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TENSOR PRODUCTS OF FUNCTION ALGEBRAS

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For appropriate topological spaces X,Y,Z the algebra $C_c(X \times_Z Y)$ of $\mathbb{C}$-valued continuous functions on the fibre product $X \times_Z Y$ in the compact-open topology, describes the completed biprojective $C_c(Z)$-tensor product of $C_c(X)$, $C_c(Y)$.

The purpose of this note is to show the following:

THEOREM. *Let* X,Y,Z *be completely regular spaces with* X,Y *σ-compact. Moreover, let* $X \times_Z Y$ *be the fibre product of* X,Y *over* Z . *Then,*

$$C_c(X \times_Z Y) = C_c(X) \underset{C_c(Z)}{\hat{\hat{\otimes}}} C_c(Y) \tag{1}$$

within an isomorphism of Fréchet locally m-convex $C_c(Z)$*-algebras.*

Concerning the definition of the topological $C_c(Z)$-algebra in the second member of (1) see [5: Definition 1.1 and also (1.6)]. Relations analogous to (1) are valid too for algebras of complex-valued holomorphic functions on Stein manifolds and C^∞-functions on compact C^∞-manifolds (see Scholium).

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We first comment on the necessary terminology. Thus let X, Z be topological spaces and $\mu : X \to Z$ a continuous map. The algebra $C_c(X)$ of $\mathbb{C}$-valued continuous functions on X with the compact-open topology becomes a locally m-convex $C_c(Z)$-algebra via a "μ-convolution" given by

$$a *_\mu f := (a \circ \mu).f \tag{2}$$

for $a \in C_c(Z)$, $f \in C_c(X)$ (see [5: Section 1]).

On the other hand, we say that a topological algebra M admits a functional representation whenever one has $C_c(\mathfrak{M}(M)) = M$, within an isomorphism of topological algebras [7 : p.474, Theorem 3.1]. So first we have.

LEMMA 1. *Let* X, Y, Z *be completely regular spaces and* $X \times_Z Y$ *the fibre product of the maps* $\mu : X \to Z$ *and* $\nu : Y \to Z$. *Then, in the category of topological algebras admitting functional representations, the algebra* $C_c(X \times_Z Y)$ *is the pushout of the maps* $\mu_* : C_c(Z) \to C_c(X)$ *and* $\nu_* : C_c(Z) \to C_c(Y)$ *defined by* $\mu_*(a) := a \circ \mu$, $\nu_*(a) := a \circ \nu$ $(a \in C_c(Z))$.

Proof. Let $\tilde{p}, \tilde{q}$ be the canonical projections of $X \times_Z Y := \{(x,y) \in X \times Y : \mu(x) = \nu(y)\}$ onto X, Y respectively. Then, one has

$$\mu \circ \tilde{p} = \nu \circ \tilde{q} \tag{3}$$

such that the following diagram is commutative

$$\begin{array}{ccc} C_c(Z) & \xrightarrow{\mu_*} & C_c(X) \\ \nu_* \downarrow & & \downarrow \tilde{p}_* \\ C_c(Y) & \xrightarrow[\tilde{q}_*]{} & C_c(X \times_Z Y) \end{array} \tag{4}$$

Moreover, let (M, r, s) be a triad consisting of a locally m-convex algebra M admitting a functional representation and continuous algebra morphisms

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$r : \mathrm{C}_c(X) \to M,\ s : \mathrm{C}_c(Y) \to M$ such that

$$r \circ \mu_* = s \circ \nu_* \tag{5}$$

The "transpose" continuous maps of r, s on the spectra of the respective algebras, that is $r^* : \mathrm{M}(M) \to \mathrm{M}(\mathrm{C}_c(X))$, $s^* : \mathrm{M}(M) \to \mathrm{M}(\mathrm{C}_c(Y))$ make the next diagram commutative.

$$\begin{array}{ccc} \mathrm{M}(M) & \xrightarrow{\ r^*\ } & X \\ {\scriptstyle s^*}\downarrow & & \downarrow{\scriptstyle \mu} \\ Y & \xrightarrow[\ \nu\ ]{} & Z \end{array} \tag{6}$$

