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# ON THE ORDERS OF GENERATORS OF CAPABLE $p$-GROUPS

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A group is called capable if it is a central factor group. For each prime $p$ and positive integer $c$, we prove the existence of a capable $p$-group of class $c$ minimally generated by an element of order $p$ and an element of order $p^{1+\lfloor c-1/p-1 \rfloor}$. This is best possible.

## 1. INTRODUCTION

Recall that a group $G$ is said to be *capable* if and only if $G$ is isomorphic to $K/Z(K)$ for some group $K$, where $Z(K)$ is the centre of $K$. There are groups which are not capable (nontrivial cyclic groups being a well-known example), so capability places restrictions on the structure of a group; see for example [**3, 4**]. As noted by Hall in his landmark paper on the classification of $p$-groups ([**2**]), the question of which $p$-groups are capable is interesting and plays an important role in their classification.

Hall observed that if $G$ is a capable $p$-group of class $c$, with $c < p$, and $\{x_1, \dots, x_n\}$ is a minimal set of generators with $o(x_1) \leqslant o(x_2) \leqslant \cdots \leqslant o(x_n)$ (where $o(g)$ denotes the order of the element $g$), then $n > 1$ and $o(x_{n-1}) = o(x_n)$.

In [**5**] we used commutator calculus to derive a similar necessary condition after dropping the hypothesis $c < p$: if $G$ is a capable $p$-group of class $c > 0$, minimally generated by $\{x_1, \dots, x_n\}$, where $o(x_1) \leqslant \cdots \leqslant o(x_n)$, then we must have $n > 1$ and letting $o(x_{n-1}) = p^a$ and $o(x_n) = p^b$, then $a$ and $b$ must satisfy

$$b \leqslant a + \left\lfloor \frac{c-1}{p-1} \right\rfloor, \tag{1.1}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$ ([**5**, Theorem 3.19]). The dihedral group of order $2^{c+1}$ shows that (1.1) is best possible when $p = 2$. The purpose of this note is to show that the inequality is best possible for all primes $p$, thus answering in the affirmative [**5**, Question 3.22].

Notation will be standard; all groups will be written multiplicatively, and we shall denote the identity by $e$. We use the convention that the commutator of two elements $x$ and $y$ is $[x, y] = x^{-1}y^{-1}xy$. The lower central series of $G$ is defined recursively by letting

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$G_1 = G$, and $G_{n+1} = [G_n, G]$. We say $G$ is nilpotent of class (at most) $c$ if and only if $G_{c+1} = \{e\}$. It is well known that if $G$ is of class exactly $c$, then $G_c \subset Z(G)$, and $G/Z(G)$ is nilpotent of class exactly $c-1$.

We let $C_n$ denote the cyclic group of order $n$, and $\mathbf{Z}$ the infinite cyclic group, both written multiplicatively.

## 2. The case $c = 1 + (r-1)(p-1)$

The construction in this section is based on the example given by Easterfield in [**1**, Section 4].

Let $p$ be a prime, $r$ a positive integer. We construct a $p$-group $K$ of class $c+1 = 2 + (r-1)(p-1)$, minimally generated by an element $y$ of order $p$, and an element $x_0$ of order $p^r$. We shall show that the images of $y$ and $x_0$ have the same order in $K/Z(K)$, thus exhibiting a capable group of class $c = 1 + (r-1)(p-1)$, minimally generated by an element of order $p$ and one of order $p^{1+\lfloor c-1/p-1 \rfloor}$.

Let $H$ be the Abelian group

$$H = C_{p^r} \times C_{p^r} \times \underbrace{C_{p^{r-1}} \times \cdots \times C_{p^{r-1}}}_{p-2 \text{ factors}}.$$

Denote the generators of the cyclic factors of $H$ by $x_0, x_1, \ldots, x_{p-1}$, respectively. If $r = 1$, then $x_2, \ldots, x_{p-1}$ are trivial. Let $y$ generate a cyclic group of order $p$, and let $y$ act on $H$ by $y^{-1}x_iy = x_ix_{i+1}$ for $0 \leqslant i \leqslant p-2$ (so $[x_i, y] = x_{i+1}$), and

$$y^{-1}x_{p-1}y = x_1^{-\binom{p}{1}} x_2^{-\binom{p}{2}} \cdots x_{p-2}^{-\binom{p}{p-2}} x_{p-1}^{1-\binom{p}{p-1}};$$

as usual, $\binom{n}{k}$ is the binomial coefficient $n$ choose $k$. Let $K = H \rtimes \langle y \rangle$.

