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THE FACTORIZABLE BRAID MONOID

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Abstract In this paper we study the factorizable braid monoid (also known as the merge-and-part braid monoid) introduced by Easdown, East and FitzGerald in 2004. We find several presentations of this monoid, and uncover an interesting connection with the singular braid monoid. This leads to the definition of the flexible singular braid monoid, which consists of 'flexible-vertex-isotopy' classes of singular braids. We conclude by defining and studying the pure factorizable braid monoid, the maximal subgroups of which are (isomorphic to) quotients of the pure braid group.

Keywords: braids; singular braids; factorizable inverse monoids; presentations

2000 *Mathematics subject classification:* Primary 20F36
Secondary 20F05

1. Introduction and notation

The factorizable braid monoid $\mathfrak{F}\mathcal{B}_n$ (also known as the merge-and-part braid monoid and denoted $\tilde{B}$) was introduced in [**8**] as a 'braid analogue' of $\mathfrak{F}_n$, the monoid of uniform block bijections on an n-set [**10, 11**]. By this we mean that $\mathfrak{F}\mathcal{B}_n$ is a natural pre-image of $\mathfrak{F}_n$ in the same way that the braid group is a pre-image of the symmetric group. In [**8**] it was shown that $\mathfrak{F}\mathcal{B}_n$ belongs to a class of factorizable inverse monoids that embed in the coset monoid of their group of units (see [**8, 16**] or the proof of Lemma 7.4 for a description of the coset monoid). In this article we study $\mathfrak{F}\mathcal{B}_n$ and another (singular) braid analogue of $\mathfrak{F}_n$, by means of presentations. In § 2 we outline a method, described in [**9**], for constructing factorizable inverse monoids and finding their presentations. Then in § 3 we review the definition, and geometric interpretation, of $\mathfrak{F}\mathcal{B}_n$ from [**8**]. In § 4 we find a presentation of $\mathfrak{F}\mathcal{B}_n$ from which we deduce, in § 5, the presentation of $\mathfrak{F}_n$ discovered by FitzGerald [**10**]. In § 6 we provide a second presentation of $\mathfrak{F}\mathcal{B}_n$ which reflects more of the symmetry possessed by $\mathfrak{F}\mathcal{B}_n$. This second presentation also highlights an interesting connection with the singular braid monoid [**3**,**4**]. This leads to the definition of the flexible singular braid monoid $\mathfrak{F}\mathcal{S}\mathcal{B}_n$. In § 7 we find a presentation of $\mathfrak{F}\mathcal{S}\mathcal{B}_n$ and show that, despite many similarities, the monoids $\mathfrak{F}\mathcal{B}_n$ and $\mathfrak{F}\mathcal{S}\mathcal{B}_n$ are not isomorphic. In doing so, we will

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show that one of the more 'peculiar' relations in our presentation of $\mathfrak{FB}_n$ is not redundant. In § 8 we define the pure factorizable braid monoid $\mathfrak{FP}_n$. We study $\mathfrak{FP}_n$ by analysing presentations of its maximal subgroups. Each of these subgroups is (isomorphic to) a quotient of the pure braid group $\mathcal{P}_n$, and we prove that these quotients are semidirect products of free groups and free abelian groups, generalizing a well-known result of Artin [**2**]. This leads to an algorithm for deciding whether two braids are merge-and-part equivalent.

Without causing confusion, we will generally denote the identity of any monoid by 1. Let M be a monoid. An equivalence relation $\sim$ on M is a *congruence* if $ab \sim cd$ whenever $a, b, c, d \in M$ with $a \sim c$ and $b \sim d$. The quotient $M/{\sim}$ of all $\sim$-classes is itself a monoid under the inherited operation. If L is another monoid and $\phi : M \to L$ a homomorphism, then the *kernel* of ϕ is the congruence

$$\ker(\phi) = \{(a, b) \in M \times M \mid a\phi = b\phi\}.$$

The Fundamental Homomorphism Theorem (for monoids) states that $M/\ker(\phi)$ is isomorphic to the image of ϕ. If G is a group, then a congruence $\sim$ on G is completely determined by the $\sim$-class N of the identity $1 \in G$. In this case, N is a normal subgroup of G and the quotients $G/{\sim}$ and G/N are precisely the same object. When referring to a group homomorphism ϕ, we will sometimes use the definition of $\ker(\phi)$ as a congruence, and sometimes as a normal subgroup.

We now establish the notation we will be using for monoid presentations. Let X be a set, and denote by X^* the free monoid on X. If $W \subseteq X^*$, then W^* denotes the submonoid of X^* generated by W. For $R \subseteq X^* \times X^*$, let $R^\sharp$ denote the smallest congruence on X^* containing R. We say that a monoid M has monoid presentation $\langle X|R\rangle$ if $M \cong X^*/R^\sharp$ or, equivalently, if there is an epimorphism $f : X^* \to M$ with $\ker f = R^\sharp$. In this case we say that M has presentation $\langle X|R\rangle$ via f. An element $(w_1, w_2) \in R$ is called a relation, and is often written as $w_1 = w_2$. If $(w_1, w_2) \in R^\sharp$ then we write $w_1 \sim_R w_2$. Even though R need not be symmetric, we will often use a phrase such as 'suppose that $(w_1, w_2) \in R$' to mean 'suppose that either $(w_1, w_2) \in R$ or $(w_2, w_1) \in R$'.

2. Factorizable inverse monoids

Throughout this article, a monoid of commuting idempotents is referred to as a *semilattice*. The reason for this terminology is that if M is a monoid of commuting idempotents, then we may define a partial order on M by

$$e \leqslant f \quad \text{if and only if} \quad ef = f$$

with respect to which each pair of elements $e, f \in M$ have a least upper bound, namely the product ef.

An inverse monoid M is said to be *factorizable* if $M = EG$ where $E = E(M)$ is the semilattice of idempotents of M, and $G = G(M)$ is the group of units of M. (For more details regarding factorizable inverse monoids, see [**6**].) In this section we review a method, outlined in [**9**], for constructing factorizable inverse monoids, and finding their

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presentations. Suppose that G is a group, and that E is a semilattice. Suppose also that for each $g \in G$ we have an automorphism $\varphi_g : E \to E : e \mapsto e^g$ such that the map $\varphi : G \to \mathrm{Aut}(E) : g \mapsto \varphi_g$ is an anti-homomorphism. Then we may form the semidirect product

$$E \rtimes G = E \rtimes_\varphi G = \{(e, g) \mid e \in E,\ g \in G\}$$

with multiplication defined by

$$(e_1, g_1)(e_2, g_2) = (e_1 e_2^{g_1}, g_1 g_2).$$

Suppose now that, for each $e \in E$, we are given a subgroup G_e of G such that $G_1 = \{1\}$ and

$$gG_e g^{-1} = G_{e^g}, \quad \forall e \in E,\ g \in G, \tag{G_e1}$$

$$G_e \vee G_f = G_{ef}, \quad \forall e, f \in E, \tag{G_e2}$$

$$e^g = e, \qquad \forall e \in E,\ g \in G_e. \tag{G_e3}$$

Here, for subgroups H and H' of G we have used the notation $H \vee H' = \langle H \cup H' \rangle$. We may then define a congruence $\sim$ on $E \rtimes G$ by

$$(e_1, g_1) \sim (e_2, g_2) \quad \text{if and only if} \quad e_1 = e_2 \text{ and } g_1 g_2^{-1} \in G_{e_1}.$$

Thus, we may form the quotient $(E \rtimes G)/{\sim}$. Denote the $\sim$-class of $(e, g) \in E \rtimes G$ by $[e, g]$, and let $[E, 1] = \{[e, 1] \mid e \in E\}$ and $[1, G] = \{[1, g] \mid g \in G\}$.

Theorem 2.1 (Easdown *et al.* [9]). *The monoid $(E \rtimes G)/{\sim}$ is a factorizable inverse monoid with semilattice of idempotents $[E, 1] \cong E$ and group of units $[1, G] \cong G$.*

Suppose that E and G have monoid presentations $\langle X_E | R_E \rangle$ via λ, and $\langle X_G | R_G \rangle$ via μ, respectively. We may assume that X_E and X_G are disjoint and that λ and μ are injective when restricted to X_E and X_G, respectively. For each $e \in E$, $g \in G$, choose $\hat{e} \in X_E^*$, $\hat{g} \in X_G^*$ such that $\hat{e}\lambda = e$ and $\hat{g}\mu = g$. Let

$$R_\rtimes = \{(yx, \widehat{x\lambda^{y\mu}} y) \mid x \in X_E,\ y \in X_G\}.$$

Suppose that for each $x \in X_E$ we have a subset $S_x \subseteq G$ such that $G_{x\lambda}$ is generated, as a submonoid, by S_x. Put

$$R_\sim = \{(x\hat{g}, x) \mid x \in X_E,\ g \in S_x\}.$$

Theorem 2.2 (Easdown *et al.* [9]). *The factorizable inverse monoid $(E \rtimes G)/{\sim}$ has monoid presentation*

$$\langle X_G \cup X_E | R_G \cup R_E \cup R_\rtimes \cup R_\sim \rangle$$

via $\nu : (X_G \cup X_E)^* \to (E \rtimes G)/{\sim}$ defined by

$$x\nu = \begin{cases} [x\lambda, 1] & \text{if } x \in X_E, \\ [1, x\mu] & \text{if } x \in X_G. \end{cases}$$

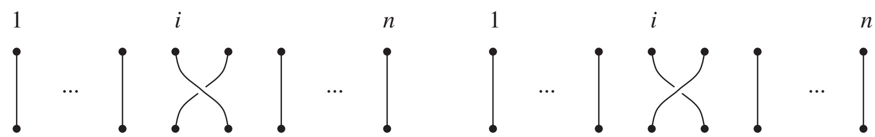

Figure 1. The braids ς_i (left) and ς_i^{-1} (right) in B.

3. The factorizable braid monoid

We now review the construction of $\mathfrak{F}\mathcal{B}_n$ described in [**8**]. Fix n, a positive integer, and denote by $\mathbf{n}$ the set $\{1,\dots,n\}$. The braid group $B=\mathcal{B}_n$ is the group of homotopy classes of geometric braids on n strings. For details regarding braids refer to [**4**]. Without causing confusion, we will generally identify a braid with its homotopy class, although in § 7 it will be convenient to draw a clear distinction. For $i\in\{1,\dots,n-1\}$ we denote by ς_i (respectively, ς_i^{-1}) the (homotopy class of the) braid in which the ith string crosses over (respectively, under) the $(i+1)$th, all other strings passing vertically downwards (see Figure 1).

Let $S=\mathrm{Sym}(\mathbf{n})$ be the symmetric group on $\mathbf{n}$. For $\beta\in B$ we denote by $\bar\beta\in S$ the permutation associated to β so that $\beta\mapsto\bar\beta$ is the natural epimorphism $B\to S$ under which $\varsigma_i^{\pm1}$ is mapped to the simple transposition s_i, which interchanges i and $i+1$. The kernel of this epimorphism is the pure braid group $P=\mathcal{P}_n=\{\beta\in B\mid\bar\beta=1\}$.

Let $E=\mathfrak{Eq}_n$ denote the set of all equivalence relations on $\mathbf{n}$. Here we regard an element of E as a subset of $\mathbf{n}\times\mathbf{n}$ satisfying reflexivity, symmetry and transitivity. The join, $\mathcal{E}_1\vee\mathcal{E}_2$, of two equivalence relations $\mathcal{E}_1,\mathcal{E}_2\in E$, is defined as the smallest equivalence relation on $\mathbf{n}$ containing $\mathcal{E}_1\cup\mathcal{E}_2$, and E forms a semilattice under $\vee$.

For $\mathcal{E}\in E$ and $\beta\in B$ we define

$$\mathcal{E}^\beta=\{(i,j)\mid(i,j)\bar\beta\in\mathcal{E}\}=\{(i,j)\bar\beta^{-1}\mid(i,j)\in\mathcal{E}\},$$

where, for $i,j\in\mathbf{n}$ and $\pi\in S$, we have written $(i,j)\pi$ for $(i\pi,j\pi)$. It is easy to check that, for each $\beta\in B$, the map $\varphi_\beta:\mathcal{E}\mapsto\mathcal{E}^\beta$ is an automorphism of E, and that $\varphi:\beta\mapsto\varphi_\beta$ is an anti-homomorphism $B\to\mathrm{Aut}(E)$. Thus, we may form the semidirect product $E\rtimes B$ as above. For $\mathcal{E}\in E$ we let $B_\mathcal{E}$ denote the subgroup of B generated by the set

$$\{\beta^{-1}\varsigma_i\beta\mid(i,i+1)\bar\beta\in\mathcal{E}\}.$$

Since this set is empty if $\mathcal{E}=1$ is the identity of E, we have $B_1=\{1\}$. In [**8**] it was shown that these subgroups satisfy

$$\beta B_\mathcal{E}\beta^{-1}=B_{\mathcal{E}^\beta}\qquad\forall\mathcal{E}\in E,\ \beta\in B,\tag{$B_\mathcal{E}1$}$$

$$B_\mathcal{E}\vee B_{\mathcal{E}_0}=B_{\mathcal{E}\vee\mathcal{E}_0}\qquad\forall\mathcal{E},\mathcal{E}_0\in E,\tag{$B_\mathcal{E}2$}$$

$$\mathcal{E}^\beta=\mathcal{E}\qquad\forall\mathcal{E}\in E,\ \beta\in B_\mathcal{E}.\tag{$B_\mathcal{E}3$}$$

As a result we may form the quotient $(E\rtimes B)/{\sim}$, which we call the *factorizable braid monoid* and denote by $\mathfrak{F}\mathcal{B}_n$.

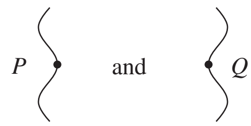

Figure 2. The strings $\mathfrak{s}$ (left) and $\mathfrak{t}$ (right).

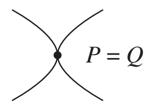

Figure 3. The strings merging.

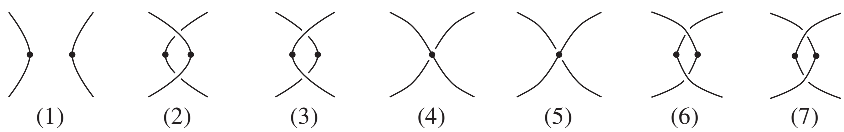

Figure 4. Possible configurations before and after merge-and-part.

