $x \in B$, x has a unique expression as a $\mathbf{Z}_2$ linear combination of nonzero elements of M,

$$x = \sum \lambda_m(x) \cdot m \qquad m \in M,\ m \neq 0$$

where $\lambda_m(x) \in \mathbf{Z}_2$ and almost all $\lambda_m(x) = 0$. Denote by $n(x)$ the number $\#\{m \in M \mid \lambda_m(x) \neq 0\}$. Notice that $n(0) = 0$, and $n(x) = 1$ if and only if $x \in M\backslash\{0\}$.

Of interest later is the set $P_0 = \{x \in B \mid n(x) \text{ is even}\}$. For later use the properties of P_0 are summarized here. First $0 \in P_0$ and if $m \in M$ with $m \neq 0$ then $m \notin P_0$. Also $P_0 + P_0 \subseteq P_0$ and P_0 is an ideal of B if and only if the nonzero elements of M form a filter in M. Thus if M has no least element P_0 is an ideal of B. Next, if P_0 is an ideal of B it is a prime ideal. Finally $M\backslash P_0 = M$ in the case M has no least element while $M\backslash P_0 = M\backslash\{0\}$ if M has a least element.

For any set X, $P(X) = \{D \mid D \subseteq X\}$ is a Boolean topological space under the topology of set theoretic order convergence (a net $(D_\lambda)_{\lambda \in \Lambda}$ converges to D if and only if for all $x \in X$: $x \in D$ implies that eventually $x \in D_\lambda$ while $x \notin D$ implies that eventually $x \notin D_\lambda$). For a Boolean ring B, $S(B) = \{P \mid P$ prime ideal of $B\}$ ($= \{P \mid P$ maximal ideal of $B\}$) inherits the above topology from $P(B)$. There is another formally different topology on $S(B)$, the spectral (or Zariski) topology, wherein a subset E of $S(B)$ is closed if and only if there is a subset D of B so that $E = \{P \in S(B) \mid D \subseteq P\}$. The lattice of ring ideals of B is order isomorphic to the lattice of subsets of $S(B)$ which are open in the spectral topology. The point to be made here is that for any Boolean ring B (even without 1) these topologies on $S(B)$ coincide. [The restriction of the power set's topology is generated by sets of the form $C_x = \{P \in S(B) \mid x \in P\}$ and $N_x = \{P \in S(B) \mid x \notin P\}$ for all $x \in B$. The spectral topology on $S(B)$ is generated by sets of the form N_x, $x \in B$. But for any $x \in B$, $C_x = \bigcup_{y > x} N_{x+y}$. Hence the topologies coincide.]

For any meet semilattice M, a filter F of M is a subset of M satisfying (i) $x, y \in F$ implies $x \wedge y \in F$ and (ii) $x \in F$, $x \leqq y$ imply $y \in F$. Let $\mathscr{F}(M) = \{F \mid F$ is a filter of $M\}$. Notice that ϕ and M are in $\mathscr{F}(M)$. (Whereas $S(B)$ has only proper ideals in it). The set $\mathscr{F}(M)$ becomes a topological space inheriting the topology of $P(M)$. We can easily establish a relation between the spaces $\mathscr{F}(M)$ and $S(B[M])$. For $P \in S(B[M])$ observe that $M \backslash P = \{x \in M \mid x \notin P\}$ is in $\mathscr{F}(M)$. So we get a mapping $\Phi: S(B[M]) \to \mathscr{F}(M)$ whereby $P \mapsto M \backslash P$. This map was first described by Mostowski and Tarski (1939) in the case that M is a chain. The next theorem, which sums up the facts about Φ, is a generalization of the results of Mostowski and Tarski.

THEOREM 3.1. *For any meet semilattice M the map $\Phi: S(B[M]) \to \mathscr{F}(M)$ given by $\Phi(P) = M \backslash P$ is a homeomorphism between $S(B[M])$ and* $\operatorname{im} \Phi$. *(The latter inherits its topology from $\mathscr{F}(M)$.) Furthermore each proper filter F of M* (i.e. *$F \neq \phi$, $F \neq M$) is in* $\operatorname{im} \Phi$.

