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Stable finiteness does not imply linear soficity

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Abstract

We prove that there exist finitely generated, stably finite algebras which are not linear sofic. This was left open by Arzhantseva and Păunescu in 2017.

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1. *Introduction*

The question of whether a non-sofic group exists is one of the most tantalising open problems in the field of metric approximation properties with applications to various fields, including algebra, functional analysis, dynamics and more. Recall that a group is *sofic* if it can be approximated by almost homomorphisms to symmetric groups, equivalently, if it embeds into a metric ultraproduct of finite symmetric groups endowed with the normalised Hamming distance. Other important variants of soficity include *hyperlinearity*, in which the symmetric groups are replaced by unitary groups, endowed with the normalised Hilbert–Schmidt norm (this is closely related to Connes' Embedding Conjecture), and *linear soficity*, in which one considers metric ultraproducts of general linear groups endowed with the normalised rank function. Every sofic group is both hyperlinear [4] and linear sofic [1]. See also [11].

Arzhantseva and Păunescu [1] studied linear soficity of groups and algebras and discovered an interesting connection between them – namely, they proved that a group G is linear sofic if and only if its group algebra $\mathbb{C}[G]$ is linear sofic. Let us recall the required definition from [1].

Fix an arbitrary field F. Let $\mathcal{U}$ be a non-principal ultrafilter on the natural numbers and $(n_k)_k$ a sequence of natural numbers tending to infinity. We define the asymptotic rank function:

$$\rho_{\mathcal{U}} : \prod_k M_{n_k}(F) \longrightarrow [0,1] \quad \text{by} \quad \rho_{\mathcal{U}}\left((A_k)_k\right) := \lim_{k\to\mathcal{U}} \frac{1}{n_k}\mathrm{rk}(A_k).$$

Then one can form the metric ultraproduct $\prod_{k\to\mathcal{U}} M_{n_k}(F)/\mathrm{Ker}(\rho_{\mathcal{U}})$.

Definition 1·1 ([1]). A countably generated algebra A over a field F is linear sofic if there exists an injective homomorphism $\Phi : A \to \prod_{k\to\mathcal{U}} M_{n_k}(F)/\mathrm{Ker}(\rho_{\mathcal{U}})$.

While no examples of non linear sofic groups are known, it is not hard to find examples of non linear sofic algebras, based on the following observation. We say that a unital ring A is *directly finite*[1] if $xy = 1$ implies $yx = 1$ for every $x, y \in A$, and *stably finite* if $M_n(A)$ is directly finite for every $n \in \mathbb{N}$. There exist examples of directly finite, not stably finite rings [**13**]. It is straightforward to check that any metric ultraproduct $\prod_{k\to\mathcal{U}} M_{n_k}(F)/\mathrm{Ker}(\rho_{\mathcal{U}})$ is stably finite, hence every linear sofic algebra is. The algebra $F\langle x, y\rangle/\langle xy - 1\rangle$ is non-directly finite, hence non linear sofic. An open conjecture of Kaplansky asserts that the group algebra of an arbitrary group is directly finite (Kaplansky proved it for fields of characteristic zero [**8**]; see also [**10**]); Elek and Szabó proved Kaplansky's conjecture for sofic groups [**5**] (a different proof is given in [**1**]).

In [**1**], the authors mention that "Such [stably finite non linear sofic] algebras seem difficult to find as counterexamples to soficity in general proved to be elusive." The aim of this paper is to prove that stably finite, non linear sofic algebras exist.

Our proof is based on an asymptotic linear algebraic analysis of certain non-commutative equations, which we then show that can be solved in various stably finite algebras. The first instance is obtained using an example of Irving [**6**], from the theory of polynomial identity (PI) algebras:

THEOREM 1·2. *Over an arbitrary field, there exists a finitely generated non linear sofic algebra which satisfies a polynomial identity and is thus stably finite.*

Another example, of a completely different flavor, arises from the Cohn–Sąsiada construction of a simple Jacobson radical ring [**3**].