Theorem 3.1 (Easdown *et al.* [8]). *The factorizable braid monoid $\mathfrak{F}\mathcal{B}_n$ is a factorizable inverse monoid with semilattice of idempotents $[E, 1] \cong E$ and group of units $[1, B] \cong B$.*

The following geometric interpretation of $\mathfrak{F}\mathcal{B}_n$ was considered in [8]. Let $\mathfrak{s}$ and $\mathfrak{t}$ be two strings descending from fixed points on an upper plane to connect to fixed points on a lower plane (see Figure 2). We say that a homotopy causes $\mathfrak{s}$ and $\mathfrak{t}$ to *merge and part* if, during the course of the homotopy, $\mathfrak{s}$ and $\mathfrak{t}$ come together just once at say P and Q (see Figure 3), and then part, reconstituting as two strings made up of respective upper and lower strands. See Figure 4 for a catalogue of the possible configurations in the neighbourhood a moment before and after merge-and-part. Note that (1), (2), and (3) can be interchanged using normal homotopy.

Let $\mathcal{E} \in E$. We say that the ith and jth strings of a braid are $\mathcal{E}$-related if and only if $(i, j) \in \mathcal{E}$. If $\beta, \gamma \in B$, then we say that β and γ are $\mathcal{E}$-equivalent if there is a homotopy from (a representative of) β to (a representative of) γ during which $\mathcal{E}$-unrelated strings never touch, and $\mathcal{E}$-related strings are allowed to merge and part (one at a time, a finite number of times).

Theorem 3.2 (Easdown *et al.* [8]). *Let $\mathcal{E}, \mathcal{E}_0 \in E$ and $\beta, \beta_0 \in B$. Then $(\mathcal{E}, \beta) \sim (\mathcal{E}_0, \beta_0)$ if and only if $\mathcal{E} = \mathcal{E}_0$ and β and β_0 are $\mathcal{E}$-equivalent.*

For $\mathcal{E} \in E$, we denote by $[\beta]_{\mathcal{E}}$ the $\mathcal{E}$-equivalence class of $\beta \in B$. We now define the *merge-and-part braid monoid*

$$\tilde{B} = \tilde{\mathcal{B}}_n = \{[\beta]_{\mathcal{E}} \mid \mathcal{E} \in E,\ \beta \in B\}$$

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with multiplication

$$[\beta_1]_{\mathcal{E}_1}[\beta_2]_{\mathcal{E}_2} = [\beta_1\beta_2]_{\mathcal{E}_1 \vee \mathcal{E}_2^{\beta_1}}.$$

It is easy to check that this multiplication is well defined and associative, and that $[1]_1$ is an identity for $\tilde{B}$. Let $[1]_E = \{[1]_{\mathcal{E}} \mid \mathcal{E} \in E\}$ and $[B]_1 = \{[\beta]_1 \mid \beta \in B\}$. The map $E \rtimes B \to \tilde{B} : (\mathcal{E}, \beta) \mapsto [\beta]_{\mathcal{E}}$ is clearly an epimorphism whose kernel, by Theorem 3.2, is $\sim$. We have shown the following.

Theorem 3.3. *The monoids $\mathfrak{F}\mathcal{B}_n$ and $\tilde{\mathcal{B}}_n$ are isomorphic via $[\mathcal{E}, \beta] \mapsto [\beta]_{\mathcal{E}}$. Thus, $\tilde{\mathcal{B}}_n$ is a factorizable inverse monoid with semilattice of idempotents $[1]_E \cong E$ and group of units $[B]_1 \cong B$.*

4. A presentation of $\mathfrak{F}\mathcal{B}_n$

In this section we will give a presentation of $\mathfrak{F}\mathcal{B}_n$. We first gather the information required to apply Theorem 2.2. Let $X_B = \{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$.

Theorem 4.1 (Artin [1]). *The braid group B has monoid presentation $\langle X_B | R_B \rangle$ via*

$$\phi_B : X_B^* \to B : \sigma_i^{\pm 1} \mapsto \varsigma_i^{\pm 1},$$

where R_B is the set of relations

$$\sigma_i^{\pm 1}\sigma_i^{\mp 1} = 1 \qquad \text{for all } i, \tag{F}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } |i-j| > 1, \tag{B1}$$

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \qquad \text{if } |i-j| = 1. \tag{B2}$$

We choose a set of words $\{\hat{\beta} \mid \beta \in B\} \subseteq X_B^*$ such that $\hat{\beta}\phi_B = \beta$ for all $\beta \in B$, and for convenience we will denote the congruence $\sim_{R_B}$ by $\sim_B$. If $w = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_k}^{\varepsilon_k} \in X_B^*$, we will denote by w^{-1} the word $\sigma_{i_k}^{-\varepsilon_k} \cdots \sigma_{i_1}^{-\varepsilon_1} \in X_B^*$ so that $ww^{-1} \sim_B w^{-1}w \sim_B 1$. We also define a homomorphism $X_B^* \to S : w \mapsto \bar{w} = \overline{w\phi_B}$.

For $1 \leqslant i < j \leqslant n$ let $\mathcal{E}_{ij} \in E$ denote the equivalence $\{(r,s) \mid r = s \text{ or } \{r,s\} = \{i,j\}\}$ and put $X_E = \{\varepsilon_{ij} \mid 1 \leqslant i < j \leqslant n\}$.

Theorem 4.2 (FitzGerald [10]). *The semilattice E has monoid presentation $\langle X_E | R_E \rangle$ via*

$$\phi_E : X_E^* \to E : \varepsilon_{ij} \mapsto \mathcal{E}_{ij},$$

where R_E is the set of relations

$$\varepsilon_{ij}^2 = \varepsilon_{ij} \qquad \text{for all } i, j, \tag{Eq1}$$

$$\varepsilon_{ij}\varepsilon_{kl} = \varepsilon_{kl}\varepsilon_{ij} \qquad \text{for all } i, j, k, l, \tag{Eq2}$$

$$\varepsilon_{ij}\varepsilon_{jk} = \varepsilon_{jk}\varepsilon_{ik} = \varepsilon_{ik}\varepsilon_{ij} \qquad \text{if } i < j < k. \tag{Eq3}$$

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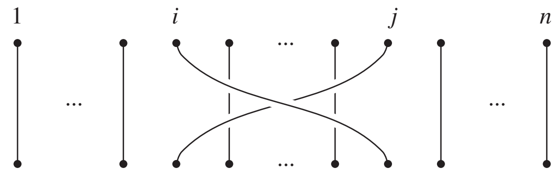

Figure 5. The braid $\varsigma_{ij} \in B$.

We choose a set of words $\{\hat{\mathcal{E}} \mid \mathcal{E} \in E\} \subseteq X_E^*$ such that $\hat{\mathcal{E}}\phi_E = \mathcal{E}$ for all $\mathcal{E} \in E$. Suppose that $1 \leqslant i < j \leqslant n$ and $1 \leqslant r \leqslant n-1$. It is immediate from the definitions that

$$\mathcal{E}_{ij}^{\varsigma_r^{\pm 1}} = \begin{cases} \mathcal{E}_{i-1,j} & \text{if } r = i-1, \\ \mathcal{E}_{i+1,j} & \text{if } r = i < j-1, \\ \mathcal{E}_{i,j-1} & \text{if } r = j-1 > i, \\ \mathcal{E}_{i,j+1} & \text{if } r = j, \\ \mathcal{E}_{ij} & \text{otherwise.} \end{cases}$$

For $1 \leqslant i < j \leqslant n$ we use the notation $\mathcal{E}_{ji} = \mathcal{E}_{ij}$ and $\varepsilon_{ji} = \varepsilon_{ij}$. We see then that $\mathcal{E}_{ij}^{\varsigma_r^{\pm 1}} = \mathcal{E}_{is_r, js_r}$. Thus, we may take $R_\rtimes$ to be the set of relations

$$\sigma_r^{\pm 1}\varepsilon_{ij} = \varepsilon_{is_r,js_r}\sigma_r^{\pm 1} \quad \text{for each } 1 \leqslant i < j \leqslant n \text{ and } 1 \leqslant r \leqslant n-1. \tag{$\rtimes$}$$

Before moving on, we find a different expression for the generators of the subgroups $B_{\mathcal{E}}$. For $1 \leqslant i < j \leqslant n$, let

$$\sigma_{ij} = (\sigma_{j-1}^{-1} \cdots \sigma_{i+1}^{-1})\sigma_i(\sigma_{i+1} \cdots \sigma_{j-1}) \in X_B^*$$

and put $\varsigma_{ij} = \sigma_{ij}\phi_B \in B$ (see Figure 5). We assume that $\hat{\varsigma}_{ij} = \sigma_{ij}$. Recall that $P = \mathcal{P}_n$ denotes the pure braid group.

Lemma 4.3. *If $\mathcal{E} \in E$, then $B_{\mathcal{E}}$ is generated by the set $\{\beta^{-1}\varsigma_{ij}\beta \mid \beta \in P,\ (i,j) \in \mathcal{E}\}$.*

Proof. Let $U = \{\beta^{-1}\varsigma_i\beta \mid (i,i+1)\bar{\beta} \in \mathcal{E}\}$ so that (by definition) $B_{\mathcal{E}} = \langle U \rangle$, and let V be the set in the statement of the lemma. It is clear that $V \subseteq U$. To show the reverse inclusion, suppose that $u = \gamma^{-1}\varsigma_i\gamma \in U$, where $(i,i+1)\bar{\gamma} = (j,k) \in \mathcal{E}$. Replacing γ by $\varsigma_i\gamma$ (if necessary) we may assume that $j < k$. Put

$$\alpha = \begin{cases} \varsigma_{j-1} \cdots \varsigma_i\varsigma_j \cdots \varsigma_{i+1} & \text{if } i < j, \\ \varsigma_{j+1} \cdots \varsigma_i\varsigma_j \cdots \varsigma_{i-1} & \text{if } j < i, \\ 1 & \text{if } i = j. \end{cases}$$

It can easily be checked that $\alpha^{-1}\varsigma_j\alpha = \varsigma_i$, and $(j,j+1)\bar{\alpha} = (i,i+1)$. Putting $\gamma_1 = \alpha\gamma$, we then have $u = \gamma_1^{-1}\varsigma_j\gamma_1$, and $(j,j+1)\bar{\gamma}_1 = (j,k)$. Now put $\gamma_2 = \varsigma_{k-1}^{-1} \cdots \varsigma_{j+1}^{-1}\gamma_1$ so that $u = \gamma_2^{-1}\varsigma_{jk}\gamma_2$ and $(j,k)\bar{\gamma}_2 = (j,k)$. Remove the jth and kth strings from γ_2^{-1} and put them back in such a way that they pass straight down and always in front of all the other strings, and call the resulting braid δ. From the construction, it is clear that δ commutes with ς_{jk}. But then if we put $\gamma_3 = \delta\gamma_2 \in P$, we see that $u = \gamma_3^{-1}\varsigma_{jk}\gamma_3$, completing the proof. □

As a result of this lemma, we may take $R_\sim$ to be the set of relations

$$\varepsilon_{ij}\hat{\beta}^{-1}\sigma_{ij}\hat{\beta} = \varepsilon_{ij} \quad \text{for all } 1 \leqslant i < j \leqslant n \text{ and } \beta \in P. \tag{$\sim$}$$

The following is now a direct consequence of Theorem 2.2.

Lemma 4.4. *The factorizable braid monoid $\mathfrak{F}\mathcal{B}_n$ has monoid presentation*

$$\langle X_B \cup X_E | R_B \cup R_E \cup R_\rtimes \cup R_\sim \rangle$$

via $\sigma_i^{\pm 1} \mapsto [1, \varsigma_i^{\pm 1}]$, $\varepsilon_{ij} \mapsto [\mathcal{E}_{ij}, 1]$.

We will now work towards simplifying this presentation. As a first step, we will remove a number of the generators. With this in mind, let $e = \varepsilon_{12}$. By ($\rtimes$) and (F) we see that, for any $1 \leqslant i < j \leqslant n$, the relation

$$\varepsilon_{ij} = (\sigma_{i-1}^{-1} \cdots \sigma_1^{-1})(\sigma_{j-1}^{-1} \cdots \sigma_2^{-1})e(\sigma_2 \cdots \sigma_{j-1})(\sigma_1 \cdots \sigma_{i-1}) \tag{$*$}$$

is in $(R_B \cup R_E \cup R_\rtimes \cup R_\sim)^\sharp$. So we remove the generators ε_{ij}, replacing their every occurrence in the relations by the word on the right-hand side of ($*$), which we denote by e_{ij} (noting in particular that $e_{12} = e$). We denote the resulting relations by (Eq1)$'$–(Eq3)$'$, ($\rtimes$)$'$, and ($\sim$)$'$. The entire sets of relations which have been modified in this way will be denoted by R'_E, $R'_\rtimes$, and $R'_\sim$.

Corollary 4.5. *The factorizable braid monoid $\mathfrak{F}\mathcal{B}_n$ has monoid presentation*

$$\langle X_B \cup \{e\} | R_B \cup R'_E \cup R'_\rtimes \cup R'_\sim \rangle$$

via $\sigma_i^{\pm 1} \mapsto [1, \varsigma_i^{\pm 1}]$, $e \mapsto [\mathcal{E}_{12}, 1]$.

Lemma 4.6. *The following relations are in $(R_B \cup R'_E \cup R'_\rtimes \cup R'_\sim)^\sharp$:*

$$e^2 = e = e\sigma_1, \tag{E1}$$

$$e\sigma_i = \sigma_i e \quad \text{if } i \neq 2, \tag{E2}$$

$$e\sigma_2 e\sigma_2 = \sigma_2 e\sigma_2 e, \tag{E3}$$

$$e\sigma_2\sigma_3\sigma_1\sigma_2 e\sigma_2\sigma_3\sigma_1\sigma_2 = \sigma_2\sigma_3\sigma_1\sigma_2 e\sigma_2\sigma_3\sigma_1\sigma_2 e, \tag{E4}$$

$$e\sigma_2^2 = \sigma_2^2 e. \tag{E5}$$

Proof. Now (E1) is part of (Eq1)$'$ and ($\sim$)$'$, while (E2) is part of ($\rtimes$)$'$. For the remainder of this proof, let $\approx$ denote the congruence $(R_B \cup R'_E \cup R'_\rtimes \cup R'_\sim)^\sharp$. To show that (E3) holds, note that by ($\rtimes$)$'$ and (Eq2)$'$ we have

$$e\sigma_2 e\sigma_2 = e_{12}\sigma_2 e_{12}\sigma_2 \approx \sigma_2 e_{13} e_{12}\sigma_2 \approx \sigma_2 e_{12} e_{13}\sigma_2 \approx \sigma_2 e_{12}\sigma_2 e_{12} = \sigma_2 e\sigma_2 e.$$

Next put $w = \sigma_2\sigma_3\sigma_1\sigma_2$. Observe that $(1, 2, 3, 4)\bar{w} = (3, 4, 1, 2)$ so that by ($\rtimes$)$'$ we have

$$e_{12}w \approx we_{34} \quad \text{and} \quad e_{34}w \approx we_{12}.$$

But then (E4) holds since, by (Eq2)′ and the observation, we have

$$ewew = e_{12}we_{12}w \approx we_{34}e_{12}w \approx we_{12}e_{34}w \approx we_{12}we_{12} = wewe.$$

For (E5), note that by $(\rtimes)'$ we have

$$e\sigma_2^2 = e_{12}\sigma_2^2 \approx \sigma_2 e_{13}\sigma_2 \approx \sigma_2^2 e_{12} = \sigma_2^2 e.$$

This completes the proof. □

Denote by R the set of relations (F), (B1), (B2), and (E1)–(E5). Our aim is to show that $\mathfrak{F}\mathcal{B}_n$ has monoid presentation $\langle X_B \cup \{e\} | R\rangle$. By Lemma 4.6, we may add relations (E1)–(E5) to the presentation stated in Corollary 4.5. We will show that relations (Eq1)′–(Eq3)′, $(\rtimes)'$ and $(\sim)'$ may be eliminated.

