Student Problems

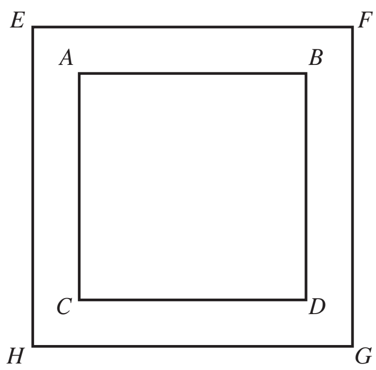
Students up to the age of 19 are invited to send s to either or both of the following problems to Tuya Sa, SCH.1.17, Schofield Building, Loughborough University, Loughborough, LE11 3TU. Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive s for either problem. It is not necessary to submit s to both. s should arrive by 20th January 2024 and will be published in the March 2024 edition.

The Mathematical Association and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions **must** be accompanied by the SPC permission form which is available on the Mathematical Association website

https:www.m-a.org.uk/the-mathematical-gazette

Note that if permission is not given, a pupil may still participate and will be eligible for a prize in the same way as others.

Problem 2023.5 **(Geoffrey Strickland)**



ABCD is a square with centre *O*. *EFGH* is an enlargement of *ABCD* with centre *O*, such that its area is twice that of *ABCD*.

Show how the border between the two squares may be dissected into no more than ten pieces which will fit in to *ABCD*.

Problem 2023.6 **(Paul Stephenson)**

The nth triangle number, $T_n = \frac{1}{2}n(n + 1)$ for $n \geqslant 1$. Determine which triangle numbers cannot be represented as the difference of two squares.

https://doi.org/10.1017/mag.2023.127 Published online by Cambridge University Press

Solutions to 2023.3 and 2023.4

Both problems were solved by Geya Wang, Emily Warren and Soham Bhadra. Problem 2023.3 was solved by Isabella Topley.

Problem 2023.3 **(Gregory Dresden)**

Show, without using series, that

$$\lim_{n \to \infty} \left(\cot \frac{x}{n+1} - \cot \frac{x}{n-1} \right) = \frac{2}{x}.$$

Solution (Gregory Dresden)

First, we use the relationship

$$\cot A - \cot B = \frac{\cos A}{\sin A} - \frac{\cos B}{\sin B} = \frac{\cos A \sin B - \cos B \sin A}{\sin A \sin B}$$
$$= \frac{\sin (B - A)}{\sin A \sin B}$$

to write our expression as

$$\cot \frac{x}{n+1} - \cot \frac{x}{n-1} = \frac{\sin\left(\frac{x}{n-1} - \frac{x}{n+1}\right)}{\sin \frac{x}{n+1} \sin \frac{x}{n-1}} = \frac{\sin \frac{2x}{n^2 - 1}}{\sin \frac{x}{n+1} \sin \frac{x}{n-1}}.$$

Now, we recall

$$\lim_{h \to 0} \frac{\sin hC}{h} = 1.$$

If we replace h with $\frac{1}{M}$, then our limit above (as $h \to 0$) becomes a limit as $M \to \infty$. To be precise, we can say that

$$\lim_{M \to 0} M \sin \frac{C}{M} = 1.$$

Back to our limit, from our work above we have

$$\lim_{n \to \infty} \left(\cot \frac{x}{n+1} - \cot \frac{x}{n-1} \right) = \lim_{n \to \infty} \frac{\sin \frac{2x}{n^2 - 1}}{\sin \frac{x}{n+1} \sin \frac{x}{n-1}}$$

and multiply top and bottom by $(n + 1)(n - 1)$, we obtain

$$\lim_{n \to \infty} \frac{(n^2 - 1) \sin \frac{2x}{n^2 - 1}}{(n + 1) \sin \frac{x}{n+1} \cdot (n - 1) \sin \frac{x}{n-1}}.$$

https://doi.org/10.1017/mag.2023.127 Published online by Cambridge University Press

We note that since $n \to \infty$, then $(n+1) \to \infty$, $(n-1) \to \infty$ and $(n^2-1) \to \infty$ do as well. This means that our limit above is actually comprised of three separate limits (two in the denominator and one in the numerator), each of the form $M \sin \frac{c}{M}$ as $M \to \infty$. This means that our limit is

$$\frac{\lim_{(n^2-1)\to\infty} \left(n^2 - 1\right) \sin \frac{2x}{n^2 - 1}}{\lim_{(n+1)\to\infty} (n + 1) \sin \frac{x}{n + 1} \cdot \lim_{(n-1)\to\infty} (n - 1) \sin \frac{x}{n - 1}} = \frac{2x}{x \cdot x}$$

which gives us $\frac{2}{x}$.

Problem 2023.4 **(Paul Stephenson)**

Show that the sum of the divisors of a positive integer (including 1 and the number itself) divides the sum of their cubes.

Solution (Geya Wang)

Let $n = 4$.

Sum of divisors: $1 + 2 + 4 = 7$.

Sum of cubed divisors: $1 + 8 + 64 = 73$.

Proven by counterexample, it is *not* generally true that sum of the divisors of a positive integer divides the sum of their cubes.

Comments (Geoffrey Strickland)

With regard to the Problem submitted as 2023.4 the difficulty of this problem comes when some of the primes are repeated. Compare $N = abc$ with divisors obtained from

$$(1 + a)(1 + b)(1 + c) \text{ as } 1,\ a,\ b,\ c,\ bc,\ ac,\ ab,\ abc.$$

And $N = ab^2$ with divisors obtained by putting c equal to b in the above i.e:

$$1, a, b, b, b^2, ab, ab, ab^2.$$

We see that two of the divisors occur twice so that $(1+a)(1+b)^2$ does not equal the sum of the divisors of N. The correct product is $\left(1 + a\right)\left(1 + b + b^2\right)$.

In general if the prime b is repeated n times, the multiplier should be $\left(1 + b + \ldots + b^n\right)$ instead of $(1 + b)^n$.

The setter, Paul Stephenson, apologises for the incorrect wording of the problem published in the July edition. The claim is only valid for certain restricted cases. Credit was given to the students who discovered this.

Prize Winners

The first prize of £25 is awarded to Geya Wang. The second prize of £20 is awarded to Soham Bhadra.

10.1017/mag.2023.127 TUYA SA

Published by Cambridge University Press on behalf of The Mathematical Association