PROOF. Write $B = B[M]$. We show first that Φ is one-one. Let $P, Q \in S(B)$ and suppose $M \backslash P = M \backslash Q$. For any $x \in B$ let $A(x)$ denote the statement: $x \in P$ if and only if $x \in Q$. Now is $n(x)$ is 0 or 1 then $A(x)$ surely holds. Suppose by way of contradiction that $P \neq Q$. Then choose x so that $A(x)$ fails and $n(x)$ is minimal making $A(x)$ fail. Note that $n(x) \geqq 1$. Write $x = m + y$ for $m \in M \backslash \{0\}$ and $y \in B$ with $n(y) < n(x)$ (also $n(m) = 1 < n(x)$). Since $A(x)$ fails we can assume $x \in P$, $x \notin Q$. Now $x \notin Q$, Q a prime ideal forces one of y or m to be in Q. But $A(m)$ and $A(y)$ hold so one of y or m is in P. But then $m + y = x \in P$ forces both y and m to be in P.

https://doi.org/10.1017/S1446788700019558 Published online by Cambridge University Press

But again $A(m)$, $A(y)$ holding puts both m and y in Q, so finally $x = m + y$ is in Q, a contradiction. Hence $P = Q$.

The continuity of Φ is apparent so we now show $\Phi\colon S(B[M]) \to \operatorname{im}\Phi$ is a homeomorphism. Suppose that for a net $(P_\lambda \mid \lambda \in \Lambda)$ in $S(B)$ the net $(M \backslash P_\lambda \mid \lambda \in \Lambda)$ converges in the topology of $\mathscr{F}(M)$ to $M \backslash P$ where $P \in S(B)$. We claim that $(P_\lambda \mid \lambda \in \Lambda)$ converges to P in the topology of $S(B)$. For each $x \in B$ let $E(x)$ denote the statement:

$$x \in P \text{ implies that eventually } x \in P_\lambda$$

and

$$x \notin P \text{ implies that eventually } x \notin P_\lambda.$$

We show $E(x)$ holds for all $x \in B$. Certainly $E(x)$ is true if $n(x) = 1$ (this is a consequence of $M \backslash P_\lambda \to M \backslash P$ in $\mathscr{F}(M)$). Suppose $E(x)$ fails for some $x \in B$. Take x in B with $n(x)$ minimal such that $E(x)$ fails. Then $n(x) > 1$ and we can write $x = m + y$ where $m \in M \backslash \{0\}$ and $n(y) < n(x)$, with $E(m)$, $E(y)$ both true.

Case 1. $x \in P$. Since it is not true that eventually $x \in P_\lambda$, then one of m or y must fail to be in P. But x being in P forces the other of m, y to fail to be in P. So eventually $m \notin P_\lambda$ and eventually $y \in P_\lambda$. But each P_λ is prime so eventually $x = m + y \in P_\lambda$. But this says $E(x)$ holds.

Case 2. $x \notin P$. Then one m or y must be in P while the other is not. So without loss of generality say $m \in P$, $y \notin P$. So eventually $m \in P_\lambda$ and $y \notin P_\lambda$. So eventually $x = m + y \notin P_\lambda$. But again we have $E(x)$.

In either case we reach a contradiction, that $E(x)$ holds. So $E(x)$ is true for all $x \in B$ and this gives $P_\lambda \to P$ in $S(B)$.

Finally we show each proper filter of M is in $\operatorname{im}\Phi$. Let F be a proper filter of M. We claim that the ideal I generated by $M \backslash F$ in B misses F. For otherwise there is an element f of F and elements $m_1, \cdots, m_t$ of $M \backslash F$ so that $f \leqq m_1 \vee \cdots \vee m_t$ and this gives $f = \bigvee_{i=1}^{t} (f \cdot m_i)$; the latter plus condition LI then force $f = f \cdot m_i$ for some i putting m_i into F, a contradiction. Now choose P to be an ideal of B maximal with respect to containing $M \backslash F$ and missing F. It is easy to show that $P \in S(B)$ and that $M \backslash P = F$. Hence $F = \Phi(P)$.

We now observe that M has a least element if and only if $\operatorname{im}\Phi$ consists precisely of the proper filters of M. If M has no least element, $\operatorname{im}\Phi$ consists of the nonempty filters, and the prime ideal of B mapping to the improper filter M is P_0.

