ON CERTAIN TYPES OF SOLUTION OF THE EQUATION OF HEAT CONDUCTION

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Let $f(x, y, z, t)$ satisfy the equation

$$f_t = \nabla^2 f = f_{xx} + f_{yy} + f_{zz}. \qquad (1)$$

For certain purposes, particularly in connection with the propagation of a boundary of fusion etc., it is of interest to discover solutions of (1) which permit the equation :

$$f = \text{constant}, \qquad (2)$$

to be solved explicitly in the form :

$$g(x, y, z) = h(t). \qquad (3)$$

This suggests the examination of solutions of the type

$$f = f(\zeta), \qquad (4)$$

where

$$\zeta \equiv \phi(x, y, z) \,.\, \psi(t), \qquad (5)$$

and f, ϕ, ψ are functions to be determined. To save repetition, Roman capitals denote arbitrary constants throughout.

1. *Linear system.* $f = f(x, t) = f[\zeta(x, t)]$; $f_t = f_{xx}$.

(*a*) Let $\zeta \equiv x^m t^n$. Then, by (1),

$$nx^2 f'(\zeta) = m(m-1)tf'(\zeta) + m^2 x^m t^{n+1} f''(\zeta). \qquad (6)$$

That is,

$$\frac{f''}{f'} = \frac{nx^{2+\frac{m}{n}} - m(m-1)\zeta^{\frac{1}{n}}}{m^2 \zeta^{1+\frac{1}{n}}}. \qquad (7)$$

Consequently $\qquad n = 0, \quad or \quad m = -2n.$

If $m = -2n \neq 0$, we may without loss of generality take $n = -\frac{1}{2}$, whereupon $m = 1$, and

$$f''/f' = -\zeta/2. \qquad (8)$$

Then

$$f' = Ae^{-\zeta^2/4}, \qquad (9)$$

and so

$$f(\zeta) = B \operatorname{erf}(\zeta/2) + C. \qquad (10)$$

$n = 0$ yields the trivial case

$$f(\zeta) = A\zeta^{\frac{1}{m}} + B = Ax + B, \qquad (11)$$

which may be regarded as a limiting form of (10).

The only solutions of (1) of the form $f(x^m t^n)$ are therefore

$$f = B \operatorname{erf}^* \left(\frac{x}{2\sqrt{t}}\right) + C, \qquad (12)$$

$$f = Ax + B. \qquad (13)$$

* $\operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du.$

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(*b*) To generalise, let now $\zeta \equiv \phi(x)\,\psi(t)$.
Here

$$f_t = \phi\dot{\psi} f' \quad \text{and} \quad f_{xx} = \ddot{\phi}\psi f' + \dot{\phi}^2\psi^2 f'',$$

where the dot signifies differentiation with respect to x or t. Consequently, by (1),

$$f''/f' = \frac{\phi\dot{\psi} - \psi\ddot{\phi}}{\dot{\phi}^2\psi^2}. \quad \text{.......(14)}$$

If we denote f''/f' by $\theta(\zeta)$, then

$$\theta_x = \theta'\psi\dot{\phi} \quad \text{and} \quad \theta_t = \theta'\phi\dot{\psi}.$$

Consequently

$$\phi\dot{\psi}\theta_x = \psi\dot{\phi}\theta_t, \quad \text{.......(15)}$$

and from (14) and (15), after some reduction, we have

$$\frac{2\phi\ddot{\phi}}{\dot{\phi}^2} \cdot \frac{\dot{\psi}}{\psi} = \left(\frac{3\dot{\psi}}{\psi} - \frac{\ddot{\psi}}{\dot{\psi}}\right) + \left(\frac{2\ddot{\phi}^2}{\dot{\phi}^2} - \frac{\dddot{\phi}}{\dot{\phi}} - \frac{\ddot{\phi}}{\phi}\right), \quad \text{.......(16)}$$

which is of the form :

$$f_1(x)\,f_2(t) = f_3(t) + f_4(x).$$

It follows that either f_1 or f_2 is a constant. That is,

$$\phi\ddot{\phi} = B\dot{\phi}^2 \quad \text{or} \quad \dot{\psi} = A^2\psi. \quad \text{.......(17)}$$