THEOREM 1·3. *Over an arbitrary field, there exists a finitely generated non linear sofic algebra which is Jacobson radical and whose unital hull*[2] *is thus a non linear sofic stably finite algebra.*

Conventions. Throughout, rings and algebras are associative but not necessarily commutative or unital; for a matrix $P \in M_n(F)$ we let $\mathrm{Im}(P)$ denote the image/column space of P, and let $\mathrm{rk}(P) = \dim_F \mathrm{Im}(P)$ denote its rank.

2. *Ring theoretic preliminaries*

Recall that a ring is *primitive* if it admits a faithful simple module, and an ideal $P \lhd R$ is primitive if the quotient ring R/P is primitive. The intersection of all primitive ideals of a ring R is called the *Jacobson radical* of R, denoted $J(R)$. Equivalently, a ring R is Jacobson radical (of itself) if it is quasi-invertible: for any $x \in R$ there exists $y \in R$ such that $x + y = xy$. This is equivalent to saying that in the unital hull R^1, the element $1 - x$ has $1 - y$ as an inverse. For more on primitive rings, radicals, and structure theory of rings, see [**7**].

An F-algebra A satisfies a *polynomial identity* (PI) if there exists some non-trivial polynomial $f(x_1, \ldots, x_n)$ in the free associative algebra $F\langle x_1, x_2, \ldots\rangle$ which vanishes under any substitution from A. For instance, any commutative algebra satisfies the (additive) commutator identity $[x, y]$. For more on PI-algebras, see [**12**].

[1] Aka 'Dedekind-finite' or 'Von Neumann finite'.

[2] The unital hull of an F-algebra R is the vector space $R^1 := F \oplus R$ with multiplication: $(\alpha + r) \cdot (\alpha' + r') := \alpha\alpha' + \alpha r' + \alpha' r + rr'$.

We now give two well-known observations regarding stable finiteness which are brought here for the reader's convenience:

LEMMA 2·1. *Let R be a ring and $J \triangleleft R$ its Jacobson radical. If R/J is stably finite then so is R.*

Proof. Suppose that $X, Y \in M_n(R)$ satisfy $XY = I$. Since R/J is stably finite then $I - YX \in M_n(J)$. Since J is a quasi-invertible ideal, so is $M_n(J) \triangleleft M_n(R)$ and hence $YX = I - (I - YX)$ is invertible, so both X, Y are invertible and since $XY = I$ it follows that $X = Y^{-1}$ and $YX = I$.

LEMMA 2·2. *Any PI-ring is stably finite.*

Proof. Suppose that R is a PI-ring and $J \triangleleft R$ is its Jacobson radical. Each primitive homomorphic image of R is a simple algebra which is finite-dimensional over its center [**9**, theorem 1], hence stably finite by linear algebra. Thus, R/J embeds into a direct product of stably finite rings, so it is stably finite itself. By Lemma 2·1, R itself is stably finite.

3. *Non linear soficity*

In this section we prove the following non-soficity machinery:

LEMMA 3·1. *Let A be an F-algebra containing elements $x, y, z \in A$ such that*:

(i) $x \in yxA$;

(ii) $z \in xA$ *and* $yz = 0$;

(iii) $z \neq 0$.

Then A is non linear sofic.