Corollary 3.2. *Let M be any meet semilattice. If M has a least element*

https://doi.org/10.1017/S1446788700019558 Published online by Cambridge University Press

then $S(B[M])$ and $\mathscr{F}_p(M) = \{F \leqq M \mid F$ proper filter of $M\}$ are homeomorphic. If M has no least element then $S(B[M])$ and $\mathscr{F}_p(M) \cup \{M\}$ are homeomorphic.

4. Comparison of congruences between M and $B[M]$

We examine the natural question of how the congruences of a semilattice M compare to those of its universal Boolean ring $B[M]$. For this some notation is required: $\theta(M) = \{\sigma \mid \sigma$ is a meet congruence of $M\}$, $\theta(B) = \{\rho \mid \rho$ is a lattice (ring) congruence of $B\}$ and $\mathscr{I}(B) = \{J \mid J$ ring ideal of $B\}$. For $J \in \mathscr{I}(B)$ and $\rho \in \theta(B)$ we say ρ and J are associated if the following condition holds: $(x, y) \in \rho$ if and only if $x + y \in J$. The relation of being associated establishes an order isomorphism between $\theta(B)$ and $\mathscr{I}(B)$.

Each meet congruence σ of M has at least one extension to a (ring) congruence of $B[M]$, namely to σ^e, the $B[M]$ congruence generated by σ. Actually $\sigma^e = \cap\{\rho \mid \rho \in \theta(B)$ and $\sigma \leqq \rho\}$. Let $I(\sigma)$ denote the ideal of $B[M]$ associated with the congruence σ^e. It is not difficult to see that $I(\sigma)$ is the ideal generated by the set $\{m + n \mid m, n \in M$ and $m\sigma n\}$. We denote this by writing $I(\sigma) = (m + n \mid m, n \in M$ and $m\sigma n]$. We call σ^e the extension of σ to $B[M]$ and $I(\sigma)$ the ideal extension of σ.

Now starting with an ideal J of $B[M]$ and its associated ring congruence ρ, let $\sigma(J)$ denote $\rho \cap (M \times M) = \rho^c$ (ρ contracted to M), which is a meet congruence of M, and call $\sigma(J)$ the contraction of J (or ρ) to M. Certainly $\sigma(J) = \{(m, n) \mid m, n \in M$ and $m + n \in J\}$.

We call a congruence σ of M contracted if for some $J \in \mathscr{I}(B[M])$, $\sigma = \sigma(J)$. A congruence ρ of $B[M]$ is called extended if $\rho = \sigma^e$ for some σ in $\theta(M)$; while an ideal J of $B[M]$ is extended if $J = I(\sigma)$ for some σ in $\theta(M)$. If ρ and J are associated then ρ is extended if and only if J is extended.

The two maps: $\theta(M) \to \mathscr{I}(B[M])$ whereby $\sigma \mapsto I(\sigma)$ and $\mathscr{I}(B[M]) \to \theta(M)$ whereby $J \mapsto \sigma(J)$ form a Galois connexion of mixed type. Namely for any $\sigma \in \theta(M)$ and any $J \in \mathscr{I}(B[M])$ we have:

$$\sigma \leqq \sigma(J) \text{ if and only if } I(\sigma) \leqq J.$$

As a consequence of this fact all the following statements hold:

(i) The map $\theta(M) \to \mathscr{I}(B[M])$ is completely join preserving (and so order preserving). Extension preserves arbitrary joins.

(ii) The map $\mathscr{I}(B[M]) \to \theta(M)$ is completely meet preserving (and so order preserving). Contraction preserves arbitrary meets.

(iii) The map $\sigma \mapsto \sigma(I(\sigma))$ is a closure operator in $\theta(M)$ whose closure family of fixed elements is the collection of contracted congruences.

(iv) The map $J \mapsto I(\sigma(J))$ is a kernel operator in $\mathscr{I}(B[M])$ whose kernel family of fixed elements is the collection of extended ideals.

https://doi.org/10.1017/S1446788700019558 Published online by Cambridge University Press

(v) The complete lattice of contracted congruences is order isomorphic to the complete lattice of extended ideals under the restriction of our mappings of extension and contraction. These restrictions are inverse to one another.