For $1 \leqslant i < j \leqslant n$ let

$$\alpha_{ij} = (\sigma_{j-1} \cdots \sigma_{i+1})\sigma_i^2(\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}),$$

and put $A = \{\alpha_{ij}^{\pm 1} \mid 1 \leqslant i < j \leqslant n\}$. The following result is well known. For proofs see, for example, [**2**, **4**].

Theorem 4.7 (Artin [2]). *If $w \in X_B^*$ and $w\phi_B \in P$, then $w \sim_B w'$ for some $w' \in A^*$.*

Lemma 4.8. *If $w \in X_B^*$ and $w\phi_B \in P$, then $we \sim_R ew$.*

Proof. Suppose that $1 \leqslant i < j \leqslant n$. If $i \geqslant 2$, then by relations (F), (E2) and (E5) we have $\alpha_{ij}e \sim_R e\alpha_{ij}$. If $i = 1$ and $j = 2$, then we have $\alpha_{12}e \sim_R e\alpha_{12}$ by (E2). If $i = 1$ and $j > 2$, then it is easy to check that $\alpha_{1j} \sim_B \sigma_1^{-1}\alpha_{2j}\sigma_1$, so that $\alpha_{1j}e \sim_R e\alpha_{1j}$ by (E2), (F), and the first calculation. Now, if $w \in X_B^*$ and $w\phi_B \in P$, then by Theorem 4.7 we have $w \sim_B w'$ for some $w' \in A^*$. The result now follows by induction on the number of generators from A involved in w'. □

For $1 \leqslant i < j \leqslant n$ let $w_{ij} = (\sigma_2 \cdots \sigma_{j-1})(\sigma_1 \cdots \sigma_{i-1})$ so that $e_{ij} = w_{ij}^{-1}ew_{ij}$.

Corollary 4.9. *If $1 \leqslant i < j \leqslant n$ and $w \in X_B^*$ with $w\phi_B \in P$, then $we_{ij} \sim_R e_{ij}w$.*

Proof. Now $(w_{ij}ww_{ij}^{-1})\phi_B = (w_{ij}\phi_B)(w\phi_B)(w_{ij}\phi_B)^{-1} \in P$, so that, by relation (F) and Lemma 4.8, we have

$$we_{ij} \sim_R w_{ij}^{-1}(w_{ij}ww_{ij}^{-1})ew_{ij} \sim_R w_{ij}^{-1}e(w_{ij}ww_{ij}^{-1})w_{ij} \sim_R e_{ij}w,$$

completing the proof. □

Lemma 4.10. *Suppose that $1 \leqslant i < j \leqslant n$, $r \in \{1, \ldots, n-1\}$, and $\eta \in \{\pm 1\}$. Then*

$$\sigma_r^{\eta}e_{ij}\sigma_r^{-\eta} \sim_R e_{is_r,js_r}.$$

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Proof. First note that if $\sigma_r^{\eta} e_{ij} \sigma_r^{-\eta} \sim_R e_{is_r, js_r}$, then we also have

$$\sigma_r^{-\eta} e_{ij} \sigma_r^{\eta} \sim_R \sigma_r^{-2\eta} \sigma_r^{\eta} e_{ij} \sigma_r^{-\eta} \sigma_r^{2\eta} \sim_R \sigma_r^{-2\eta} e_{is_r, js_r} \sigma_r^{2\eta} \sim_R e_{is_r, js_r}$$

by (F) and Corollary 4.9. Thus, it suffices to prove the lemma for any choice of η. If $r = i-1$ or $r = j-1 > i$, we use $\eta = 1$, while if $r = i < j-1$ or $r = j$, we use $\eta = -1$, and the result follows trivially. Suppose now that either $r \notin \{i-1, i, j-1, j\}$ or $r = i = j-1$. It is an easy exercise, using the braid relations, to show that

$$w_{ij}\sigma_r \sim_B \begin{cases} \sigma_{r+2} w_{ij} & \text{if } r < i-1, \\ \sigma_{r+1} w_{ij} & \text{if } i < r < j-1, \\ \sigma_r w_{ij} & \text{if } j < r, \\ \sigma_1 w_{ij} & \text{if } r = i = j-1. \end{cases}$$

It now follows that $\sigma_r^{-1} e_{ij} \sigma_r = \sigma_r^{-1} w_{ij}^{-1} e w_{ij} \sigma_r \sim_R e_{ij}$ using (F) and (E2). □

Corollary 4.11. *If $w \in X_B^*$ and $1 \leqslant i < j \leqslant n$, then $w^{-1} e_{ij} w \sim_R e_{i\bar{w}, j\bar{w}}$.*

Proof. This follows from Lemma 4.10 and induction on the length of w. □

Lemma 4.12. *If $1 \leqslant i < j \leqslant n$, then $e_{ij}^2 \sim_R e_{ij}$.*

Proof. This follows immediately from (F) and (E1). □

Lemma 4.13. *If $1 \leqslant i < j \leqslant n$ and $1 \leqslant k < l \leqslant n$, then $e_{ij} e_{kl} \sim_R e_{kl} e_{ij}$.*

Proof. We first show that $e_{12} e_{23} \sim_R e_{23} e_{12}$ and $e_{12} e_{34} \sim_R e_{34} e_{12}$. For the former we have

$$\begin{aligned} e_{12} e_{23} &= e \sigma_1^{-1} \sigma_2^{-1} e \sigma_2 \sigma_1 \\ &\sim_R \sigma_1^{-1} e \sigma_2^{-2} \sigma_2 e \sigma_2 \sigma_1 && \text{by (E2) and (F)} \\ &\sim_R \sigma_1^{-1} \sigma_2^{-2} e \sigma_2 e \sigma_2 \sigma_1 && \text{by (F) and (E5)} \\ &\sim_R \sigma_1^{-1} \sigma_2^{-2} \sigma_2 e \sigma_2 e \sigma_1 && \text{by (E3)} \\ &\sim_R \sigma_1^{-1} \sigma_2^{-1} e \sigma_2 \sigma_1 e && \text{by (F) and (E2)} \\ &= e_{23} e_{12}. \end{aligned}$$

For the latter, note that $w_{34}^{-2} \phi_B \in P$ so that

$$\begin{aligned} e_{12} e_{34} &= e w_{34}^{-1} e w_{34} \\ &\sim_R e w_{34}^{-2} w_{34} e w_{34} && \text{by (F)} \\ &\sim_R w_{34}^{-2} e w_{34} e w_{34} && \text{by Lemma 4.8} \\ &\sim_R w_{34}^{-2} w_{34} e w_{34} e && \text{by (E4)} \\ &\sim_R w_{34}^{-1} e w_{34} e && \text{by (F)} \\ &= e_{34} e_{12}. \end{aligned}$$

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Returning to the general case, suppose first that one of k, l (say k) is equal to one of i, j (say j). Choose $w \in X_B^*$ such that $(1,2,3)\bar{w} = (i,j,l)$. By Corollary 4.11, (F) and the first calculation, we then have

$$e_{ij}e_{jl} = e_{1\bar{w},2\bar{w}}e_{2\bar{w},3\bar{w}} \sim_R w^{-1}e_{12}ww^{-1}e_{23}w \sim_R w^{-1}e_{12}e_{23}w \sim_R w^{-1}e_{23}e_{12}w \sim_R e_{jl}e_{ij}.$$

Finally, if i, j, k and l are all distinct, then we choose $w \in X_B^*$ such that $(1,2,3,4)\bar{w} = (i,j,k,l)$. We use the same trick, and the second calculation, to show that $e_{ij}e_{kl} \sim_R e_{kl}e_{ij}$. □

Lemma 4.14. *If $1 \leqslant i < j < k \leqslant n$, then $e_{ij}e_{jk} \sim_R e_{jk}e_{ik} \sim_R e_{ik}e_{ij}$.*

Proof. As in the proof of the previous lemma, we need only show that the lemma holds when $(i,j,k) = (1,2,3)$. Now

$$\begin{aligned} e_{12}e_{23} &= e\sigma_1^{-1}\sigma_2^{-1}e\sigma_2\sigma_1 \\ &\sim_R e\sigma_2^{-2}\sigma_2e\sigma_2\sigma_1 && \text{by (E1) and (F)} \\ &\sim_R \sigma_2^{-2}e\sigma_2e\sigma_2\sigma_1 && \text{by (E5) and (F)} \\ &\sim_R \sigma_2^{-2}\sigma_2e\sigma_2e\sigma_1 && \text{by (E3)} \\ &\sim_R \sigma_2^{-1}e\sigma_2e && \text{by (F) and (E1)} \\ &= e_{13}e_{12}. \end{aligned}$$

Next observe that

$$\begin{aligned} e_{23}\sigma_2 &= \sigma_1^{-1}\sigma_2^{-1}e\sigma_2\sigma_1\sigma_2 \\ &\sim_R \sigma_1^{-1}\sigma_2^{-1}e\sigma_1\sigma_2\sigma_1 && \text{by (B2)} \\ &\sim_R \sigma_1^{-1}\sigma_2^{-1}e\sigma_2\sigma_1 && \text{by (E1)} \\ &= e_{23}. \end{aligned}$$

But then

$$\begin{aligned} e_{23}e_{13} &= e_{23}\sigma_2^{-1}e_{12}\sigma_2 \\ &\sim_R e_{23}e_{12}\sigma_2 && \text{by the observation and (F)} \\ &\sim_R e_{12}e_{23}\sigma_2 && \text{by Lemma 4.13} \\ &\sim_R e_{12}e_{23} && \text{by the observation again} \end{aligned}$$

and we are done. □

Lemma 4.15. *If $1 \leqslant i < j \leqslant n$ and $\beta \in P$, then $e_{ij}\hat{\beta}^{-1}\sigma_{ij}\hat{\beta} \sim_R e_{ij}$.*

Proof. Now, it may easily be checked that $\sigma_{ij} \sim_B w_{ij}^{-1}\sigma_1 w_{ij}$ for each $1 \leqslant i < j \leqslant n$. So if $\beta \in P$ and $1 \leqslant i < j \leqslant n$, we then have

$$\begin{aligned} e_{ij}\hat{\beta}^{-1}\sigma_{ij}\hat{\beta} &\sim_R \hat{\beta}^{-1}e_{ij}\sigma_{ij}\hat{\beta} && \text{by Corollary 4.9} \\ &\sim_R \hat{\beta}^{-1}w_{ij}^{-1}ew_{ij}w_{ij}^{-1}\sigma_1 w_{ij}\hat{\beta} \\ &\sim_R \hat{\beta}^{-1}w_{ij}^{-1}e\sigma_1 w_{ij}\hat{\beta} && \text{by (F)} \\ &\sim_R \hat{\beta}^{-1}w_{ij}^{-1}ew_{ij}\hat{\beta} && \text{by (E1)} \\ &= \hat{\beta}^{-1}e_{ij}\hat{\beta} \\ &\sim_R e_{ij} && \text{by Corollary 4.9 and (F)} \end{aligned}$$

and the lemma is proved. □

Lemmas 4.10 and 4.12–4.15 show that relations $R'_E \cup R'_{\rtimes} \cup R'_{\sim}$ are all implied by R. Thus, we have the following.

Theorem 4.16. *The factorizable braid monoid $\mathfrak{F}\mathcal{B}_n$ has monoid presentation*

$$\langle X_B \cup \{e\} | R \rangle$$

via $\sigma_i^{\pm 1} \mapsto [1, \varsigma_i^{\pm 1}]$, $e \mapsto [\mathcal{E}_{12}, 1]$.

5. The monoid of uniform block bijections

For $\mathcal{E} \in E$ we denote by $\mathbf{n}/\mathcal{E}$ the quotient of $\mathbf{n}$ by $\mathcal{E}$, which is the set of $\mathcal{E}$-classes of $\mathbf{n}$. A *block bijection* on $\mathbf{n}$ is a bijection $\theta : \mathbf{n}/\mathcal{E} \to \mathbf{n}/\mathcal{E}'$ where $\mathcal{E}, \mathcal{E}' \in E$. The set of all block bijections on $\mathbf{n}$ forms an inverse monoid, denoted $\mathcal{I}_n^*$, called the *dual symmetric inverse monoid* (see [**11**] for details).