(see also [6 : p.223, Theorem 1.2]). Thus, there exists a unique continuous map

$$\psi : \mathrm{M}(M) \to X \times_Z Y \tag{7}$$

such that

$$\tilde{p} \circ \psi = r^*\ ,\ \tilde{q} \circ \psi = s^* \tag{8}$$

(see [8 : p.231, Definition 12]). Hence, one gets a continuous algebra morphism

$$\psi_* : \mathrm{C}_c(X \times_Z Y) \to M \tag{9}$$

such that

$$\psi_* \circ \tilde{p}_* = r\ ,\ \psi_* \circ \tilde{q}_* = s\ , \tag{10}$$

which yields the assertion (see [8 : p.255, Definition 10]). □

Now, if E is a topological algebra and I a closed 2-sided ideal of E, one has $\mathrm{M}(E/I) = h(I)$, within a homeomorphism. Here $h(I) = \{f \in \mathrm{M}(E) : I \subseteq \ker(f)\}$ denotes the hull of I (see [6 : p.339, Theorem 4.1]). This yields the following

LEMMA 2. *Let E be a topological algebra admitting a functional representation and I a closed 2-sided ideal of E. Then, the quotient topological algebra E/I admits a functional representation.* □

Let X,Y,Z be completely regular spaces and J the closed subspace of $C_c(X \times Y)$ generated by the set $T := \{a *_\mu f - a *_\nu f : a \in C_c(Z)$ $f \in C_c(X \times Y)\}$ (see (2)). Then, J is a closed 2-sided $C_c(Z)$-ideal of $C_c(X \times Y)$ (see (2) and also [5 : Section 1]), such that $C_c(X \times Y)/J$ is a locally m-convex $C_c(Z)$-algebra (see [5], [6]). Moreover, the homeomorphism $\mathrm{M}(C_c(X \times Y)) = X \times Y$ (see [6 : p.223, Theorem 1.2]) and Lemma 2 imply the following

COROLLARY 1. *If X,Y are completely regular spaces, the algebra $C_c(X \times Y)/J$ admits a functional representation.* □

LEMMA 3. *Let X,Y,Z be completely regular spaces and $X \times_Z Y$ the fibre product of the maps $\mu : X \to Z$ and $\nu : Y \to Z$. Then, the algebra $C_c(X \times Y)/J$ is the pushout of $C_c(X)$, $C_c(Y)$ over $C_c(Z)$ in the category of topological algebras admitting functional representations.*

Proof. If p,q are the canonical projections of $X \times Y$ onto X,Y respectively, then Corollary 1 and (5) imply

$$\hat{p}_* \circ \mu_* = \hat{q}_* \circ \nu_* \tag{11}$$

where $\hat{p}_*$, $\hat{q}_*$ are the compositions of the continuous algebra morphisms

$$C_c(X) \xrightarrow[p_*]{} C_c(X \times Y) \xrightarrow[\pi]{} C_c(X \times Y)/J$$
$$C_c(Y) \xrightarrow[q_*]{} C_c(X \times Y) \xrightarrow[\pi]{} C_c(X \times Y)/J \tag{12}$$

respectively (see Lemma 1). Moreover, let (M,r,s) be a system consisting of a locally m-convex algebra M admitting a functional representation and continuous algebra morphisms r,s satisfying (5). Thus, the (uniquely defined) continuous map (7) implies the existence of a continuous map

$$\psi' : \mathrm{M}(M) \to X \times Y \tag{13}$$

such that $p \circ \psi' = r^*$, $q \circ \psi' = s^*$. Hence, one gets a continuous algebra morphism

(14) $$\psi'_* : C_c(X \times Y) \to M$$

with $\psi'_* \circ p_* = r$, $\psi'_* \circ q_* = s$ (see (12), (8), (10)). Furthermore, the relation $J = \ker(\pi) \subseteq \ker(\psi'_*)$ (see Corollary 1, (12), (14)) implies the existence of a (unique) continuous algebra morphism

(15) $$\hat{\psi} : C_c(X \times Y)/J \to M$$

with $\psi'_* = \hat{\psi} \circ \pi$. Therefore, one gets $r = \hat{\psi} \circ \hat{p}_*$, $s = \hat{\psi} \circ \hat{q}_*$ (see (14), (11), (15)), and the assumption follows [8 : p.255, Definition 10]. □