The previous results are general consequences of any Galois connexion (see Schmidt (1953, section 8)). But in our specific case we can say more. Using the universal property of $B[M]$ it is not difficult to show that for any congruence σ of M:

$$B[M/\sigma] = B[M]/\sigma^e$$

and so $(\sigma^e)^c = \sigma$. Thus:

PROPOSITION 4.1. *For each meet semilattice M each congruence of M is contracted. So for any $\sigma \in \theta(M)$, $\sigma = \sigma(I(\sigma)) = (\sigma^e)^c$. Hence $\theta(M)$ is order isomorphic to the lattice of extended ideals.*

We turn now to the other side of the coin to examine the extended ideals of $B[M]$. Write B for $B[M]$. For any ideal J of B let $D(J) = \{m + n \mid m, n \in M \text{ and } m + n \in J\}$. We have seen that $I(\sigma(J)) = (D(J)]$ (the ideal generated by the set $D(J)$). Certainly $(D(J)]$ consists of all finite sums of elements of $D(J)$. Hence an ideal J is extended if and only if $J = \{x \mid x$ is a finite sum of elements of $D(J)\}$.

There is a largest extended ideal, namely $I(M \times M) = I(\sigma(B)) = (D(B)]$, which consists of all finite sums of elements of the form $m + n$ where $m, n \in M$. Following Byrd, Mena and Troy (1975) we will call this ideal the ideal evenly generated by M and denote it E_M. Though for the above authors M was a distributive sublattice of B generating B, their results are analogous to what we find here. For our next lemma, which summarizes the easily established facts about E_M, we remind the reader of the notation $n(x)$ (for $x \in B$) and the set $P_0 = \{x \mid n(x) \text{ is even}\}$, which were introduced in section 2.

LEMMA 4.2. *For any semilattice M:* (a) $P_0 \leqq E_M$, (b) $E_M = B$ *if and only if* $0 \in M$ *(meaning M has a least element)*, (c) *if M has no least element, $E_M = P_0$ and is a prime ideal, and finally* (d) *the ideals of E_M are exactly the ideals of B contained in E_M. So extension-contraction is actually a Galois connexion between $\theta(M)$ and $\mathscr{I}(E_M)$.*

PROOF. For any ideal J of B, the ideals of J (viewed as a Boolean ring) are exactly the ideals of B contained in J. Hence (d) holds. Now (a) is obvious and (c) holds once (b) is true. So we show only (b). If $0 \in M$ then any m in M can be written $m = 0 + m$, the latter certainly in E_M. Hence $M \leqq E_M$ and we get $E_M = B$ in the case that $0 \in M$. Now suppose $0 \notin M$ and that $E_M = B$. Since M has no least element it is clear that $D(B) \leqq P_0$. But also P_0 is an ideal so $E_M = (D(B)] \leqq P_0$, hence $P_0 = B[M]$; but the latter is impossible. (We have not explicitly said it but we assume $M \neq \varnothing$. So choosing $m \in M$, m is not zero, because M has no least element, so $n(m) = 1$, forcing $m \notin P_0$.)

We emphasize the last statement of 4.2. All extended ideals of B are contained in E_M and so are ideals of the Boolean ring E_M. Also for each $\sigma \in \theta(M)$, $\sigma = \sigma(I(\sigma))$ where $I(\sigma) \in \mathscr{I}(E_M)$. This means that each M congruence is the contraction of an ideal of the ring E_M. We will hereafter treat our Galois connexion as between $\theta(M)$ and $\mathscr{I}(E_M)$, with all elements of $\theta(M)$ contracted.

Our interest now is in what conditions on M will make all ideals of E_M extended. If this happens then $\theta(M) \cong \mathscr{I}(E_M)$, which makes $\theta(M)$ distributive. So in light of the result of Papert (1964) we have: for each ideal of E_M to be extended it is necessary that M be a semilattice tree. We aim to show that M being a semilattice tree is sufficient to make all E_M ideals extended.

LEMMA 4.3. *Let B be any Boolean ring and let $x = b_1 + \cdots + b_k + y + z$ where $b_1, \cdots, b_k$, $y, z \in B$ and each $b_i \geqq y \geqq z$. Then if k is even $x = (b_1 + \cdots + b_k) \vee (y + z)$ while if k is odd then $x = (b_1 + \cdots + b_k + y) \vee z$.*

PROOF. In a Boolean ring, if $a \cdot b = 0$ then $a \vee bb = a + b$. If k is even then under the above hypotheses we have:

$$(b_1 + \cdots + b_k) \cdot (y + z) = (b_1 y + \cdots + b_k y) + (b_1 z + \cdots + b_k z)$$
$$= k \cdot y + k \cdot z = 0 + 0 = 0.$$

Hence

$$x = (b_1 + \cdots + b_k) + (y + z) = (b_1 + \cdots + b_k) \vee (y + z).$$

This proof for the k odd case is similar.