In the former case,

$$\phi\dddot{\phi} + \dot{\phi}\ddot{\phi} = 2B\dot{\phi}\ddot{\phi} = 2\phi\ddot{\phi}^2/\dot{\phi},$$

so that $f_4(x) \equiv 0$. Also

$$\dot{\phi} = C\phi^B, \quad \phi^{1-B} = (1-B)(Cx+D). \quad \text{.......(18)}$$

and we lose no generality in setting $B=0$, so that

$$\phi = Cx + D. \quad \text{.......(19)}$$

Then

$$3\dot{\psi}/\psi = \ddot{\psi}/\dot{\psi}, \quad \text{.......(20)}$$

whence

$$\dot{\psi} = -E\psi^3, \quad \psi = 1/2\sqrt{(Et+F)} \quad \text{.......(21)}$$

and we may, without loss, take $E = 2C^2$.

Finally, by (14), (18) and (21),

$$f''/f' = -\frac{E\zeta}{C^2} = -2\zeta, \quad \text{.......(22)}$$

which leads to

$$f(\zeta) = P\,\mathrm{erf}\,(\zeta) + Q. \quad \text{.......(23)}$$

Thus, when $\phi\ddot{\phi}/\dot{\phi}^2 = \text{constant}$, *the only solution* is

$$f = P\,\mathrm{erf}\left(\frac{x+R}{2\sqrt{t+S}}\right) + Q. \quad \text{.......(24)}$$

$$f = Px + Q \quad \text{.......(25)}$$

is again a trivial limiting case.

If $\dot{\psi} = A^2\psi$,

$$f_3(t) = 2A^2 \quad \text{and} \quad \psi = Be^{A^2t}. \quad \text{.......(26)}$$

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Also, by (14),

$$f''/f' = \frac{A^2\phi - \ddot{\phi}}{\dot{\phi}^2} \cdot \frac{\phi}{\zeta}, \quad \text{.......................................(27)}$$

so that

$$\ddot{\phi} - A^2\phi = C\dot{\phi}^2/\phi.$$

The term $C\dot{\phi}^2/\phi$ can be removed by setting $\phi \equiv \Phi^{\frac{1}{1-C}}$, so that we may take $C=0$, giving

$$\phi = De^{Ax} + Ee^{-Ax}. \quad \text{..(28)}$$

Then, by (27),

$$f''/f' = 0, \quad \text{..(29)}$$

which implies

$$f = F\zeta + G. \quad \text{..(30)}$$

Collecting (26), (28), and (30), we conclude that when $\dot{\psi}/\psi = \text{constant}$, *the only solution* is

$$f = e^{A^2t}[Pe^{Ax} + Qe^{-Ax}] + R. \quad \text{......................................(31)}$$

A may be imaginary, in which case P and Q are complex.

The only solutions $f[\phi(x) \,.\, \psi(t)]$ of $f_t = f_{xx}$ are therefore : (24), (25) and (31).

2. *Cylindrical system* $f = f(r, t) = f[\zeta(r, t)]$; $f_t = f_{rr} + \frac{1}{r} f_r$.

Let $\zeta \equiv \phi(r)\,\psi(t)$.

Then

$$\theta(\zeta) \equiv f''/f' = \frac{\phi\dot{\psi} - \ddot{\phi}\psi - \frac{1}{r}\dot{\phi}\psi}{\dot{\phi}^2\psi^2}, \quad \text{..................................(32)}$$

and (15) leads to

$$\frac{2\phi\ddot{\phi}}{\dot{\phi}^2} \cdot \frac{\dot{\psi}}{\psi} = \left(\frac{3\dot{\psi}}{\psi} - \frac{\ddot{\psi}}{\dot{\psi}}\right) + \left(\frac{2\ddot{\phi}^2}{\dot{\phi}^2} - \frac{\dddot{\phi}}{\dot{\phi}} - \frac{\ddot{\phi}}{\phi} + \frac{1}{r^2} + \frac{1}{r}\frac{\ddot{\phi}}{\dot{\phi}} - \frac{1}{r}\frac{\dot{\phi}}{\phi}\right), \quad \text{......................(33)}$$

whence

$$\phi\ddot{\phi} = B\dot{\phi}^2 \quad \textit{or} \quad \dot{\psi} = -A^2\psi.$$

In the former case, $\dot{\phi} = C\phi^B$, $\phi^{1-B} = C(1-B)(r+D)$, and we may take $B=0$, $C=1$, whereupon

$$\ddot{\phi} = 0, \quad \dot{\phi} = 1, \quad \phi = r + D. \quad \text{......................................(34)}$$