Proof. On the contrary, if A is linear sofic then we have an embedding:

$$\Phi : A \longrightarrow \prod_{k \to \mathcal{U}} M_{n_k}(F)/\mathrm{Ker}(\rho_{\mathcal{U}}).$$

Fix a linear lift of Φ to $\prod_k M_{n_k}(F)$, say, $\widehat{\varphi} : A \to \prod_k M_{n_k}(F)$ so:

$$\Phi(a) = 0 \iff \widehat{\varphi}(a) \in \mathrm{Ker}(\rho_{\mathcal{U}}).$$

Write $\widehat{\varphi} = \prod_k \varphi_k$ with each $\varphi_k : A \to M_{n_k}(F)$. For every $0 \neq a \in A$ there exists $\varepsilon > 0$ such that:

$$\{k : \mathrm{rk}(\varphi_k(a)) > \varepsilon n_k\} \in \mathcal{U},$$

and for every $a, b \in A$ and $\varepsilon > 0$, we have:

$$\{k : \mathrm{rk}(\varphi_k(a)\varphi_k(b) - \varphi_k(ab)) < \varepsilon n_k\} \in \mathcal{U}.$$

Since $z \neq 0$, it follows that also $x \neq 0$. In particular, since $\mathcal{U}$ is a non-principal ultrafilter, we can fix a positive real $\alpha > 0$ and a linear map $\varphi = \varphi_k : A \to M_n(F)$ for some k such that:

$$\mathrm{rk}(\varphi(x)),\ \mathrm{rk}(\varphi(z)) > \alpha n.$$

By the assumptions of the lemma, $x = yxa$ and $z = xb$ for some $a, b \in A$, and $yz = 0$. Let $T := \varphi(x) - \varphi(y)\varphi(x)\varphi(a)$ and $S := \varphi(z) - \varphi(x)\varphi(b)$. We may additionally assume that:

$$\mathrm{rk}(\varphi(y)\varphi(z)),\ \mathrm{rk}(T),\ \mathrm{rk}(S) < \frac{\alpha}{4}n.$$

Claim. We have:

$$\mathrm{rk}(\varphi(y)\varphi(x)) < \mathrm{rk}(\varphi(x)) - \frac{\alpha}{2}n.$$

Proof of claim. First, since $\varphi(z) = \varphi(x)\varphi(b) + S$:

$$\mathrm{Im}(\varphi(z)) \subseteq \mathrm{Im}(\varphi(x)) + \mathrm{Im}(S).$$

It follows that:

$$\begin{aligned}\frac{\mathrm{Im}(\varphi(z))}{\mathrm{Im}(\varphi(z)) \cap \mathrm{Im}(\varphi(x))} &\cong \frac{\mathrm{Im}(\varphi(z)) + \mathrm{Im}(\varphi(x))}{\mathrm{Im}(\varphi(x))} \\ &\subseteq \frac{\mathrm{Im}(\varphi(x)) + \mathrm{Im}(S)}{\mathrm{Im}(\varphi(x))} \\ &\cong \frac{\mathrm{Im}(S)}{\mathrm{Im}(\varphi(x)) \cap \mathrm{Im}(S)}.\end{aligned}$$

So:

$$\dim_F (\mathrm{Im}(\varphi(z)) \cap \mathrm{Im}(\varphi(x))) \geqslant \mathrm{rk}(\varphi(z)) - \mathrm{rk}(S) > \frac{3\alpha}{4}n.$$

Denote $\mathcal{V} := \mathrm{Im}(\varphi(y)\varphi(z))$ and recall that $\dim_F \mathcal{V} < \alpha n/4$. Fix a direct sum complement of $\mathrm{Im}(\varphi(z)) \cap \mathrm{Im}(\varphi(x))$ inside $\mathrm{Im}(\varphi(x))$, say, $\mathcal{W}$, and notice that:

$$\begin{aligned}\dim_F \mathcal{W} &= \mathrm{rk}(\varphi(x)) - \dim_F (\mathrm{Im}(\varphi(z)) \cap \mathrm{Im}(\varphi(x))) \\ &< \mathrm{rk}(\varphi(x)) - \frac{3\alpha}{4}n.\end{aligned}$$