A block bijection $\theta : \mathbf{n}/\mathcal{E} \to \mathbf{n}/\mathcal{E}'$ is called *uniform* if $|A| = |A\theta|$ for every $\mathcal{E}$-class $A \in \mathbf{n}/\mathcal{E}$. The set of all uniform block bijections, denoted $\mathfrak{F}_n$, is the largest factorizable inverse submonoid of $\mathcal{I}_n^*$ (see [**10, 11**]). We identify $\mathbf{n}/1$ with $\mathbf{n}$, where here 1 represents the identity of E, and in the same way we may regard a permutation $\pi : \mathbf{n} \to \mathbf{n}$ as a block bijection $\pi : \mathbf{n}/1 \to \mathbf{n}/1$. For $\mathcal{E} \in E$ denote by $\mathrm{id}_{\mathcal{E}} : \mathbf{n}/\mathcal{E} \to \mathbf{n}/\mathcal{E}$ the identity map on $\mathbf{n}/\mathcal{E}$. We have $E(\mathfrak{F}_n) = \{\mathrm{id}_{\mathcal{E}} \mid \mathcal{E} \in E\} \cong E$ since $\mathrm{id}_{\mathcal{E}}\,\mathrm{id}_{\mathcal{E}'} = \mathrm{id}_{\mathcal{E}\vee\mathcal{E}'}$ for each $\mathcal{E}, \mathcal{E}' \in E$, and $G(\mathfrak{F}_n) = S$. Thus, every element $\theta \in \mathfrak{F}_n$ has a factorization

$$\theta = \mathrm{id}_{\mathcal{E}}\,\pi$$

for some $\mathcal{E} \in E$ and some $\pi \in S$. In this factorization, $\mathcal{E}$ is uniquely determined, but π need not be. In fact, we have $\mathrm{id}_{\mathcal{E}}\,\pi = \mathrm{id}_{\mathcal{E}}\,\tau$ if and only if $\pi\tau^{-1} \in S_{\mathcal{E}}$, where $S_{\mathcal{E}}$ is the subgroup of S defined by

$$S_{\mathcal{E}} = \{\pi \in S \mid (i, i\pi) \in \mathcal{E},\ \forall i \in \mathbf{n}\}.$$

The subgroup $S_{\mathcal{E}}$ is generated by the set $\{t_{ij} \mid 1 \leqslant i < j \leqslant n,\ (i,j) \in \mathcal{E}\}$, where t_{ij} denotes the transposition which interchanges i and j. As an application of Theorem 4.16 we will provide an alternative proof of the presentation of $\mathfrak{F}_n$ given by FitzGerald [**10**].

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Theorem 5.1 (FitzGerald [10]). *The monoid $\mathfrak{F}_n$ has monoid presentation $\langle X_F | R_F \rangle$, where $X_F = \{\sigma_1, \dots, \sigma_{n-1}, e\}$ and R_F is the set of relations (B1), (B2), (E1)–(E4), and*

$$\sigma_i^2 = 1 \quad \textit{for all } i. \tag{S}$$

Proof. Define $\Theta : (X_B \cup \{e\})^* \to \mathfrak{F}_n$ by $\sigma_i^{\pm 1}\Theta = s_i$ for each i, and $e\Theta = \mathrm{id}_{\mathcal{E}_{12}}$. Then Θ is an epimorphism since $\mathfrak{F}_n$ is generated by the s_i and $\mathrm{id}_{\mathcal{E}_{ij}} = e_{ij}\Theta$. Let $\sim_S$ be the congruence on $(X_B \cup \{e\})^*$ generated by R together with relations (S). It is easy to check that $w_1\Theta = w_2\Theta$ for all $(w_1, w_2) \in R$ and, since $s_i^2 = 1$ for each i, we have $\sim_S \subseteq \ker\Theta$. To show the reverse inclusion, suppose that $w_1, w_2 \in (X_B \cup \{e\})^*$ and $w_1\Theta = w_2\Theta$. We have

$$w_1 \sim_R \hat{\mathcal{E}}_1\hat{\beta}_1 \quad \text{and} \quad w_2 \sim_R \hat{\mathcal{E}}_2\hat{\beta}_2$$

for some $\mathcal{E}_1, \mathcal{E}_2 \in E$ and $\beta_1, \beta_2 \in B$. Put $\pi_1 = \bar{\beta}_1$ and $\pi_2 = \bar{\beta}_2$. Then

$$\mathrm{id}_{\mathcal{E}_1}\pi_1 = w_1\Theta = w_2\Theta = \mathrm{id}_{\mathcal{E}_2}\pi_2.$$

Thus, $\mathcal{E}_1 = \mathcal{E}_2$ and $\pi_1\pi_2^{-1} \in S_{\mathcal{E}_1}$, so that $\hat{\mathcal{E}}_1 = \hat{\mathcal{E}}_2$ and

$$\pi_1\pi_2^{-1} = t_{i_1j_1} \cdots t_{i_kj_k}$$

for some $k \in \mathbb{N}$, and some $i_1, \dots, i_k, j_1, \dots, j_k \in \mathbf{n}$ with $i_s < j_s$ and $(i_s, j_s) \in \mathcal{E}_1$ for each $s \in \mathbf{k}$. But then

$$\bar{\beta}_1 = \pi_1 = t_{i_1j_1} \cdots t_{i_kj_k}\pi_2 = \overline{\varsigma_{i_1j_1} \cdots \varsigma_{i_kj_k}\beta_2}$$

so that $\beta_1 = \varsigma_{i_1j_1} \cdots \varsigma_{i_kj_k}\beta_2\gamma$ for some $\gamma \in P$. Now, by Theorem 4.7, we have

$$\hat{\gamma} \sim_B \alpha_{p_1q_1}^{\pm 1} \cdots \alpha_{p_hq_h}^{\pm 1}$$

for some $h \in \mathbb{N}$, and $p_1, \dots, p_h, q_1, \dots, q_h \in \mathbf{n}$ with $p_s < q_s$ for each $s \in \mathbf{h}$, and so

$$\begin{aligned}
w_1 &\sim_S \hat{\mathcal{E}}_1\hat{\beta}_1 \\
&\sim_S \hat{\mathcal{E}}_1\sigma_{i_1j_1} \cdots \sigma_{i_kj_k}\hat{\beta}_2\alpha_{p_1q_1}^{\pm 1} \cdots \alpha_{p_hq_h}^{\pm 1} \\
&\sim_S \hat{\mathcal{E}}_1\hat{\beta}_2\alpha_{p_1q_1}^{\pm 1} \cdots \alpha_{p_hq_h}^{\pm 1} && \text{by Lemma 4.15} \\
&\sim_S \hat{\mathcal{E}}_1\hat{\beta}_2 && \text{by (S)} \\
&= \hat{\mathcal{E}}_2\hat{\beta}_2 \\
&\sim_S w_2
\end{aligned}$$

so that $\ker\Theta \subseteq {\sim_S}$. Thus, $\ker\Theta = {\sim_S}$ and so $\mathfrak{F}_n \cong (X_B \cup \{e\})^*/{\sim_S}$. It finally remains to observe, by rewriting the presentation using (S), that $(X_B \cup \{e\})^*/{\sim_S} \cong X_F^*/R_F^\sharp$. $\square$

Remark 5.2. Since $\mathfrak{F}_n$ itself is a factorizable inverse monoid, an approach similar to that used in § 4 may be used to obtain Theorem 5.1 directly.

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6. A second presentation of $\mathfrak{F}\mathcal{B}_n$

While the presentation of $\mathfrak{F}\mathcal{B}_n$ we derived in § 4 was economical in terms of the number of generators involved, the relations do not display a great deal of symmetry. The aim of this section will be to introduce a number of new generators, thereby obtaining a presentation which reflects the symmetry possessed by $\mathfrak{F}\mathcal{B}_n$. This presentation will also highlight an interesting connection between $\mathfrak{F}\mathcal{B}_n$ and $\mathcal{SB}_n$, the singular braid monoid (introduced in [**3**, **5**]). We will explore this connection in the next section.

We begin with the presentation $\langle X_B \cup \{e\} | R\rangle$ of $\mathfrak{F}\mathcal{B}_n$ obtained in Theorem 4.16. We now rename $e = e_1$, and add generators $e_2, \dots, e_{n-1}$ to the presentation along with relations

$$e_i = (\sigma_{i-1}^{-1} \cdots \sigma_1^{-1})(\sigma_i^{-1} \cdots \sigma_2^{-1})e_1(\sigma_2 \cdots \sigma_i)(\sigma_1 \cdots \sigma_{i-1}) \quad \text{for all } i, \tag{D}$$

which define them in terms of the original generators. In fact, in the notation of § 4, we have $e_i = e_{i,i+1}$. So, by Corollary 4.9, and Lemmas 4.10, 4.12, 4.13 and 4.15, the relations

$$e_i^2 = e_i = e_i\sigma_i = \sigma_i e_i \quad \text{for all } i, \tag{E1$'$}$$
$$e_i e_j = e_j e_i \quad \text{for all } i, j, \tag{E2$'$}$$
$$e_i \sigma_j = \sigma_j e_i \quad \text{if } |i-j| > 1, \tag{E3$'$}$$
$$e_i \sigma_j \sigma_i = \sigma_j \sigma_i e_j \quad \text{if } |i-j| = 1, \tag{E4$'$}$$
$$e_i \sigma_j^2 = \sigma_j^2 e_i \quad \text{if } |i-j| = 1 \tag{E5$'$}$$

follow from R. Thus, we add relations (E1)$'$–(E5)$'$ to the presentation. Now relations (E1), (E2) and (E5) may clearly be removed since they are part of relations (E1)$'$, (E3)$'$ and (E5)$'$. Next we will show that relations (E3), (E4), and (D) may also be removed. Put $Y = X_B \cup \{e_1, \dots, e_{n-1}\}$ and denote by R' the set of relations (F), (B1), (B2) and (E1)$'$–(E5)$'$.

Lemma 6.1. *We have* $e_1\sigma_2 e_1 \sigma_2 \sim_{R'} \sigma_2 e_1 \sigma_2 e_1$.

Proof. Observe first that if $1 \leqslant i, j \leqslant n-1$ and $|i-j| = 1$, then by (F) and (E4)$'$ we have

$$\sigma_i e_j \sigma_i^{-1} \sim_{R'} \sigma_j^{-1}\sigma_j\sigma_i e_j \sigma_i^{-1} \sim_{R'} \sigma_j^{-1} e_i \sigma_j \sigma_i \sigma_i^{-1} \sim_{R'} \sigma_j^{-1} e_i \sigma_j.$$

But then

$$\begin{aligned} e_1\sigma_2 e_1\sigma_2 &\sim_{R'} e_1\sigma_2 e_1 \sigma_2^{-1}\sigma_2^2 && \text{by (F)} \\ &\sim_{R'} e_1(\sigma_1^{-1} e_2 \sigma_1 \sigma_2^2) && \text{by the observation} \\ &\sim_{R'} (\sigma_1^{-1} e_2 \sigma_1 \sigma_2^2) e_1 && \text{by (E1)}', \text{(F)}, \text{(E2)}' \text{ and (E5)}' \\ &\sim_{R'} \sigma_2 e_1 \sigma_2^{-1} \sigma_2^2 e_1 && \text{by the observation again} \\ &\sim_{R'} \sigma_2 e_1 \sigma_2 e_1 && \text{by (F)} \end{aligned}$$

and we are done. □

Lemma 6.2. *We have* $e_1 w e_1 w \sim_{R'} w e_1 w e_1$, *where* $w = \sigma_2\sigma_3\sigma_1\sigma_2$.

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Proof. First observe that by (B1) and (E4)′ we have

$$e_1 w = e_1\sigma_2\sigma_3\sigma_1\sigma_2 \sim_{R'} e_1\sigma_2\sigma_1\sigma_3\sigma_2 \sim_{R'} \sigma_2\sigma_1 e_2\sigma_3\sigma_2$$
$$\sim_{R'} \sigma_2\sigma_1\sigma_3\sigma_2 e_3 \sim_{R'} \sigma_2\sigma_3\sigma_1\sigma_2 e_3 = we_3,$$

By a similar calculation we also have $e_3 w \sim_{R'} we_1$. But then by (E2)′ and these observations we have

$$e_1 w e_1 w \sim_{R'} we_3 e_1 w \sim_{R'} we_1 e_3 w \sim_{R'} we_1 we_1,$$

completing the proof. □

Lemma 6.3. *If* $1 \leqslant i \leqslant n-1$, *then*

$$e_i \sim_{R'} (\sigma_{i-1}^{-1}\cdots\sigma_1^{-1})(\sigma_i^{-1}\cdots\sigma_2^{-1})e_1(\sigma_2\cdots\sigma_i)(\sigma_1\cdots\sigma_{i-1}).$$

Proof. We prove the lemma by induction on i. If $i = 1$, then there is nothing to prove, so suppose that the lemma holds for some $1 \leqslant i \leqslant n-2$. We then have

$$\begin{aligned} e_{i+1} &\sim_{R'} \sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_{i+1}\sigma_i e_{i+1} && \text{by (F)}\\ &\sim_{R'} \sigma_i^{-1}\sigma_{i+1}^{-1}e_i\sigma_{i+1}\sigma_i && \text{by (E4)}'\\ &\sim_{R'} \sigma_i^{-1}\sigma_{i+1}^{-1}(\sigma_{i-1}^{-1}\cdots\sigma_1^{-1})(\sigma_i^{-1}\cdots\sigma_2^{-1})e_1(\sigma_2\cdots\sigma_i)(\sigma_1\cdots\sigma_{i-1})\sigma_{i+1}\sigma_i \\ &\sim_{R'} \sigma_i^{-1}(\sigma_{i-1}^{-1}\cdots\sigma_1^{-1})\sigma_{i+1}^{-1}(\sigma_i^{-1}\cdots\sigma_2^{-1})e_1(\sigma_2\cdots\sigma_i)\sigma_{i+1}(\sigma_1\cdots\sigma_{i-1})\sigma_i && \text{by (B1), and (F)} \end{aligned}$$

and we are done. □

The last three lemmas have shown that relations (E3), (E4) and (D) are implied by R'. Thus, we have the following.

Theorem 6.4. *The factorizable braid monoid* $\mathfrak{F}\mathcal{B}_n$ *has monoid presentation* $\langle Y|R'\rangle$ *via* $\sigma_i^{\pm 1} \mapsto [1, \varsigma_i^{\pm 1}]$, $e_i \mapsto [\mathcal{E}_{i,i+1}, 1]$.

Remark 6.5. We may also derive a second presentation of $\mathfrak{F}_n$ from the presentation of $\mathfrak{F}\mathcal{B}_n$ given in Theorem 6.4. First we add the relations

$$\sigma_i^2 = 1 \quad \text{for all } i \tag{S}$$

to the presentation $\langle Y|R'\rangle$ to obtain, by the same method as in the proof of Theorem 5.1, an intermediate presentation of $\mathfrak{F}_n$. This presentation then simplifies to $\langle X_F'|R_F'\rangle$, where $X_F' = \{\sigma_1,\dots,\sigma_{n-1},e_1,\dots,e_{n-1}\}$ and R_F' is the set of relations (B1), (B2), (E1)′–(E4)′, and (S).

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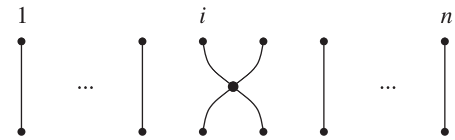

Figure 6. The singular braid $\tau_i \in \mathbf{SB}_n$.