REMARK 2.1. The group constructed by Easterfield is the subgroup of $K$ generated by $y$ and $x_1, \ldots, x_r$. We can also realise $K$ as the semidirect product of this subgroup by $\langle x_0 \rangle$, letting $x_0$ act on $y$ by $x_0^{-1}yx_0 = yx_1^{-1}$, and act trivially on the $x_i$.

Note that $K$ is metabelian of class exactly $2 + (r-1)(p-1)$. To verify the class, note that $[K,K] = \langle x_1, \ldots, x_{p-1}\rangle$. We then have:

$$\begin{aligned}
K_3 &= \langle x_1^p, x_2, \ldots, x_{p-1}\rangle;\\
K_4 &= \langle x_1^p, x_2^p, x_3, \ldots, x_{p-1}\rangle;\\
&\vdots\\
K_{2+(p-1)} &= \langle x_1^p, x_2^p, \ldots, x_{p-1}^p\rangle;\\
K_{2+(p-1)+1} &= \langle x_1^{p^2}, x_2^p, \ldots, x_{p-1}^p\rangle;\\
&\vdots\\
K_{2+k(p-1)} &= \langle x_1^{p^k}, x_2^{p^k}, \ldots, x_{p-1}^{p^k}\rangle;\\
&\vdots\\
K_{2+(r-1)(p-1)} &= \langle x_1^{p^{r-1}}, x_2^{p^{r-1}}, \ldots, x_{p-1}^{p^{r-1}}\rangle = \langle x_1^{p^{r-1}}\rangle.
\end{aligned}$$

Finally, note that $x_1^{p^{r-1}}$ is central: $y^{-1}x_1^{p^{r-1}}y = (x_1x_2)^{p^{r-1}} = x_1^{p^{r-1}}$. Therefore $K$ is of class exactly $2+(r-1)(p-1)$.

The group $G = K/Z(K)$ will therefore be of class $1+(r-1)(p-1)$, minimally generated by $yZ(K)$ and $x_0Z(K)$. The order of $yZ(K)$ is of course equal to $p$. As for $x_0Z(K)$, note that no nontrivial power of $x_0$ is central: if $x_0^k$ is central, then

$$x_0^k = y^{-1}x_0^k y = (y^{-1}x_0y)^k = (x_0x_1)^k = x_0^k x_1^k;$$

therefore $x_1^k = e$, which implies that $p^r \mid k$, so $x_0^k = e$. Therefore, the order of $x_0Z(K)$ is $p^r$. Thus, $G$ is a capable group of class $c$, with $c = 1+(r-1)(p-1)$, minimally generated by an element of order $p$ and an element of order $p^r = p^{1+\lfloor c-1/p-1\rfloor}$.

We note the following fact about $K$, which we shall use in the following section:

**Lemma 2.2.** *Let $p$ be any prime, and let $r$ be an arbitrary positive integer. There exists a group $K$ of class $2+(r-1)(p-1)$, generated by elements $y$ and $x_0$ of orders $p$ and $p^r$, respectively, such that $x_0^{p^{r-1}}$ does not commute with $y$.*

## 3. General case

Again, let $p$ be a prime, and let $c > 1$ be an arbitrary integer. We want to exhibit a capable group $G$ of class exactly $c$, generated by an element of order $p$ and an element of order $p^{1+\lfloor c-1/p-1\rfloor}$.

Our construction in this section will be based on the nilpotent product of groups; we specialise the definition to the case we are interested in:

Definition 3.1: Let $A_1, \ldots, A_n$ be cyclic groups, and let $c > 0$. The $c$-nilpotent product of the $A_i$, denoted $A_1 \amalg^{\mathfrak{N}_c} \cdots \amalg^{\mathfrak{N}_c} A_n$ is defined to be the group $F/F_{c+1}$, where $F$ is the free product of the $A_i$, $F = A_1 * \cdots * A_n$, and $F_{c+1}$ is the $(c+1)$-st term of the lower central series of $F$.

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It is easy to verify that the $c$-nilpotent product of the $A_i$ is of class exactly $c$, and that it is their coproduct (in the sense of category theory) in the variety $\mathfrak{N}_c$ of all nilpotent groups of class at most $c$. The 1-nilpotent product is simply the direct sum of the $A_i$.

