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CORRECTION TO: CONJUGACY CLASSES OF MAXIMAL TORI IN SIMPLE REAL ALGEBRAIC GROUPS AND APPLICATIONS

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ABSTRACT. An error in *Conjugacy classes of maximal tori in simple real algebraic groups and applications*, Canad. J. Math. **46**(1994), 699–717 is corrected.

We wish to inform the reader that there is an error in the proof of Theorem 4.5 of our paper [2]. The statement of this theorem remains valid except that the row B I of Table VII should contain $\mathrm{Spin}(2n, 1)$, $n \geq 2$, instead of $\mathrm{SO}(2n, 1)^{\circ}$, $n \geq 1$. (In our notation, $\mathrm{Spin}(p, q)$ is a connected Lie group.) Unless stated otherwise, all references below are to [2].

We recall some of the notation: $\tilde{G}$ is a connected and simply connected almost simple $\mathbf{R}$-group, σ is the corresponding anti-holomorphic involution of $\tilde{G}$, T is a maximal $\mathbf{R}$-torus of $\tilde{G}$ containing a maximal split $\mathbf{R}$-torus S of $\tilde{G}$, Φ is the root system of $\tilde{G}$ relative to T, and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ a base of Φ. For other non-explained notation, see the original paper.

Let X^* be the character group of T and X_* the group of (algebraic) one-parameter subgroups of T, both written additively, and $\langle\, ,\, \rangle: X^* \times X_* \longrightarrow \mathbf{Z}$ the canonical pairing. Thus we have $\alpha\big(h(z)\big) = z^{\langle \alpha, h\rangle}$ for $\alpha \in X^*$, $h \in X_*$, and $z \in \mathbf{C}^*$. For $\alpha \in \Phi$, we denote by h_α the corresponding coroot considered as an element of X_*. Since $\sigma(T) = T$, σ acts on X^* and X_* as follows:

$$\sigma(\alpha)(t) := \overline{\alpha\big(\sigma(t)\big)}, \quad t \in T,\ \alpha \in X^*;$$
$$\sigma(h)(z) := \sigma\big(h(\bar{z})\big), \quad z \in \mathbf{C}^*,\ h \in X_*.$$

We write h_i for h_{α_i}. Let $\alpha = k_1\alpha_1 + \cdots + k_l\alpha_l \in X^*$, $k_i \in \mathbf{Z}$, and set $S_\alpha = h = k_1h_1 + \cdots + k_lh_l \in X_*$. The torus $S(\alpha) \subset T$, as defined on p. 711, is the image of h. The notation S_α and $S(\alpha)$ (and its generalization $S(\beta_1, \ldots, \beta_r)$) were ill-chosen and were partially responsible for our error. However, in order to avoid possible confusion, we shall continue to use that notation.

If $\sigma(h) = h$ then $\sigma\big(h(z)\big) = h(\bar{z})$, and so $\mathrm{Im}(h)$ is a split $\mathbf{R}$-torus. If $\sigma(h) = -h$, then $\sigma\big(h(z)\big) = h(\bar{z}^{-1})$, and so $\mathrm{Im}(h)$ is an anisotropic $\mathbf{R}$-torus. Hence the assertion:

(*) If $\sigma(\alpha) = \epsilon\alpha$, $\epsilon = \pm 1$, then $\sigma S_\alpha(t)\sigma^{-1} = S_\alpha(\bar{t}^{\epsilon})$,

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made on p. 712, is false and should be replaced by the following:

(**) If $\alpha = k_1\alpha_1 + \cdots + k_l\alpha_l$, $k_i \in \mathbf{Z}$, $h = k_1h_1 + \cdots + k_lh_l$, and $\sigma(h) = h$ (resp. $\sigma(h) = -h$), then the torus $S(\alpha)$ is defined over $\mathbf{R}$ and is split (resp. anisotropic) over $\mathbf{R}$.

If all roots of Φ have the same length, then one can identify the dual root system $\Phi^\vee := \{h_\alpha : \alpha \in \Phi\}$ with Φ so that the action of σ is preserved. In that case (*) is valid, but if Φ has roots of different lengths, then (*) fails.

For $\alpha \in \Phi$ let $T_\alpha := \mathrm{Ker}(\alpha)^\circ$, $Z_\alpha = Z_{\tilde{G}}(T_\alpha)$, and $G_\alpha = [Z_\alpha, Z_\alpha]$, the commutator subgroup of Z_α. Then $G_\alpha \cong \mathrm{SL}_2(\mathbf{C})$, $\mathrm{Im}(h_\alpha) = G_\alpha \cap T$, and $\langle \alpha, h_\alpha \rangle = 2$. These properties characterize h_α, and consequently we have

$$(***) \qquad \sigma(h_\alpha) = h_{\sigma(\alpha)},\ \alpha \in \Phi.$$

Our determination of the maximal split $\mathbf{R}$-torus S has to be corrected in the cases B I and F II only.

CASE B I (P. 714). We have to correct the treatment of the case $q = 1$. In the Satake diagram of $G(\mathbf{R}) = \mathrm{SO}(p,1)^\circ$, $p = 2n$, $n > 1$, only the first vertex α_1 is white while all others are black. Hence we have $\sigma(\alpha_i) = -\alpha_i$ for $i > 1$. By a simple computation (or see [1]) one can show that

$$\sigma(\alpha_1) = \tilde{\alpha} = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_n)$$

is the highest root of Φ.

We claim that the torus $S := S(\alpha)$, where

$$\alpha := 2(\alpha_1 + \cdots + \alpha_{n-1}) + \alpha_n \in X^*,$$

is split over $\mathbf{R}$. The corresponding $h \in X_*$ is given by

$$h = 2(h_1 + \cdots + h_{n-1}) + h_n.$$

Since Φ is the root system of type B_n, we have

$$h_{\tilde{\alpha}} = h_1 + 2(h_2 + \cdots + h_{n-1}) + h_n.$$

By using (***) we find that

$$\sigma(h) = 2h_{\tilde{\alpha}} - 2(h_2 + \cdots + h_{n-1}) - h_n = h.$$

In view of (**), our claim is proved.

We have $Z_G(S) = SH$, where $\Delta_H = \Delta \setminus \{\alpha_1\}$. As $S^f = \langle h(-1) \rangle = \langle \epsilon_n \rangle \subset H$, $T(\mathbf{R})$ is connected and $G^* = \mathrm{Spin}(2n, 1)$ is weakly exponential.

CASE F II (P. 717). We only indicate that, on line 11, $S(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4)$ should be replaced by $S(2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4)$, and $\langle \epsilon_1\epsilon_3 \rangle$ by $\langle \epsilon_3 \rangle$.

We close with two other minor corrections.

1) In the proof of Proposition 1.3, p. 701, line −3, $T_2(\mathbf{R})$ and $T_2'(\mathbf{R})$ should be replaced by $S_1(\mathbf{R})$ and $S_1'(\mathbf{R})$, respectively. On the same page, last two lines, $H(\mathbf{R})$ should be replaced by $(T_0H)(\mathbf{R})$.

2) In Table IV, line 15 (b), $\mathrm{SO}^*(2)$ should be replaced by $\mathrm{SO}^*(4)$.

REFERENCES

1. S. Araki, *On root systems and on infinitesimal classification of irreducible symmetric spaces*, Osaka J. Math. **13**(1962), 1–34.

2. D. Ž. Đoković and Nguyen Q. Thang, *Conjugacy classes of maximal tori in simple real algebraic groups and applications*, Canad. J. Math. **46**(1994), 699–717.

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