7. Flexible singular braids and relation (E5)′

A *singular braid* is a collection of strings, much like a braid, with the exception that there may exist a finite number of *double points* (or *singular points*) where a pair of strings intersect. Let $\mathbf{SB}_n$ denote the set of all singular braids with n strings. The concatenation of two singular braids $\beta, \gamma \in \mathbf{SB}_n$ is the singular braid $\beta\gamma$ obtained by joining the 'bottom' of β to the 'top' of γ. Thus, $\mathbf{SB}_n$ is a groupoid under concatenation. The *singular braid monoid*, denoted $\mathcal{SB}_n$, is the monoid of *rigid-vertex-isotopy classes* of singular braids on n strings. (For more details on singular braids, see [**3**, **4**].) In this section it will be useful to draw a clear distinction between a singular braid $\beta \in \mathbf{SB}_n$ and its rigid-vertex-isotopy-class which we will denote by $[\beta] \in \mathcal{SB}_n$. The singular braid monoid is generated by $[\varsigma_1^{\pm 1}], \dots, [\varsigma_{n-1}^{\pm 1}]$ together with $[\tau_1], \dots, [\tau_{n-1}]$. The singular braid $\tau_i \in \mathbf{SB}_n$ is pictured in Figure 6.

The following was first proved in [**4**] (see also [**3**, **13**]).

Theorem 7.1 (Birman [4]). *The singular braid monoid $\mathcal{SB}_n$ has monoid presentation $\langle Y | R_{\mathrm{SB}} \rangle$ via*

$$\phi_{\mathrm{SB}} : Y^* \to \mathcal{SB}_n : \begin{cases} \sigma_i^{\pm 1} \mapsto [\varsigma_i^{\pm 1}] \\ e_i \mapsto [\tau_i] \end{cases}$$

where R_{SB} is the set of relations (F), (B1), (B2), and

$$e_i\sigma_i = \sigma_i e_i \quad \text{for all } i, \tag{SB1}$$
$$e_i e_j = e_j e_i \quad \text{if } |i-j| > 1, \tag{SB2}$$
$$e_i \sigma_j = \sigma_j e_i \quad \text{if } |i-j| > 1, \tag{SB3}$$
$$e_i \sigma_j \sigma_i = \sigma_j \sigma_i e_j \quad \text{if } |i-j| = 1. \tag{SB4}$$

Notice that relations (SB1)–(SB4) are part of relations (E1)′–(E4)′, so that, in particular, $\mathfrak{F}\mathcal{B}_n$ is (isomorphic to) a quotient of $\mathcal{SB}_n$. If $\beta, \gamma \in \mathbf{SB}_n$ are singular braids, then we write:

(i) $\beta \asymp_{(\mathrm{i})} \gamma$ if β and γ are equivalent under rigid-vertex-isotopy;

(ii) $\beta \asymp_{(\mathrm{ii})} \gamma$ if β and γ are identical except for a neighbourhood which contains any of the fragments catalogued in Figure 7; or

(iii) $\beta \asymp_{(\mathrm{iii})} \gamma$ if β and γ are identical except for a neighbourhood which contains any of the fragments catalogued in Figure 8.

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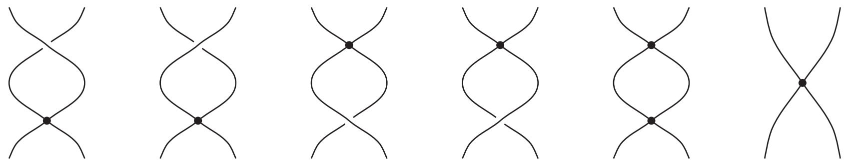

Figure 7. Move (ii).

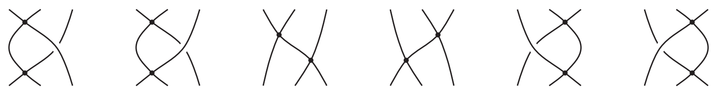

Figure 8. Move (iii).

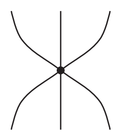

Figure 9. An intermediate triple singular point created during move (iii).

Remark 7.2. Move (ii) can be thought of as allowing a singular point to 'swallow up' or 'produce' another twist or singular point directly above or below it, involving the same two strings. Move (iii) can be achieved by allowing a *triple* singular point (see Figure 9) to be momentarily created and then destroyed, as the strings pass from one of the configurations of Figure 8 to another.

If $\beta, \gamma \in \mathbf{SB}_n$ are singular braids, then we say that β and γ are *flexible-vertex-isotopic*, and write $\beta \asymp \gamma$ if there is a sequence of singular braids $\beta = \beta_0, \beta_1, \dots, \beta_k = \gamma$ such that, for each j, we have either $\beta_j \asymp_{(\mathrm{i})} \beta_{j+1}$, $\beta_j \asymp_{(\mathrm{ii})} \beta_{j+1}$, or $\beta_j \asymp_{(\mathrm{iii})} \beta_{j+1}$. We denote the $\asymp$-class of a singular braid $\beta \in \mathbf{SB}_n$ by $[\beta]_\asymp$. It is clear that $\asymp$ is a (groupoid) congruence on $\mathbf{SB}_n$, and that $\beta(\gamma\delta) \asymp (\beta\gamma)\delta$ and $1\beta \asymp \beta 1 \asymp \beta$ for all $\beta, \gamma, \delta \in \mathbf{SB}_n$. Thus, we may form the quotient monoid $\mathbf{SB}_n/\!\asymp = \{[\beta]_\asymp \mid \beta \in \mathbf{SB}_n\}$, which we call the *flexible singular braid monoid* and denote by $\mathfrak{F}\mathcal{SB}_n$.

Theorem 7.3. *The flexible singular braid monoid $\mathfrak{F}\mathcal{SB}_n$ has monoid presentation $\langle Y | R'' \rangle$ via*

$$\Phi : \sigma_i^{\pm 1} \mapsto [\varsigma_i^{\pm 1}]_\asymp, \quad e_i \mapsto [\tau_i]_\asymp,$$

where R'' is the set of relations (F), (B1), (B2) and (E1)′–(E4)′.

Proof. Now $\Phi = \phi_{\mathrm{SB}}\nu : Y^* \to \mathfrak{F}\mathcal{SB}_n$, where $\nu : \mathcal{SB}_n \to \mathfrak{F}\mathcal{SB}_n$ is the natural map $[\beta] \mapsto [\beta]_\asymp$. So Φ is an epimorphism and, since $w_1\Phi = w_2\Phi$ for each $(w_1, w_2) \in R''$, as may easily be checked, we have $\sim_{R''} \subseteq \ker\Phi$. To show the reverse inclusion, suppose that $w_1, w_2 \in Y^*$ such that $w_1\Phi = w_2\Phi$. Choose $\beta, \gamma \in \mathbf{SB}_n$ such that $[\beta] = w_1\phi_{\mathrm{SB}}$ and $[\gamma] = w_2\phi_{\mathrm{SB}}$. We then have $\beta \asymp \gamma$, and we must show that $w_1 \sim_{R''} w_2$. By induction it suffices to assume that $\beta \asymp_{(\mathrm{i})} \gamma$, $\beta \asymp_{(\mathrm{ii})} \gamma$ or $\beta \asymp_{(\mathrm{iii})} \gamma$. If $\beta \asymp_{(\mathrm{i})} \gamma$, then $w_1 \sim_{R''} w_2$, using the singular braid relations $R_{\mathrm{SB}} \subseteq R''$. Suppose next that $\beta \asymp_{(\mathrm{ii})} \gamma$. There then exists

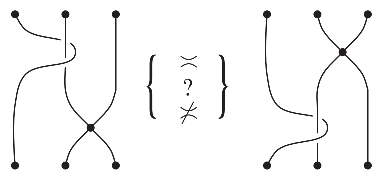

Figure 10. The relation $\varsigma_i^2\tau_{i+1} \asymp \tau_{i+1}\varsigma_i^2$ does not appear to hold in $\mathfrak{F}\mathcal{SB}_n$.

$1 \leqslant i \leqslant n-1$ such that $w_1 = wuw'$ and $w_2 = wvw'$ for some $w, w' \in Y^*$ and some

$$u, v \in \{\sigma_i^{\pm 1}e_i, e_i\sigma_i^{\pm 1}, e_i^2, e_i\}.$$

But then we have $w_1 \sim_{R''} w_2$ by (E1)$'$ and (F). Finally, suppose that $\beta \asymp_{\text{(iii)}} \gamma$. There then exist $1 \leqslant i, j \leqslant n-1$ with $|i-j| = 1$ such that $w_1 = wuw'$ and $w_2 = wvw'$ for some $w, w' \in Y^*$ and some

$$u, v \in \{e_ie_j, e_je_i, e_i\sigma_j^{\pm 1}e_i, e_j\sigma_i^{\pm 1}e_j\}.$$

The proof will be complete if we can show that all of the words in this set are R''-equivalent. Now $e_ie_j \sim_{R''} e_je_i$ by (E2)$'$, while if $\{k, l\} = \{i, j\}$, then

$$\begin{aligned} e_k\sigma_l^{\pm 1}e_k &\sim_{R''} \sigma_l^{\pm 1}\sigma_l^{\mp 1}e_k\sigma_l^{\pm 1}e_k && \text{by (F)} \\ &\sim_{R''} \sigma_l^{\pm 1}\sigma_k^{\pm 1}e_l\sigma_k^{\mp 1}e_k && \text{by the observation in the proof of Lemma 6.1} \\ &\sim_{R''} \sigma_l^{\pm 1}\sigma_k^{\pm 1}e_le_k && \text{by (E1)}' \text{ and (F)} \\ &\sim_{R''} e_le_k && \text{by several applications of (E1)}', \text{(E2)}' \text{ and (F),} \end{aligned}$$

completing the proof. □

Since the presentation of $\mathfrak{F}\mathcal{SB}_n$ in Theorem 7.3 differs from the presentation of $\mathfrak{F}\mathcal{B}_n$ in Theorem 6.4 only by the absence of relation (E5)$'$, it is natural to wonder whether in fact $\mathfrak{F}\mathcal{SB}_n$ and $\mathfrak{F}\mathcal{B}_n$ are isomorphic. The existence of such an isomorphism would be guaranteed if relation (E5)$'$ was a consequence of relations R'', which, by Theorem 7.3, would be equivalent to knowing that $\tau_i\varsigma_j^2 \asymp \varsigma_j^2\tau_i$ for each $1 \leqslant i, j \leqslant n-1$ with $|i-j| = 1$. Figure 10 gives a good indication that this relation 'ought not' to hold, and Lemma 7.4, below, proves that it does not.

Lemma 7.4. *Suppose that* $1 \leqslant i, j \leqslant n-1$ *and* $|i-j| = 1$*. Then* $e_i\sigma_j^2 \not\sim_{R''} \sigma_j^2e_i$*.*

Proof. Let $\mathcal{C}(B)$ be the set of all cosets of all subgroups of $B = \mathcal{B}_n$. If H and K are subgroups of B and $\beta, \gamma \in B$, then the product of the cosets $H\beta$ and $K\gamma$ is defined to be

$$(H\beta) * (K\gamma) = (H \vee (\beta K\beta^{-1}))\beta\gamma.$$

This product turns $\mathcal{C}(B)$ into a (factorizable inverse) monoid known as the *coset monoid* of the braid group (for more details see [**8**, **16**]). We define a homomorphism $\Psi : Y^* \to \mathcal{C}(B)$ by

$$\sigma_i^{\pm 1}\Psi = \{1\}\varsigma_i^{\pm 1} \quad \text{and} \quad e_i\Psi = \langle\varsigma_i\rangle \quad \text{for each } i.$$

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One may easily check that $w_1\Psi = w_2\Psi$ for each $(w_1, w_2) \in R''$ so that $\sim_{R''} \subseteq \ker\Psi$. Suppose now that there exist $1 \leqslant i, j \leqslant n-1$ with $|i-j| = 1$ such that $e_i\sigma_j^2 \sim_{R''} \sigma_j^2 e_i$. We must then have the coset equality

$$\{\varsigma_i^m\varsigma_j^2 \mid m \in \mathbb{Z}\} = \langle\varsigma_i\rangle\varsigma_j^2 = (e_i\sigma_j^2)\Psi = (\sigma_j^2 e_i)\Psi = \varsigma_j^2\langle\varsigma_i\rangle = \{\varsigma_j^2\varsigma_i^m \mid m \in \mathbb{Z}\}.$$

In particular we must have $\varsigma_i\varsigma_j^2 = \varsigma_j^2\varsigma_i^m$ for some $m \in \mathbb{Z}$, and so $\sigma_i\sigma_j^2 \sim_B \sigma_j^2\sigma_i^m$. But since two $\sim_B$-equivalent words over X_B have the same exponent sum, we must have $m = 1$ so that $\sigma_i\sigma_j^2 \sim_B \sigma_j^2\sigma_i$. But then we must be able to transform $\sigma_i\sigma_j^2$ into $\sigma_j^2\sigma_i$ using only relations (B1) and (B2) (see, for example, [**12**]). But this is clearly impossible, and we have the required contradiction. □

This lemma shows that the map $\mathfrak{FSB}_n \to \mathfrak{FB}_n : [\varsigma_i^{\pm 1}]_{\asymp} \mapsto [1, \varsigma_i^{\pm 1}], [\tau_i]_{\asymp} \mapsto [\mathcal{E}_{i,i+1}, 1]$ is not an isomorphism. We now work towards showing that *no* isomorphism exists from $\mathfrak{FSB}_n$ to $\mathfrak{FB}_n$. We first state a well-known result concerning automorphisms of B. For $\beta \in B$, we denote by $\chi_\beta \in \mathrm{Aut}(B)$ the inner automorphism defined by $\gamma\chi_\beta = \beta^{-1}\gamma\beta$ for all $\gamma \in B$. We also let $\iota \in \mathrm{Aut}(B)$ be the automorphism of B determined by $\varsigma_i\iota = \varsigma_i^{-1}$ for each i.

Theorem 7.5 (Dyer and Grossman [7]). *Suppose that $\rho \in \mathrm{Aut}(B)$. Then*

(i) $\rho = \chi_\beta$ *for some* $\beta \in B$;

(ii) $\rho = \iota$; *or*

(iii) $\rho = \chi_\beta\iota$ *for some* $\beta \in B$.