Note that if $G$ is the $c$-nilpotent product of the $A_i$, then $G/G_{k+1}$ is the $k$-nilpotent product of the $A_i$ for all $k$, $1 \leqslant k \leqslant c$.

We consider $\mathcal{G} = C_p \amalg^{\mathfrak{N}_{c+1}} \mathbf{Z}$, the $(c+1)$-nilpotent product of a cyclic group of order $p$ and the infinite cyclic group. Denote the generator of the finite cyclic group by $a$, and the generator of the infinite cyclic group by $z$. Let $G = \mathcal{G}/Z(\mathcal{G})$. Then $G$ is capable of class $c$. We want to show that $zZ(\mathcal{G})$ has the required order.

**Proposition 3.2.** *Let $a$ generate $C_p$ and $z$ generate the infinite cyclic group $\mathbf{Z}$. If $\mathcal{G} = C_p \amalg^{\mathfrak{N}_{c+1}} \mathbf{Z}$, then*

$$Z(\mathcal{G}) \cap \langle z \rangle = \left\langle z^{p^{1+\lfloor c-1/p-1 \rfloor}} \right\rangle.$$

Proof: The fact that $z^{p^{1+\lfloor c-1/p-1 \rfloor}}$ is central follows from [**5**, Theorem 3.16], so we just need to prove the other inclusion. We proceed by induction on $c$. The claim is true if $c = 1$ since the commutator bracket is bilinear in a group of class two. Assume the inclusion holds for $c - 1$, with $c > 1$. Note that $\langle z \rangle \cap \mathcal{G}_2 = \{e\}$.

Consider $\mathcal{G}/\mathcal{G}_{c+1}$; this is the $c$-nilpotent product of $C_p$ and $\mathbf{Z}$, so by the induction hypothesis, the intersection of the centre and the subgroup generated by $z$ is generated by the $p^{1+\lfloor c-2/p-1 \rfloor}$-st power of $z$. Since the center of $\mathcal{G}$ is contained in the pullback of the centre of $\mathcal{G}/\mathcal{G}_{c+1}$, we deduce that the smallest power of $z$ that could possibly be in $Z(\mathcal{G})$ is the $p^{1+\lfloor c-2/p-1 \rfloor}$-st power.

If $\lfloor c - 2/p - 1 \rfloor = \lfloor c - 1/p - 1 \rfloor$, then we are done. So the only case that needs to be dealt with is the case considered in the previous section, when $c = 1 + (r-1)(p-1)$ for some positive integer $r > 1$.

Here we use the universal property of the coproduct. Let $K$ be the group from Lemma 2.2. Since $\mathcal{G}$ is the coproduct of $C_p$ and $\mathbf{Z}$ in $\mathfrak{N}_{c+1}$, the morphisms $C_p \to K$ given by $a \mapsto y$, and $\mathbf{Z} \to K$ given by $z \mapsto x_0 \in K$, induce a unique homomorphism $\varphi \colon \mathcal{G} \to K$. The image of $Z(\mathcal{G})$ must lie in $Z(K)$ (since the map is surjective). Since $\varphi(z^{p^{r-1}}) = x_0^{p^{r-1}}$ does not commute with $y$, we conclude that $z^{p^{r-1}} \notin Z(\mathcal{G})$. This proves that the smallest power of $z$ that could lie in $Z(\mathcal{G})$ is $z^{p^r}$, which gives the desired inclusion. ∎

Now let $G = \mathcal{G}/Z(\mathcal{G})$. This is a group of class $c$, minimally generated by $aZ(\mathcal{G})$ and $zZ(\mathcal{G})$. The former has order $p$, and the latter element has order $p^{1+\lfloor c-1/p-1 \rfloor}$ by the proposition above. Thus $G$ is a capable group of class $c$, minimally generated by two elements whose orders satisfy the equality in (1.1), showing that the inequality is indeed best possible.

Remark 3.3. I believe that in general inequality (1.1) will be both necessary and sufficient for the capability of a $c$-nilpotent product of cyclic $p$-groups. This is indeed the case when $c < p$ and when $p = c = 2$; see [5]. However, I have not been able to establish this for arbitrary $p$ and $c$, which forced the somewhat indirect approach taken in this note.

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