Now:

$$\begin{aligned}\mathrm{rk}(\varphi(y)\varphi(x)) &= \dim_F \mathrm{Im}(\varphi(y)\varphi(x)) \\ &= \dim_F \varphi(y)\mathrm{Im}(\varphi(x)) \\ &\leqslant \dim_F \varphi(y)\,(\mathrm{Im}(\varphi(z)) \cap \mathrm{Im}(\varphi(x))) + \dim_F \varphi(y)\mathcal{W} \\ &\leqslant \dim_F \mathcal{V} + \dim_F \mathcal{W} \\ &< \frac{\alpha}{4}n + \left(\mathrm{rk}(\varphi(x)) - \frac{3\alpha}{4}n\right) \\ &= \mathrm{rk}(\varphi(x)) - \frac{\alpha}{2}n.\end{aligned}$$

Return to the proof of the lemma. Since $\varphi(x) = \varphi(y)\varphi(x)\varphi(a) + T$ for some $T \in M_n(F)$ with $\mathrm{rk}(T) < \alpha n/4$, we have:

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$$\begin{aligned}\operatorname{Im}\varphi(x) &= \operatorname{Im}(\varphi(y)\varphi(x)\varphi(a)+T)\\ &\subseteq \operatorname{Im}(\varphi(y)\varphi(x))+\operatorname{Im}(T)\end{aligned}$$

whose dimension is at most $\mathrm{rk}(\varphi(y)\varphi(x))+\mathrm{rk}(T)$ which is, by the above claim, at most:

$$\mathrm{rk}(\varphi(x))-\frac{\alpha}{2}n+\mathrm{rk}(T)<\mathrm{rk}(\varphi(x))-\frac{\alpha}{4}n,$$

a contradiction. Hence A is non linear sofic.

4. *Stably finite non linear sofic algebras*

Proof of Theorem 1·2. Let $A=F\langle x,y\rangle/\langle x^2, yxy-x\rangle$. This algebra was introduced by Irving [6] as an example of a finitely presented PI algebra which is not embeddable into any matrix algebra over a field.

The set of monomials in x, y which avoid occurrences of x^2 and yxy forms a linear basis for A; this fact was established in [6]. Indeed, this is a direct consequence of Bergman's Diamond Lemma [2], since the only overlap between the reductions $yxy \mapsto x$, $x^2 \mapsto 0$ is:

$$0=xxy=(yxy)xy=yx(yxy)=yxx=0.$$

By [6, theorem 2], A satisfies a polynomial identity (and has linear growth). Explicitly, since $\langle x\rangle \lhd A$ satisfies $A/\langle x\rangle \cong F[y]$ and $\langle x\rangle^3=0$, the identity:

$$[X_1,Y_1][X_2,Y_2][X_3,Y_3]=0$$

holds in A. In particular, by Lemma 2·2, A is stably finite.

Finally, A fulfills the requirements of Lemma 3·1 with $z=xyx$. Indeed, xyx contains no occurrences of x^2 or yxy, and is thus non-zero; obviously, $xyx\in xA$ and $x=yxy\in yxA$; and finally,

$$yz=yxyx=x^2=0.$$

Hence A is non linear sofic.

Proof of Theorem 1·3. Let $A=F\langle\langle x,y\rangle\rangle$ be the ring of noncommutative formal power series and let A^+ be the ideal of A consisting of all power series with zero constant term. Let $I=\langle yx^2y-x\rangle \lhd A^+$. Since A^+ is Jacobson radical, the quotient ring $R:=A^+/I$ is also Jacobson radical. By [3, section 2, "basic lemma"], the image of x in R (for simplicity, we identify elements in A^+ with their images modulo I, by abuse of notation) is non-zero. Consider the ring:

$$S=\begin{pmatrix}R & R\\ 0 & 0\end{pmatrix}$$

and consider the ideal:

$$K:=\left\langle\begin{pmatrix}0 & yx^2\\ 0 & 0\end{pmatrix}\right\rangle \lhd S.$$