Lemma 7.6. *Suppose that $\rho \in \mathrm{Aut}(B)$. There then exists $\tilde{\rho} \in \mathrm{Aut}(\mathfrak{FSB}_n)$ such that $[\gamma]_{\asymp}\tilde{\rho} = [\gamma\rho]_{\asymp}$ for all $\gamma \in B$.*

Proof. Suppose first that $\rho = \chi_\beta$ for some $\beta \in B$. Then we may take $\tilde{\rho}$ to be the automorphism of $\mathfrak{FSB}_n$ defined by $[\gamma]_{\asymp}\tilde{\rho} = [\beta^{-1}\gamma\beta]_{\asymp}$ for all $\gamma \in \mathfrak{FSB}_n$. Suppose next that $\rho = \iota$. Then we define $\tilde{\rho} : \mathfrak{FSB}_n \to \mathfrak{FSB}_n$ by $[\varsigma_i^{\pm 1}]_{\asymp}\tilde{\rho} = [\varsigma_i^{\mp 1}]_{\asymp}$ and $[\tau_i]_{\asymp}\tilde{\rho} = [\tau_i]_{\asymp}$ for each i. One may easily check, with the aid of Theorem 7.3, that $\tilde{\rho}$ is a well-defined homomorphism, which is clearly an involution and hence an automorphism. The result now follows from Theorem 7.5. □

Corollary 7.7. *If the monoids $\mathfrak{FSB}_n$ and $\mathfrak{FB}_n$ are isomorphic, then there is an isomorphism $\phi : \mathfrak{FSB}_n \to \mathfrak{FB}_n$ such that $[\beta]_{\asymp}\phi = [1, \beta]$ for all $\beta \in B$.*

Proof. Suppose that $\psi : \mathfrak{FSB}_n \to \mathfrak{FB}_n$ is an isomorphism. Since $[\beta]_{\asymp}\psi$ is invertible for all $\beta \in B$, we must have $[\beta]_{\asymp}\psi = [1, \beta\rho]$ for some $\beta\rho \in B$. But then $\rho : \beta \mapsto \beta\rho$ is easily seen to be an automorphism of B. By Lemma 7.6, we may extend ρ to an automorphism $\tilde{\rho}$ of $\mathfrak{FSB}_n$ such that $[\gamma]_{\asymp}\tilde{\rho} = [\gamma\rho]_{\asymp}$ for all $\gamma \in B$. The result now follows with $\phi = \tilde{\rho}^{-1}\psi$. □

Theorem 7.8. *The monoids $\mathfrak{FB}_n$ and $\mathfrak{FSB}_n$ are not isomorphic.*

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Proof. Suppose that $\mathfrak{F}\mathcal{B}_n$ and $\mathfrak{F}\mathcal{S}\mathcal{B}_n$ are isomorphic. Then, by Corollary 7.7, there is an isomorphism $\phi : \mathfrak{F}\mathcal{S}\mathcal{B}_n \to \mathfrak{F}\mathcal{B}_n$ such that $[\beta]_{\asymp}\phi = [1, \beta]$ for all $\beta \in B$. By Theorem 7.3 and Lemma 6.3 (the proof of which uses only the singular braid relations), we have

$$\tau_i \asymp (\varsigma_{i-1}^{-1} \cdots \varsigma_1^{-1})(\varsigma_i^{-1} \cdots \varsigma_2^{-1})\tau_1(\varsigma_2 \cdots \varsigma_i)(\varsigma_1 \cdots \varsigma_{i-1})$$

for each i. This shows that ϕ is completely determined by $[\tau_1]_{\asymp}\phi$, and also that $\mathfrak{F}\mathcal{S}\mathcal{B}_n$ is generated by $\{[\beta]_{\asymp} \mid \beta \in B\} \cup \{[\tau_1]_{\asymp}\}$. It follows that the monoid $\mathfrak{F}\mathcal{B}_n$ is generated by $[1, B] \cup \{[\tau_1]_{\asymp}\phi\}$, and since $[\tau_1]_{\asymp}\phi$ is an idempotent, we must have $[\tau_1]_{\asymp}\phi = [\mathcal{E}_{ij}, 1]$ for some $1 \leqslant i < j \leqslant n$. But then $[\mathcal{E}_{ij}, \varsigma_1] = [\tau_1\varsigma_1]_{\asymp}\phi = [\tau_1]_{\asymp}\phi = [\mathcal{E}_{ij}, 1]$ so that $(i, j) = (1, 2)$. Therefore, we also must have

$$\begin{aligned}[\tau_1\varsigma_2^2]_{\asymp}\phi &= ([\tau_1]_{\asymp}\phi)([\varsigma_2^2]_{\asymp}\phi) = [\mathcal{E}_{12}, 1][1, \varsigma_2^2] \\ &= [1, \varsigma_2^2][\mathcal{E}_{12}, 1] = ([\varsigma_2^2]_{\asymp}\phi)([\tau_1]_{\asymp}\phi) = [\varsigma_2^2\tau_1]_{\asymp}\phi.\end{aligned}$$

Since ϕ is an isomorphism, it follows that $\tau_1\varsigma_2^2 \asymp \varsigma_2^2\tau_1$. But then, by Theorem 7.3, we have $e_1\sigma_2^2 \sim_{R''} \sigma_2^2 e_1$, which contradicts Lemma 7.4. This completes the proof. □

8. The pure factorizable braid monoid

The results of this section will generally be concerned with group presentations, so we now take the time to establish the notation we will be using. Let X be a set, and let $X^{-1} = \{x^{-1} \mid x \in X\}$ be a set of formal inverses for the elements of X. Put $R_{F,X} = \{(x^{\pm 1}x^{\mp 1}, 1) \mid x \in X\}$. The free group on X, denoted $F(X)$, is defined to be the quotient $(X \cup X^{-1})^*/R_{F,X}^\sharp$. In practice, we will denote elements of $F(X)$ simply as words over $X \cup X^{-1}$, identifying two words w_1 and w_2 if $(w_1, w_2) \in R_{F,X}^\sharp$. If $R \subseteq F(X) \times F(X)$, then we denote by $R^\sharp$ the smallest congruence on $F(X)$ containing R. We say that a group G has group presentation $\langle X | R\rangle$ if $G \cong F(X)/R^\sharp$ or, equivalently, if there is an epimorphism $f : F(X) \to G$ with $\ker f = R^\sharp$. In this case we say that G has presentation $\langle X | R\rangle$ via f. If $(w_1, w_2) \in R^\sharp$, we write $w_1 \sim_R w_2$.

We now state two general results concerning group presentations. A proof of the first may be found in [**15**].

Lemma 8.1 (Magnus *et al.* [15]). *Suppose that G is a group with presentation $\langle X | R\rangle$ via f. Suppose also that $W \subseteq F(X)$ is a set of words and that N is the normal closure of Wf in G. Then G/N has presentation $\langle X | R \cup R_W\rangle$ via $f' : F(X) \to G/N : w \mapsto N(wf)$, where R_W is the set of relations*

$$w = 1 \quad \text{for all } w \in W. \qquad \square$$

Lemma 8.2. *Let X and Y be two disjoint sets, and define*

$$\bar{\ } : F(X \cup Y) \to F(X)$$

by $\bar{x} = x$ and $\bar{y} = 1$ for each $x \in X$ and $y \in Y$. Suppose that G is a group with presentation $\langle X \cup Y | R \cup S\rangle$ via f, where $R \subseteq F(X) \times F(X)$ and $\bar{S} \subseteq R^\sharp$. Then $H = (F(X))f$ has presentation $\langle X | R\rangle$ via $\phi = f|_{F(X)} : F(X) \to H : w \mapsto wf$.

Proof. Now, by definition, we know that ϕ is an epimorphism, and that $\sim_R \subseteq \ker\phi$. To prove the reverse inclusion, suppose that $w, w' \in F(X)$ and $w\phi = w'\phi$. There is then a sequence of words $w = w_1, w_2, \dots, w_k = w' \in F(X \cup Y)$ such that, for each i, $w_i = x_i u y_i$ and $w_{i+1} = x_i v y_i$ for some $x_i, y_i \in F(X \cup Y)$ and $(u, v) \in R \cup S$. Since $\bar{w} = w$ and $\bar{w}' = w'$, the result will follow if we can show that $\bar{w}_i \sim_R \bar{w}_{i+1}$ for each i. Now, if $(u, v) \in R$, then

$$\bar{w}_i = \bar{x}_i u \bar{y}_i \sim_R \bar{x}_i v \bar{y}_i = \bar{w}_{i+1},$$

while if $(u, v) \in S$, then $\bar{u} \sim_R \bar{v}$ by assumption, so that

$$\bar{w}_i = \bar{x}_i \bar{u} \bar{y}_i \sim_R \bar{x}_i \bar{v} \bar{y}_i = \bar{w}_{i+1},$$

completing the proof. □

We now return to our study of $\mathfrak{F}\mathcal{B}_n$. Recall that the pure braid group $P = \mathcal{P}_n$ is the normal subgroup of B that consists of all braids β such that $\bar{\beta} = 1$. We identify P with the subgroup $[1, P] = \{[1, \beta] \mid \beta \in P\}$ of $\mathfrak{F}\mathcal{B}_n$. With this in mind, we define the *pure factorizable braid monoid*

$$\mathfrak{F}\mathcal{P}_n = \{[\mathcal{E}, \beta] \mid \mathcal{E} \in E,\ \beta \in P\} = \bigcup_{\mathcal{E} \in E} [\mathcal{E}, P].$$

For $\mathcal{E} \in E$ we define a subgroup

$$P_{\mathcal{E}} = \{\beta \in B \mid (i, i\bar{\beta}) \in \mathcal{E}\ (\forall i \in \mathbf{n})\}.$$

Note that we have $P \subseteq P_{\mathcal{E}}$ for each $\mathcal{E} \in E$, with equality if and only if $\mathcal{E} = 1$.

Lemma 8.3. *Suppose that $\mathcal{E} \in E$. Then $[\mathcal{E}, P] = [\mathcal{E}, P_{\mathcal{E}}]$. Further, $B_{\mathcal{E}}$ is a normal subgroup of $P_{\mathcal{E}}$ and $P_{\mathcal{E}}/B_{\mathcal{E}} \cong [\mathcal{E}, P_{\mathcal{E}}]$.*

Proof. Let $\mathcal{E} \in E$. It is clear that $[\mathcal{E}, P] \subseteq [\mathcal{E}, P_{\mathcal{E}}]$, since $P \subseteq P_{\mathcal{E}}$. To show the reverse inclusion, suppose that $\beta \in P_{\mathcal{E}}$. Since $(i, i\bar{\beta}) \in \mathcal{E}$ for each $i \in \mathbf{n}$, we have $\bar{\beta} = t_{i_1 j_1} \cdots t_{i_k j_k}$ for some $i_1, \dots, i_k, j_1, \dots, j_k \in \mathbf{n}$ with $i_s < j_s$ and $(i_s, j_s) \in \mathcal{E}$ for each $s \in \mathbf{k}$. Thus, $\beta = \gamma \varsigma_{i_1 j_1} \cdots \varsigma_{i_k j_k}$ for some $\gamma \in P$. But then $[\mathcal{E}, \beta] = [\mathcal{E}, \gamma] \in [\mathcal{E}, P]$ since $\beta\gamma^{-1} = \varsigma_{i_1 j_1} \cdots \varsigma_{i_k j_k} \in B_{\mathcal{E}}$ by Lemma 4.3, and so $[\mathcal{E}, P] = [\mathcal{E}, P_{\mathcal{E}}]$. Finally, it is easy to check that $\beta \mapsto [\mathcal{E}, \beta]$ defines a group epimorphism $P_{\mathcal{E}} \to [\mathcal{E}, P_{\mathcal{E}}]$ with kernel $B_{\mathcal{E}}$. □

In particular, $\mathfrak{F}\mathcal{P}_n = \bigcup_{\mathcal{E} \in E} [\mathcal{E}, P_{\mathcal{E}}]$ is the disjoint union of the groups $[\mathcal{E}, P_{\mathcal{E}}]$, which are therefore the maximal subgroups of $\mathfrak{F}\mathcal{P}_n$. To make further progress towards understanding the structure of $\mathfrak{F}\mathcal{P}_n$ we will study the quotients $P_{\mathcal{E}}/B_{\mathcal{E}} \cong [\mathcal{E}, P_{\mathcal{E}}]$. We say that two equivalences $\mathcal{E}_1, \mathcal{E}_2 \in E$ are *conjugate* if $\mathcal{E}_2 = \mathcal{E}_1^{\beta}$ for some $\beta \in B$.

Lemma 8.4. *If $\mathcal{E}_1, \mathcal{E}_2 \in E$ are conjugate, then $[\mathcal{E}_1, P_{\mathcal{E}_1}] \cong [\mathcal{E}_2, P_{\mathcal{E}_2}]$.*

Proof. If $\mathcal{E}_2 = \mathcal{E}_1^{\beta}$ for some $\beta \in B$, then it is easy to check that $[\mathcal{E}_1, \gamma] \mapsto [\mathcal{E}_2, \beta\gamma\beta^{-1}]$ defines a group isomorphism $[\mathcal{E}_1, P_{\mathcal{E}_1}] \to [\mathcal{E}_2, P_{\mathcal{E}_2}]$. □

Suppose now that $\mathcal{E} \in E$ and that $\mathbf{n}/\mathcal{E} = \{N_1, \ldots, N_k\}$ with $\min(N_1) < \cdots < \min(N_k)$. Put $\lambda_i = |N_i|$ for each i. We say that $\mathcal{E}$ is *convex* if $r < s$ whenever $r \in N_i$ and $s \in N_j$ with $1 \leqslant i < j \leqslant k$. If $\mathcal{E}$ is convex, then we say that $\mathcal{E}$ is *standard* if we also have $\lambda_1 \leqslant \cdots \leqslant \lambda_k$. Note that every equivalence $\mathcal{E}' \in E$ is conjugate to a (unique) standard equivalence $\mathcal{E}$ and so, by Lemmas 8.3 and 8.4, we have $P_{\mathcal{E}'}/B_{\mathcal{E}'} \cong P_{\mathcal{E}}/B_{\mathcal{E}}$. From now on we fix $\mathcal{E} \in E$, a standard equivalence, with the N_i and λ_i as defined above. The remainder of this section will be devoted to analysing the structure of the quotient $P_{\mathcal{E}}/B_{\mathcal{E}}$. For $i \in \mathbf{k}$ we put $N_i^\flat = N_i \setminus \{\max(N_i)\}$, and let

$$\mathbf{n}^\flat = N_1^\flat \cup \cdots \cup N_k^\flat.$$

Let $X_P = \{a_{ij} \mid 1 \leqslant i < j \leqslant n\}$. The following result is well known. For proofs see, for example, [**2**] or [**4**].