Now, by the uniqueness of the pushout in a given category [8 : p.255] in connection with Lemmas 1,3 one gets the following

PROPOSITION. *Let X, Y, Z be completely regular spaces and $X \times_Z Y$ the fibre product of X, Y over Z (see Lemma 1). Then,*

(16) $$C_c(X \times_Z Y) = C_c(X \times Y)/J$$

within an isomorphism of locally m-convex algebras. In particular, (16) yields an isomorphism of locally m-convex $C_c(Z)$-algebras (see (2)). □

Let X, Y be completely regular k-spaces with $X \times Y$ a k-space as well (take, for example, X, Y to be locally compact spaces; see [1]). Thus one has the following isomorphism of locally m-convex algebras.

(17) $$C_c(X) \hat{\underset{\varepsilon}{\otimes}} C_c(Y) = C_c(X \times Y)$$

(see [6 : p.392. Corollary 1.1]). Moreover, if Z is a completely regular space, (17) preserves the respective topological $C_c(Z)$-algebra structures (see (2)), such that (17) yields an isomorphism of locally m-convex $C_c(Z)$-algebras. Furthermore, let I be the closed 2-sided $C_c(Z)$-ideal of $C_c(X) \underset{\varepsilon}{\otimes} C_c(Y)$ defined by the set $S := \{(a *_\mu f) \otimes g - f \otimes (a *_\nu g) :$ $a \in C_c(Z),\ f \in C_c(X),\ g \in C_c(Y)\}$. By (17) I is a dense subset of J (see Corollary 1), hence

(18) $$C_c(X) \hat{\underset{\varepsilon}{\otimes}} C_c(Y)/\hat{I} = C_c(X \times Y)/J \,,$$

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within an isomorphism of (complete) locally m-convex $C_c(Z)$-algebras.

We are now in the position to give the

Proof of Theorem. We first remark that $C_c(X) \hat{\hat{\otimes}} C_c(Y)$ is a Fréchet locally m-convex $C_c(Z)$-algebra (see [6 : p.392, Corollary 1.1], [3 : p.345, Proposition 2] and (17)). Hence ([5 : (1.6)] and [2 : p.113 and also p.138, Theorem 2]) one has the next topological-algebraic isomorphism

$$C_c(X) \hat{\hat{\otimes}}_{C_c(Z)} C_c(Y) = C_c(X) \hat{\hat{\otimes}} C_c(Y)/\hat{I} ,$$

such that the assumption now follows from (18) and the Proposition. □

SCHOLIUM. We get relations analogous to (1) by considering (complex-valued) holomorphic and C^∞-functions. Thus, if X, Y, Z are Stein spaces [4], the fibre product $X \times_Z Y$ is a Stein space [4 : p.225, E.51b]; so we get results analogous to Lemmas 1,3 in the category of Stein algebras (see also [6 : p.229; (3.2)] for this type of algebras). Thus, one has

$$(19) \qquad O(X \times_Z Y) = O(X) \hat{\hat{\otimes}}_{O(Z)} O(Y)$$

within an isomorphism of Fréchet locally m-convex algebras (see the Proposition and [6 : p.402; (4.10)].

Moreover, suppose that X, Y, Z are compact C^∞-manifolds, with $X \times_Z Y$ a (compact) C^∞-manifold too (this happens, for example, if one of the C^∞-functions μ, ν (see Lemma 1) is a submersion). Hereafter $C^\infty(X)$ stands for the algebra of complex-valued C^∞-functions on a smooth manifold X endowed with the (canonical) C^∞-topology (see for example [6 : Chapter IV, 4.(2)]). Thus consider now topological algebras "admitting differentiable representations"; that is, topological algebras M such that $C^\infty(M(M)) = M$, within a topological-algebraic isomorphism (see [6 : p.227, Theorem 2.1]). So, by adapting Lemmas 1,3 and then the Proposition and Theorem, we have

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$$C^{\infty}(X \times_Z Y) = C^{\infty}(X) \hat{\hat{\otimes}}_{C^{\infty}(Z)} C^{\infty}(Y) \tag{20}$$

within an isomorphism of Fréchet locally m-convex $C^{\infty}(Z)$-algebras.

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