PROPOSITION 4.4. *Let M be a semilattice tree with least element. In $B[M]$ form the set $D = \{m + n \mid m, n \in M\}$. Then each element of $B[M]$ is the finite join of elements of D.*

PROOF. Let x be an element of $B = B[M]$ which is the sum of s nonzero elements of M and suppose for each $y \in B$: if y is the sum of fewer than s nonzero elements of M then y is the join of finitely many elements of D. Since $M \subseteq D$ we may as well assume $s \geqq 2$. Write $x = m_1 + \cdots + m_s$, $m_i \in M \backslash \{0\}$.

Let $i \in \{1, \cdots, s\}$. The set $\{m_1 m_i, m_2 m_i, \cdots, m_s m_i\}$ is a subset of M which is bounded above by m_i. Since M is a tree this set must be totally ordered. Let $m_t m_i$ be the least element of this set and choose $m_k m_i$ to be the least in $\{m_1 m_i, \cdots, m_{t-1} m_i, m_{t+1} m_i, \cdots, m_s m_i\}$. Then

$$m_i x = \sum_{j \neq t,k} m_j m_i + m_k m_i + m_t m_i.$$

If s is even the above lemma says that

$$m_i x = \Big(\sum_{j \neq t,k} m_j m_i \Big) \vee (m_k m_i + m_t m_i).$$

By the choice of s, $\Sigma_{j \neq t,k}\, m_j m_i$ is the finite join of elements of D. So apparently $m_i x$ is the finite join of elements of D. If s is odd then the above lemma, plus the fact that

$$m_i x = \sum_{j \neq t} m_j m_i + m_t m_i$$

give us:

$$m_i x = \Big(\sum_{j \neq t} m_j m_i \Big) \vee (m_t m_i).$$

Again the choice of s makes $\Sigma_{j \neq t}\, m_j m_i$ the finite join of elements of D and so with $m_t m_i$ in D we get $m_i x$ to be the finite join of elements of D.

Thus for each $i = 1, \cdots, s$, $x \cdot m_i$ is the finite join of elements of D. But $x = m_1 + \cdots + m_s \leqq m_1 \vee \cdots \vee m_s$ implies $x = (xm_1) \vee (xm_2) \vee \cdots \vee (xm_s)$ and so x itself is the finite join of elements of D.

COROLLARY 4.5. *Let M be a semilattice tree. Let*

$$D = \{a + b \mid a, b \in M \cup \{0\}\} \qquad (\textit{formed in } B[M]).$$

Then each element of $B[M]$ is the finite join of elements of D.

PROOF. If M has a least element this corollary is identical to 4.4. Suppose M has no least element. In $B[M]$ let $N = M \cup \{0\}$. Then N is a semilattice tree with a least element (0) and $B[M] = B[N]$. The claim of 4.5 then follows from 4.4 applied to N and $B[N]$.

We come to our main result.

PROPOSITION 4.6. *Let M be a meet semilattice. The following statements are equivalent*:

(i) *M is a semilattice tree*

(ii) *each ideal of E_M is extended.*

Hence $\theta(M) \cong \mathscr{I}(E_M)$ for any semilattice tree M. The congruences of a semilattice tree are order isomorphic to the congruences of some Boolean ring.

PROOF. We need only show (i) $\Rightarrow$ (ii). Let $J \in \mathscr{I}(E_M)$. Then $J \in \mathscr{I}(B[M])$ and $J \subseteq E_M$. We must show $J \subseteq I(\sigma(J))$. Let $j \in J$. By corollary 4.5 write $j = \bigvee_{i=1}^{n} d_i$ where each $d_i = a_i + b_i$ with $a_i, b_i \in M \cup \{0\}$. Notice that each $a_i + b_i \in J$. If M has a least element then each $a_i, b_i \in M$ and so $a_i + b_i \in \{m + n \mid m, n \in M$ and $m + n \in J\}$. Thus each $a_i + b_i \in I(\sigma(J))$ and so $j \in I(\sigma(J))$. If M has no least element then $E_M = P_0$ and each $a_i + b_i \in J \subseteq E_M$ forces each $a_i + b_i$ into P_0. Since $n(a_i + b_i)$ is even either both a_i, b_i are in M or both are zero. In either of these cases $a_i + b_i \in (m + n \mid m, n \in M, m + n \in J] \subseteq I(\sigma(J))$. So $j \in I(\sigma(J))$. So in any case $J \subseteq I(\sigma(J))$. The other statements follow easily.