Theorem 8.5 (Artin [2]). *The pure braid group P has group presentation $\langle X_P | R_P\rangle$ via*

$$\pi : F(X_P) \to P : a_{ij} \mapsto \alpha_{ij}\phi_B,$$

where R_P is the set of relations

$$a_{rs}a_{ij}a_{rs}^{-1} = a_{ij} \qquad \text{if } i < r \text{ or } i > s, \tag{P1}$$
$$a_{rs}a_{sj}a_{rs}^{-1} = (a_{sj}^{-1}a_{rj}^{-1})a_{sj}(a_{rj}a_{sj}), \tag{P2}$$
$$a_{rs}a_{rj}a_{rs}^{-1} = a_{sj}^{-1}a_{rj}a_{sj}, \tag{P3}$$
$$a_{rs}a_{ij}a_{rs}^{-1} = (a_{sj}^{-1}a_{rj}^{-1}a_{sj}a_{rj})a_{ij}(a_{rj}^{-1}a_{sj}^{-1}a_{rj}a_{sj}) \quad \text{if } r < i < s, \tag{P4}$$

with $1 \leqslant r < s \leqslant n$, $1 \leqslant i < j \leqslant n$, and $s < j$ in each case.

For convenience we will simply write $\sim_{R_P}$ as $\sim_P$. For $1 \leqslant i < j \leqslant n$ let $\tilde{\alpha}_{ij} = a_{ij}\pi \in P$, and put $\widetilde{X}_P = \{\tilde{\alpha}_{ij} \mid 1 \leqslant i < j \leqslant n\}$. Also, put $\Sigma_{\mathcal{E}} = \{\sigma_i \mid i \in \mathbf{n}^\flat\}$ and $\tilde{\Sigma}_{\mathcal{E}} = \{\varsigma_i \mid i \in \mathbf{n}^\flat\}$. A proof of the next lemma is included for the reader's convenience, although it follows from general facts about parabolic subgroups of Coxeter groups (see, for example, [**14**]).

Lemma 8.6. *Suppose that $\beta \in P_{\mathcal{E}}$. Then $\bar{\beta} = s_{i_1} \cdots s_{i_\ell}$ for some $i_1, \ldots, i_\ell \in \mathbf{n}^\flat$.*

Proof. Suppose that $c = (a_1, \ldots, a_r)$ is a cycle from the cycle decomposition of $\bar{\beta}$. Now $c = t_{a_{r-1}a_r} \cdots t_{a_1a_2}$, and we have $a_1, \ldots, a_r \in N_j$ for some $j \in \mathbf{k}$ since $(i, i\bar{\beta}) \in \mathcal{E}$ for all $i \in \mathbf{n}$. Now for each $i \in \{1, \ldots, r-1\}$ we have

$$t_{a_ia_{i+1}} = \begin{cases} (s_{a_i} \cdots s_{a_{i+1}-2})s_{a_{i+1}-1}(s_{a_{i+1}-2} \cdots s_{a_i}) & \text{if } a_i < a_{i+1}, \\ (s_{a_{i+1}} \cdots s_{a_i-2})s_{a_i-1}(s_{a_i-2} \cdots s_{a_{i+1}}) & \text{if } a_i > a_{i+1}. \end{cases}$$

Each of the subscripts in this expression are in $\mathbf{n}^\flat$ since $(a_i, a_{i+1}) \in \mathcal{E}$ and $\mathcal{E}$ is convex. □

Lemma 8.7. *The subgroup $P_{\mathcal{E}}$ is generated (as a group) by $\widetilde{X}_P \cup \tilde{\Sigma}_{\mathcal{E}}$.*

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Proof. We clearly have $\widetilde{X}_P \cup \tilde{\Sigma}_{\mathcal{E}} \subseteq P_{\mathcal{E}}$, so that $\langle \widetilde{X}_P \cup \tilde{\Sigma}_{\mathcal{E}} \rangle \subseteq P_{\mathcal{E}}$. To prove the reverse inclusion, suppose that $\beta \in P_{\mathcal{E}}$. Then, by Lemma 8.6, we have $\bar{\beta} = s_{i_1} \cdots s_{i_\ell}$ for some $i_1, \dots, i_\ell \in \mathbf{n}^\flat$. Putting $\gamma = \varsigma_{i_1} \cdots \varsigma_{i_\ell}$, we have $\beta = (\beta\gamma^{-1})\gamma$. Choose $w \in X_B^*$ such that $w\phi_B = \beta\gamma^{-1}$. Now $\beta\gamma^{-1} \in P$ and so, by Theorem 4.7, we have

$$w \sim_B \alpha_{p_1 q_1}^{\pm 1} \cdots \alpha_{p_h q_h}^{\pm 1}$$

for some $h \in \mathbb{N}$, and $p_1, \dots, p_h, q_1, \dots, q_h \in \mathbf{n}$ with $p_t < q_t$ for each $t \in \mathbf{h}$. But then

$$\beta = \tilde{\alpha}_{p_1 q_1}^{\pm 1} \cdots \tilde{\alpha}_{p_h q_h}^{\pm 1} \varsigma_{i_1} \cdots \varsigma_{i_\ell} \in \langle \widetilde{X}_P \cup \tilde{\Sigma}_{\mathcal{E}} \rangle,$$

and we are done. □

Now put $X_{P_{\mathcal{E}}} = X_P \cup \Sigma_{\mathcal{E}}$.

Lemma 8.8. *The group $P_{\mathcal{E}}$ has group presentation $\langle X_{P_{\mathcal{E}}} | R_{P_{\mathcal{E}}} \rangle$ via*

$$\pi_{\mathcal{E}} : F(X_{P_{\mathcal{E}}}) \to P_{\mathcal{E}} : \begin{cases} a_{ij} \mapsto \tilde{\alpha}_{ij}, \\ \sigma_h \mapsto \varsigma_h, \end{cases}$$

where $R_{P_{\mathcal{E}}}$ is the following set of relations:

$$\text{braid relations among the } \sigma_i, \tag{B}$$

$$\text{pure braid relations among the } a_{ij}, \tag{P}$$

$$a_{ij} = (\sigma_{j-1} \cdots \sigma_{i+1}) \sigma_i^2 (\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}) \quad \text{if } (i,j) \in \mathcal{E}, \tag{D}$$

$$\sigma_r a_{ij} \sigma_r^{-1} = \begin{cases} a_{ij}^{-1} a_{i-1,j} a_{ij} & \text{if } r = i-1, \\ a_{i+1,j} & \text{if } r = i < j-1, \\ a_{ij}^{-1} a_{i,j-1} a_{ij} & \text{if } r = j-1 > i, \\ a_{i,j+1} & \text{if } r = j, \\ a_{ij} & \text{otherwise.} \end{cases} \tag{C}$$

Proof. By Lemma 8.7 we see that $\pi_{\mathcal{E}}$ is an epimorphism. For the remainder of this proof, let $\approx$ denote the congruence $R_{P_{\mathcal{E}}}^\sharp$ on $F(X_{P_{\mathcal{E}}})$. One may easily check that $R_{P_{\mathcal{E}}} \subseteq \ker \pi_{\mathcal{E}}$ so that $\approx \subseteq \ker \pi_{\mathcal{E}}$. To show the reverse inclusion, suppose that $w \in F(X_{P_{\mathcal{E}}})$ and $w\pi_{\mathcal{E}} = 1$. It is sufficient to show that $w \approx 1$. Now, by relations (C) we have

$$w \approx w_1 w_2$$

for some $w_1 \in F(X_P)$ and $w_2 \in F(\Sigma_{\mathcal{E}})$. By Lemma 8.6, we have $\bar{w}_2 = \bar{w} = s_{i_1} \cdots s_{i_\ell}$ for some $i_1, \dots, i_\ell \in \mathbf{n}^\flat$. We also assume that this expression is of minimal length. Now by relations (B) and (D), and Theorem 4.7, we have

$$w_2 \approx w_2' \sigma_{i_1} \cdots \sigma_{i_\ell}$$

for some $w_2' \in F(X_P)$, and so $w \approx w_1 w_2' \sigma_{i_1} \cdots \sigma_{i_\ell}$. Since $w\pi_{\mathcal{E}} = 1$ we must have $\ell = 0$ and $w_1 w_2' \sim_P 1$ so that $w \approx 1$. □

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Lemma 8.9. *The subgroup $B_{\mathcal{E}}$ is the normal closure in $P_{\mathcal{E}}$ of $\tilde{\Sigma}_{\mathcal{E}}$.*

Proof. Since $\tilde{\Sigma}_{\mathcal{E}} \subseteq B_{\mathcal{E}}$ and $B_{\mathcal{E}}$ is normal in $P_{\mathcal{E}}$, we see that the normal closure of $\tilde{\Sigma}_{\mathcal{E}}$ is contained in $B_{\mathcal{E}}$. Conversely, by Lemma 4.3 we know that $B_{\mathcal{E}}$ is generated by elements of the form $\beta^{-1}\varsigma_{ij}\beta$ with $\beta \in P$, $1 \leqslant i < j \leqslant n$, and $(i,j) \in \mathcal{E}$. In particular, we have $i, i+1, \dots, j-1 \in \mathbf{n}^\flat$. Now

$$\beta^{-1}\varsigma_{ij}\beta = (\varsigma_{i+1}\cdots\varsigma_{j-1}\beta)^{-1}\varsigma_i(\varsigma_{i+1}\cdots\varsigma_{j-1}\beta),$$

and since $\varsigma_{i+1}\cdots\varsigma_{j-1}\beta \in P_{\mathcal{E}}$, the proof is complete. □

For $\beta \in P_{\mathcal{E}}$ we will denote the coset $B_{\mathcal{E}}\beta \in P_{\mathcal{E}}/B_{\mathcal{E}}$ by $[\beta]_{\mathcal{E}}$. There is no conflict with our use of this notation in § 3 since, by Theorem 3.2, the set of braids which are $\mathcal{E}$-equivalent to β is precisely the coset $B_{\mathcal{E}}\beta$.

Corollary 8.10. *The quotient $P_{\mathcal{E}}/B_{\mathcal{E}}$ has presentation $\langle X_{P_{\mathcal{E}}} | R_{P_{\mathcal{E}}} \cup R_{\mathrm{O}}\rangle$ via*

$$\pi'_{\mathcal{E}} : F(X_{P_{\mathcal{E}}}) \to P_{\mathcal{E}}/B_{\mathcal{E}} : \begin{cases} a_{ij} \mapsto [\tilde{\alpha}_{ij}]_{\mathcal{E}}, \\ \sigma_h \mapsto [\varsigma_h]_{\mathcal{E}}, \end{cases}$$

where R_{O} is the set of relations

$$\sigma_i = 1 \quad \text{for all } i \in \mathbf{n}^\flat. \tag{O}$$

Proof. This follows from Lemmas 8.1, 8.8 and 8.9. □

We now examine the manner in which the presentation of $P_{\mathcal{E}}/B_{\mathcal{E}}$ given in Corollary 8.10 simplifies. Denote by $\sim_{\mathcal{E}}$ the congruence $(R_{P_{\mathcal{E}}} \cup R_{\mathrm{O}})^\sharp$.

Lemma 8.11. *If $(r,s) \in \mathcal{E}$ with $r < s$, then*

(i) *$a_{ir} \sim_{\mathcal{E}} a_{is}$ for all $1 \leqslant i < r$,*

(ii) *$a_{rj} \sim_{\mathcal{E}} a_{sj}$ for all $s < j \leqslant n$.*

Proof. To prove (i), suppose that $r, s \in N_\ell$ for some ℓ and $1 \leqslant i < r < s$. Then, since $\mathcal{E}$ is convex, we must have $r, \dots, s-1 \in N_\ell^\flat$ and it follows, by (O) and (C), that

$$a_{ir} \sim_{\mathcal{E}} (\sigma_{s-1}\cdots\sigma_r)a_{ir}(\sigma_r^{-1}\cdots\sigma_{s-1}^{-1}) \sim_{\mathcal{E}} a_{is}.$$

Statement (ii) is proved in an almost identical manner. □

Corollary 8.12. *If $(i,j),(r,s) \in \mathcal{E}$ with $i < r$ and $j < s$, then $a_{ir} \sim_{\mathcal{E}} a_{js}$.*

Proof. Using the previous lemma, we have $a_{ir} \sim_{\mathcal{E}} a_{jr} \sim_{\mathcal{E}} a_{js}$ if $j < r$ and $a_{ir} \sim_{\mathcal{E}} a_{is} \sim_{\mathcal{E}} a_{js}$ if $r \leqslant j$. □

Lemma 8.13. *If $(s,j) \in \mathcal{E}$ with $s < j$, then*

$$a_{ij}a_{rs} \sim_{\mathcal{E}} a_{rs}a_{ij} \quad \text{for all } 1 \leqslant i < j \text{ and } 1 \leqslant r < s.$$

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Proof. First observe that by (O) and (D), we also have $a_{pq} \sim_{\mathcal{E}} 1$ if $(p,q) \in \mathcal{E}$. Now if $i < r$ or $i > s$, then the commuting relation already exists as part of (P1). If $i = s$, then $a_{ij} = a_{sj} \sim_{\mathcal{E}} 1$ by the observation, and the relation is trivial. If $i = r$, then by (P3) and the observation we have $a_{rs}a_{rj}a_{rs}^{-1} \sim_{\mathcal{E}} a_{sj}^{-1}a_{rj}a_{sj} \sim_{\mathcal{E}} a_{rj}$. If $r < i < s$, then using (P4) and the observation again, we have

$$a_{rs}a_{ij}a_{rs}^{-1} \sim_{\mathcal{E}} (a_{sj}^{-1}a_{rj}^{-1}a_{sj}a_{rj})a_{ij}(a_{rj}^{-1}a_{sj}^{-1}a_{rj}a_{sj}) \sim_{\mathcal{E}} (a_{rj}^{-1}a_{rj})a_{ij}(a_{rj}^{-1}a_{rj}) \sim_{\mathcal{E}} a_{ij}.$$

□

Corollary 8.14. *If $1 \leqslant i < r < j$ and $j \in N_\ell$ for some $\ell \in \mathbf{k}$ with $\lambda_\ell > 1$, then*

$$a_{ij}a_{rj} \sim_{\mathcal{E}} a_{rj}a_{ij}.$$

Proof. Choose $s \in N_\ell \setminus \{j\}$. If $r < s$, then by Lemmas 8.11 (i) and 8.13 we have $a_{ij}a_{rj} \sim_{\mathcal{E}} a_{ij}a_{rs} \sim_{\mathcal{E}} a_{rs}a_{ij} \sim_{\mathcal{E}} a_{rj}a_{ij}$. If $s \leqslant r$, then since $\mathcal{E}$ is convex we must have $(r,j) \in \mathcal{E}$ so that $a_{rj} \sim_{\mathcal{E}} 1$ and the commuting relation is trivial. □

Corollary 8.15. *If $1 \leqslant i < j \leqslant n$, $1 \leqslant r < s < j$, and $j \in N_\ell$ for some $\ell \in \mathbf{k}$ with $\lambda_\ell > 1$, then*

$$a_{rs}a_{ij}a_{rs}^{-1} \sim_{\mathcal{E}} a_{ij}.$$

Proof. By Theorem 8.5, we have $a_{rs}a_{ij}a_{rs}^{-1} \sim_{\mathcal{E}} wa_{ij}w^{-1}$ for some word w in the $a_{hj}^{\pm 1}$. By Corollary 8.14, we have $wa_{ij}w^{-1} \sim_{\mathcal{E}} a_{ij}ww^{-1} \sim_{\mathcal{E}} a_{ij}$ and we are done. □

For $i \in \mathbf{k}$, let $\mu_i = \min(N_i)$, and denote by $k_0 \in \mathbf{k}$ the index such that

(i) $\lambda_j = 1$ for all $1 \leqslant j \leqslant k_0$,

(ii) $\lambda_j > 1$ for all $k_0 < j \leqslant k$.