Then, by (33)

$$0 = \left(\frac{3\dot{\psi}}{\psi} - \frac{\ddot{\psi}}{\dot{\psi}}\right) + \frac{1}{r^2} - \frac{1}{r(r+D)},$$

so that $D=0$, giving $\phi = r$. Also

$$\psi\ddot{\psi} = 3\dot{\psi}^2, \quad \dot{\psi} = -E\psi^3, \quad \psi = 1/\sqrt{2E(t+R)}.$$

We may take $E=2$, whence

$$\zeta = \frac{r}{2\sqrt{t+R}}. \quad \text{..(35)}$$

Finally, by (32),

$$f''/f' = -2\zeta - \frac{1}{\zeta}, \quad \text{..(36)}$$

which leads to *

$$f = PEi\left(\frac{-r^2}{4(t+R)}\right) + Q. \quad \text{.......................................(37)}$$

* $Ei(x) \equiv \int_{\infty}^{-x} e^{-u} du/u$.

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Again, if $=\dot{\psi}-A^2\psi$, $\psi=Be^{-A^2t}$ and

$$f''/f'=-\frac{\phi}{\zeta}\cdot\frac{A^2\phi+\ddot{\phi}+\dot{\phi}/r}{\dot{\phi}^2}, \quad\text{......(38)}$$

ɜo that

$$\ddot{\phi}+\frac{1}{r}\dot{\phi}+A^2\phi=C\dot{\phi}^2/\phi. \quad\text{......(39)}$$

The term, $C\dot{\phi}^2/\phi$ can be removed by substituting $\phi=\Phi^{\frac{1}{1-C}}$, so that we may without loss ;ake $C=0$. The solution of (39) is then

$$\phi=DJ_0(Ar)+EY_0(Ar), \quad\text{......(40)}$$

vhere J_0 and Y_0 are zero-order Bessel functions of first and second kinds respectively.

Finally, by (38),

$$f''/f'=0,$$

o that

$$f=G\zeta+R. \quad\text{......(41)}$$

Collecting the results,

$$f=e^{-A^2t}[PJ_0(Ar)+QY_0(Ar)]+R. \quad\text{......(42)}$$

1 may again be imaginary, with P, Q complex.

When $A=0$, we have the trivial limiting case

$$f=P\ln r+Q. \quad\text{......(43)}$$

37), (42) and (43) represent *the only solutions* $f[\phi(r)\,.\,\psi(t)]$ of $f_t=f_{rr}+\frac{1}{r}f_r$.

3. *Spherical system* $f=f(r,t)=f[\zeta(r,t)]$; $f_t=f_{rr}+\frac{2}{r}f_r$. As before, let $\zeta\equiv\phi(r)\psi(t)$. The ylindrical solution applies, except that $1/r$ must be replaced by $2/r$ in (32) and (33). (35) folɔws as before, but (36) becomes

$$f''/f'=-2\zeta-\frac{2}{\zeta}, \quad\text{......(44)}$$

vhich leads to

$$f=P\left[\frac{\sqrt{t+R}}{r}e^{-\frac{r^2}{4(t+R)}}+\frac{\sqrt{\pi}}{2}\operatorname{erf}\frac{r}{2\sqrt{t+R}}\right]+Q. \quad\text{......(45)}$$

Again, if $\dot{\psi}=A^2\psi$, (39) becomes

$$\ddot{\phi}+\frac{2}{r}\dot{\phi}-A^2\phi=C\dot{\phi}^2/\phi, \quad\text{......(46)}$$

nd the term $C\dot{\phi}^2/\phi$ may be removed as before, so that we take $C=0$, whereupon

$$\phi=[De^{Ar}+Ee^{-Ar}]/r. \quad\text{......(47)}$$

(41) still applies, and so

$$f=\frac{e^{A^2t}}{r}[Pe^{Ar}+Qe^{-Ar}]+R. \quad\text{......(48)}$$

! may again be imaginary, with complex P, Q.

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When $A=0$, we have the trivial limiting case

$$f=\frac{P}{r}+Q. \qquad (49)$$

(45), (48) and (49) are *the only solutions*

$$f[\phi(r)\,.\,\psi(t)] \quad \text{of} \quad f_t=f_{rr}+\frac{2}{r}f_r.$$

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