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We claim that:

$$(*) \quad \begin{pmatrix} 0 & x^2 \\ 0 & 0 \end{pmatrix} \notin K.$$

Indeed, a straightforward calculation shows that $KS = 0$, so if $(*)$ was not true then:

$$\begin{pmatrix} 0 & x^2 \\ 0 & 0 \end{pmatrix} = \alpha \cdot \begin{pmatrix} 0 & yx^2 \\ 0 & 0 \end{pmatrix} + \sum_{i=1}^{n} \begin{pmatrix} a_i & b_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & yx^2 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \alpha yx^2 + \sum_{i=1}^{n} a_i yx^2 \\ 0 & 0 \end{pmatrix},$$

(where $\alpha \in F$ is some – possibly zero – scalar.) Considering the upper right corner of the above equation, we obtain $x^2 = fx^2$ where $f = \alpha y + \sum_{i=1}^{n} a_i y \in R$. Let g be the quasi-inverse of f, namely, $gf = f + g$. Then $gx^2 = gfx^2 = fx^2 + gx^2$, so $x^2 = fx^2 = 0$, hence $x = yx^2 y = 0$. This contradicts that $x \neq 0$. It follows that $(*)$ holds.

Consider $T := S/K$ and consider the following elements, identified with their images modulo K:

$$X := \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \; Y := \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}, \; Z := \begin{pmatrix} 0 & x^2 \\ 0 & 0 \end{pmatrix}$$

Notice that in T, it holds that $X = YX^2Y \in YX \cdot T$ and:

$$Z = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in X \cdot T$$

and in addition:

$$YZ = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x^2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & yx^2 \\ 0 & 0 \end{pmatrix}$$

which belongs to K, hence equal to 0 in T. Moreover, by $(*)$ it follows that Z is non-zero in T.

By Lemma 3·1 applied to T with respect to X, Y, Z, the algebra T is non linear sofic; since S is Jacobson radical, so is $T = S/K$. The unital hull T^1 is non linear sofic and stably finite by Lemma 2·1.

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REFERENCES

[1] G. ARZHANTSEVA and L. PĂUNESCU. Linear sofic groups and algebras. *Trans. Amer. Math. Soc.* **369** (2017), 2285–2310.

[2] G. M. BERGMAN. The diamond lemma for ring theory. *Adv. Math.* **29** (2), 2017, 2285–2310.

[3] P. M. COHN and E. SĄSIADA. An example of a simple radical ring. *J. Algebra* **5** (1967), 373–377.

[4] G. ELEK and E. SZABÓ. Hyperlinearity, essentially free actions and L^2-invariants. The sofic property. *Math. Ann.* **332** (2005), 421–441.

[5] G. ELEK and E. SZABÓ. Sofic groups and direct finiteness. *J. Algebra* **280** (3) (2004), 426–434.

[6] R. S. IRVING. Affine PI-algebras not embeddable in matrix rings. *J. Algebra* **82** (1) (1983), 94–101.

[7] N. JACOBSON. Structure of rings. Revised edition *Amer. Math. Soc. Colloq. Publ.*, vol. 37 (American Mathematical Society, Providence, R.I., 1964), pp.ix+299.

[8] I. KAPLANSKY. *Fields and Rings* (University of Chicago Press, Chicago, Ill., 1972), (second edition).

[9] I. KAPLANSKY. Rings with a polynomial identity. *Bull. Amer. Math. Soc.* **54** (1948), 575–580.

[10] S. M. MONTGOMERY. Left and right inverses in group algebras. *Bull. Amer. Math. Soc.* **75** (1969), 539–540.

[11] V. G. PESTOV. Hyperlinear and sofic groups: a brief guide. *Bull. Symbolic Logic* **14** (4) (2008), 449–480.

[12] L. H. ROWEN. *Polynomial Identities in Ring Theory* (Academic Press, New York, 1980), pp. xx+365.

[13] J. C. SHEPHERDSON. Inverses and zero-divisors in matrix rings. *Proc. London. Math. Soc.* s**3-1** (1) (1951), 71–85.

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