Note that $\mu_j = j$ if $1 \leqslant j \leqslant k_0 + 1$, while $\mu_j > j$ if $k_0 + 1 < j \leqslant k$.

Corollary 8.16. *If $1 \leqslant r < s \leqslant k$, $1 \leqslant i < j \leqslant k$, and $j > k_0$, then*

$$a_{\mu_r\mu_s}a_{\mu_i\mu_j}a_{\mu_r\mu_s}^{-1} \sim_{\mathcal{E}} a_{\mu_i\mu_j}.$$

By Corollary 8.12, and the observation in the proof of Lemma 8.13, we have

$$a_{rs} \sim_{\mathcal{E}} \begin{cases} 1 & \text{if } (r,s) \in \mathcal{E}, \\ a_{\mu_i\mu_j} & \text{if } r \in N_i \text{ and } s \in N_j \text{ with } 1 \leqslant i < j \leqslant k. \end{cases}$$

So we remove all generators $a_{rs} \in X_P$ unless $r = \mu_i$ and $s = \mu_j$ for some i and j. We replace any occurrence of $a_{rs}^{\pm 1}$ in the relations by $a_{\mu_i\mu_j}^{\pm 1}$ if $r \in N_i$ and $s \in N_j$ with $1 \leqslant i < j \leqslant k$, or by 1 if $(r,s) \in \mathcal{E}$. By (O) we may remove each $\sigma_i \in \Sigma_{\mathcal{E}}$ with $i \in \mathbf{n}^\flat$. We also remove relations (O), (D), (B), and (C), which are now trivial.

Put $X_{\mathcal{E}} = \{a_{\mu_i\mu_j} \mid 1 \leqslant i < j \leqslant k\}$. By the previous paragraph, and Corollary 8.16, we have the following.

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Theorem 8.17. *The quotient $P_{\mathcal{E}}/B_{\mathcal{E}}$ has presentation $\langle X_{\mathcal{E}} | R_{\mathcal{E}}\rangle$ via*

$$\varPi_{\mathcal{E}} : F(X_{\mathcal{E}}) \to P_{\mathcal{E}}/B_{\mathcal{E}} : a_{\mu_i\mu_j} \mapsto [\tilde{\alpha}_{\mu_i\mu_j}]_{\mathcal{E}},$$

where $R_{\mathcal{E}}$ is the set of relations

$$a_{rs}a_{ij}a_{rs}^{-1} = a_{ij} \qquad \text{if } i < r \text{ or } i > s, \qquad (R_{\mathcal{E}}1)$$
$$a_{rs}a_{sj}a_{rs}^{-1} = (a_{sj}^{-1}a_{rj}^{-1})a_{sj}(a_{rj}a_{sj}), \qquad (R_{\mathcal{E}}2)$$
$$a_{rs}a_{rj}a_{rs}^{-1} = a_{sj}^{-1}a_{rj}a_{sj}, \qquad (R_{\mathcal{E}}3)$$
$$a_{rs}a_{ij}a_{rs}^{-1} = (a_{sj}^{-1}a_{rj}^{-1}a_{sj}a_{rj})a_{ij}(a_{rj}^{-1}a_{sj}^{-1}a_{rj}a_{sj}) \quad \text{if } r < i < s, \qquad (R_{\mathcal{E}}4)$$

with $1 \leqslant r < s \leqslant k_0$, $1 \leqslant i < j \leqslant k_0$, and $s < j$ in each case, together with

$$a_{\mu_r\mu_s}a_{\mu_i\mu_j}a_{\mu_r\mu_s}^{-1} = a_{\mu_i\mu_j} \qquad \text{if } j > k_0, \qquad (R_{\mathcal{E}}5)$$

with $1 \leqslant r < s \leqslant k$, $1 \leqslant i < j \leqslant k$, and $s \leqslant j$ in each case.

For $\ell \in \mathbf{k}$ let $\mathcal{U}_{N_\ell}$ be the subgroup of $P_{\mathcal{E}}/B_{\mathcal{E}}$ generated by $\{[\tilde{\alpha}_{\mu_i\mu_\ell}]_{\mathcal{E}} \mid 1 \leqslant i < \ell\}$, and let $(P_{\mathcal{E}}/B_{\mathcal{E}})'$ be the subgroup generated by $\{[\tilde{\alpha}_{\mu_i\mu_j}]_{\mathcal{E}} \mid 1 \leqslant i < j \leqslant k-1\}$.

Lemma 8.18. *We have the decomposition $P_{\mathcal{E}}/B_{\mathcal{E}} = \mathcal{U}_{N_k} \rtimes (P_{\mathcal{E}}/B_{\mathcal{E}})'$.*

Proof. Now if $\lambda_k = 1$, then $P_{\mathcal{E}}/B_{\mathcal{E}} \cong P$ and the result is well known (see, for example, [**2**, **4**]). So suppose that $\lambda_k > 1$. It is immediate from Theorem 8.17 that $\mathcal{U}_{N_k}$ is normal in $P_{\mathcal{E}}/B_{\mathcal{E}}$, and $P_{\mathcal{E}}/B_{\mathcal{E}}$ is clearly generated by $\mathcal{U}_{N_k} \cup (P_{\mathcal{E}}/B_{\mathcal{E}})'$. Suppose now that $\beta \in P_{\mathcal{E}}$ such that $[\beta]_{\mathcal{E}} \in \mathcal{U}_{N_k} \cap (P_{\mathcal{E}}/B_{\mathcal{E}})'$. Since $[\beta]_{\mathcal{E}} \in \mathcal{U}_{N_k}$, and since $\mathcal{U}_{N_k}$ is commutative by $(R_{\mathcal{E}}5)$, we have

$$[\beta]_{\mathcal{E}} = [\tilde{\alpha}_{\mu_1\mu_k}^{m_1} \cdots \tilde{\alpha}_{\mu_{k-1}\mu_k}^{m_{k-1}}]_{\mathcal{E}}$$

for some $m_1, \dots, m_{k-1} \in \mathbb{Z}$. By Theorem 8.17 we see that, for each $1 \leqslant i < j \leqslant k$, there is a well-defined homomorphism

$$\exp_{ij} : P_{\mathcal{E}}/B_{\mathcal{E}} \to \mathbb{Z} : [\tilde{\alpha}_{\mu_r\mu_s}]_{\mathcal{E}} \mapsto \begin{cases} 1 & \text{if } r = i \text{ and } s = j, \\ 0 & \text{otherwise.} \end{cases}$$

Since $[\beta]_{\mathcal{E}} \in (P_{\mathcal{E}}/B_{\mathcal{E}})'$, we have $m_i = \exp_{ik}([\beta]_{\mathcal{E}}) = 0$ for each $1 \leqslant i < k$ so that $[\beta]_{\mathcal{E}} = [1]_{\mathcal{E}}$. This shows that $\mathcal{U}_{N_k} \cap (P_{\mathcal{E}}/B_{\mathcal{E}})' = \{[1]_{\mathcal{E}}\}$ and completes the proof. □

Lemma 8.19. *Let $\mathcal{E}' \in \mathfrak{Eq}_{n-\lambda_k}$ be the equivalence relation such that*

$$\{1, \dots, n - \lambda_k\}/\mathcal{E}' = \{N_1, \dots, N_{k-1}\}.$$

Then $(P_{\mathcal{E}}/B_{\mathcal{E}})' \cong P_{\mathcal{E}'}/B_{\mathcal{E}'}$, where here we regard $P_{\mathcal{E}'}$ and $B_{\mathcal{E}'}$ as subgroups of $\mathcal{B}_{n-\lambda_k}$.

Proof. This follows from Lemma 8.2 and Theorem 8.17. □

Theorem 8.20. *We have the decomposition*

$$P_{\mathcal{E}}/B_{\mathcal{E}} = \mathcal{U}_{N_k} \rtimes (\mathcal{U}_{N_{k-1}} \rtimes (\cdots \rtimes (\mathcal{U}_{N_3} \rtimes \mathcal{U}_{N_2}) \ldots)).$$

Furthermore,

(i) *if $\lambda_i = 1$, then $\mathcal{U}_{N_i}$ is the free group with basis $[\tilde{\alpha}_{\mu_1\mu_i}]_{\mathcal{E}}, \ldots, [\tilde{\alpha}_{\mu_{i-1}\mu_i}]_{\mathcal{E}}$;*

(ii) *if $\lambda_i > 1$, then $\mathcal{U}_{N_i}$ is the free abelian group with basis $[\tilde{\alpha}_{\mu_1\mu_i}]_{\mathcal{E}}, \ldots, [\tilde{\alpha}_{\mu_{i-1}\mu_i}]_{\mathcal{E}}$.*

Proof. The semidirect product decomposition follows from Lemmas 8.18 and 8.19 and a simple induction.

(i) If $\lambda_i = 1$, then the subgroup of $P_{\mathcal{E}}/B_{\mathcal{E}}$ generated by $\{[\tilde{\alpha}_{\mu_r\mu_s}]_{\mathcal{E}} \mid 1 \leqslant r < s \leqslant i\}$ is isomorphic to $\mathcal{P}_i$ via the map

$$[\tilde{\alpha}_{\mu_r\mu_s}]_{\mathcal{E}} \mapsto \tilde{\alpha}_{\mu_r\mu_s} = \tilde{\alpha}_{rs} \quad \text{for each } 1 \leqslant r < s \leqslant i.$$

The image of $\mathcal{U}_{N_i}$ under this isomorphism is $\mathcal{U}_i$, the subgroup of $\mathcal{P}_i$ generated by $\tilde{\alpha}_{1i}, \ldots, \tilde{\alpha}_{i-1,i}$. It is well known (see, for example, [**2**, **4**]) that $\mathcal{U}_i$ is a free group of rank $i - 1$.

(ii) If $\lambda_i > 1$, then the map $\mathcal{U}_{N_i} \to \mathbb{Z}^{i-1}$ defined by

$$[\beta]_{\mathcal{E}} \mapsto (\exp_{1i}([\beta]_{\mathcal{E}}), \ldots, \exp_{i-1,i}([\beta]_{\mathcal{E}}))$$

is clearly an isomorphism.

□

Theorem 8.21. *The problem of deciding whether two elements $[\mathcal{E}_1, \beta_1], [\mathcal{E}_2, \beta_2] \in \mathfrak{F}\mathcal{B}_n$ are equal is decidable.*

Proof. Now $[\mathcal{E}_1, \beta_1] = [\mathcal{E}_2, \beta_2]$ if and only if $\mathcal{E}_1 = \mathcal{E}_2$ and $\beta_1\beta_2^{-1} \in B_{\mathcal{E}_1}$. An algorithm to determine whether or not this is the case is as follows.

(i) If $\mathcal{E}_1 \neq \mathcal{E}_2$, then $[\mathcal{E}_1, \beta_1] \neq [\mathcal{E}_2, \beta_2]$. If $\mathcal{E}_1 = \mathcal{E}_2$, then go to step (ii).

(ii) Choose $\gamma \in B$ such that $\mathcal{E} = \mathcal{E}_1^{\gamma}$ is a standard equivalence, and put $\beta = \gamma\beta_1\beta_2^{-1}\gamma^{-1}$. Then $\beta_1\beta_2^{-1} \in B_{\mathcal{E}_1}$ if and only if $\beta \in \gamma B_{\mathcal{E}_1}\gamma^{-1} = B_{\mathcal{E}}$. Now if $\beta \notin P_{\mathcal{E}}$, then $\beta \notin B_{\mathcal{E}}$, and so $[\mathcal{E}_1, \beta_1] \neq [\mathcal{E}_2, \beta_2]$. If $\beta \in P_{\mathcal{E}}$, then go to step (iii).

(iii) If $\beta \in P_{\mathcal{E}}$, then we have $\hat{\beta} \sim_B w\sigma_{i_1} \cdots \sigma_{i_r}$ for some $w \in A^*$ and $i_1, \ldots, i_r \in \mathbf{n}^\flat$. Let $w' \in F(X_{\mathcal{E}})$ be the word obtained from w by replacing each $\alpha_{rs}^{\pm 1}$ by $\alpha_{\mu_i\mu_j}^{\pm 1}$, where $r \in N_i$ and $s \in N_j$ (and then deleting any resulting $\alpha_{\mu_i\mu_i}^{\pm 1}$). Let $w'' \in F(X_{\mathcal{E}})$ be the word obtained from w' by replacing each $\alpha_{\mu_i\mu_j}^{\pm 1}$ by $a_{\mu_i\mu_j}^{\pm 1}$. Now, by Theorem 8.17, we have $\beta \in B_{\mathcal{E}}$ if and only if $w' \sim_{R_{\mathcal{E}}} 1$. Again by Theorem 8.17, we have

$$w' \sim_{R_{\mathcal{E}}} w_k \cdots w_2,$$

where each w_j is a word over $\{a_{\mu_i\mu_j}^{\pm 1} \mid 1 \leqslant i < j\}$. By Theorem 8.20, $w'' \sim_{R_{\mathcal{E}}} 1$ if and only if each word w_j either freely reduces to the empty word (in the case $j \leqslant k_0$), or has a zero exponent sum for each $a_{\mu_i\mu_j}$ (in the case $j < k_0$).

□

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