Notice that if M is a tree with no least element then $\theta(M) \cong \mathscr{I}(P_0)$. If M is a tree with a least element then $E_M = B$, so we have the following.

Corollary 4.7. *Let T be a semilattice tree with a least element. Then $\theta(T) \cong \mathscr{I}(B[T])$ under the mappings of extension and contraction. So the compact congruences of T, $c(\theta(T))$, form a Boolean ring isomorphic to $B[T]$. Also $\theta(T)$ is isomorphic to the lattice of open subsets of the space $\mathscr{F}_p(T)$.*

Proof. Only the last statement needs clarification. For any topological space X let $\mathscr{O}(X)$ denote the lattice of open subsets of X. For any Boolean ring B we have:

$$\mathscr{I}(B) \cong \mathscr{O}(S(B))$$

and so

$$\mathscr{I}(B[T]) \cong \mathscr{O}(S(B[T])).$$

But $S(B[T])$ is homeomorphic to $\mathscr{F}_p(T)$ and so $\mathscr{O}(S(B[T])) \cong \mathscr{O}(F_p(T))$ and hence $\mathscr{I}(B[T]) \cong \mathscr{O}(\mathscr{F}_p(T))$. But $\theta(T) \cong \mathscr{I}(B[T])$ and so the last statement of the corollary follows.

We can rephrase the results of 4.6 in terms of congruences by passing from ideals of E_M to the associated congruences of the ring E_M. So if T is a semilattice tree then $\theta(T) \cong \theta(E_T)$ and if further T has a least element $\theta(T) \cong \theta(B[T])$.

Finally we have an application. Let T_1, T_2 be semilattice trees with a least element. If $\mathscr{F}_p(T_1)$ and $\mathscr{F}_p(T_2)$ are homeomorphic then their open set lattices are isomorphic and then it follows from 4.7 that $\theta(T_1) \cong \theta(T_2)$. On the other hand suppose $\theta(T_1) \cong \theta(T_2)$. Then the lattices of compact elements are isomorphic: $B[T_1] \cong B[T_2]$. The latter are then isomorphic as rings and so as $\mathbf{Z}_2$ vector spaces. It then follows that $S(B[T_1])$ and $S(B[T_2])$ are homeomorphic, hence $\mathscr{F}_p(T_1)$ is homeomorphic to $\mathscr{F}_p(T_2)$. But the isomorphic $\mathbf{Z}_2$ spaces $B[T_1]$ and $B[T_2]$ must have the same $\mathbf{Z}_2$ dimension so that $\# T_1 = \# T_2$. Summing up these observations we get:

Corollary 4.8. *Let T_1, T_2 be semilattice trees each with a least element. Then $\theta(T_1) \cong \theta(T_2)$ if and only if $\mathscr{F}_p(T_1)$ is homeomorphic to $\mathscr{F}_p(T_2)$. If this happens then $\# T_1 = \# T_2$.*

Acknowledgement

This is an adaptation of the dissertation done by the author under the direction of Professor J. Schmidt at the University of Houston.

References

R. D. Byrd, R. A. Mena and L. A. Troy (1975), 'Generalized Boolean Lattices', *J. Austral. Math. Soc.* **19**, 225–237.

A. H. Clifford and G. B. Preston (1961), *The Algebraic Theory of Semigroups*, v. 1 (American Mathematical Society, Providence).

H. M. MacNeille (1939), 'The Extension of a distributive lattice to a Boolean ring', *Bull. Amer. Math. Soc.* **45**, 452–455.

A. Mostowski and A. Tarski (1939), 'Boolesche Ringe mit Geordneter Basis', *Fund. Math.* **32**, 69–86.

D. Papert (1964), 'Congruence relations in Semilattices', *J. London Math. Soc.* **39**, 723–729.

J. Schmidt (1953), 'Beiträge sur Filtertheorie II', *Math. Nachr.* **10**, 197–232.

Department of Mathematics,
University of Tennessee,
Knoxville, 37916,
U.S.A.