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The unirationality of the moduli spaces of curves of genus 14 or lower

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Abstract

We prove the unirationality of the moduli space of complex curves of genus 14. The method essentially relies on linkage of curves. In particular it is shown that a general curve of genus 14 admits a projective model D, in a six-dimensional projective space, which is linked to a general curve C of degree 14 and genus 8 by a complete intersection of quadrics. Using this property we are able to obtain, after a reasonable amount of further work, the unirationality result in the case of genus 14. Moreover, some variations of the same method, involving the Hilbert schemes of curves of very low genus, are used to obtain the same result for the known cases of genus 11, 12, 13.

Introduction

In this paper we prove that the moduli space of complex curves of genus 14 is unirational. Our method applies more in general to the moduli space $\mathcal{M}_g$ with $g \leqslant 14$, so we use it to give new proofs of the unirationality of $\mathcal{M}_g$ for $g = 11, 12, 13$.

The proof relies on linkage of curves in the projective space and on Mukai's description of canonical curves, of certain low genera, as linear sections of a homogeneous space [Muk88]. From these results of Mukai we deduce the unirationality of the Hilbert schemes of *non-special,* smooth, irreducible curves of degree d and genus $g \leqslant 10$ in $\mathbf{P}^r$ (see § 1). Then we use this property, together with linkage, for proving our results.

To add some historical remarks we recall that the proof of the unirationality of $\mathcal{M}_g$ goes back to Severi for $g \leqslant 10$ (see [Sev21]). The cases of genus $11, 12, 13$ were first proved by Sernesi [Ser81] for $g = 12$ and Chang and Ran [CR84] for $g = 11, 13$. Quite recently a proof which is in part computational was given by Schreyer and Tonoli for $g = 11, 12, 13$ (see [ST02]).

A conjecture of Harris and Morrison implies that $\mathcal{M}_g$ has negative Kodaira dimension for $g \leqslant 22$. Our result implies such a property for $\mathcal{M}_{14}$, this was known up to now for $g \leqslant 13$ and $g = 15, 16$, cf. [HM90, CR91, FP02, MM83]. Of course things are different in higher genus: due to the fundamental results of Eisenbud, Harris and Mumford [EH83, HM82], $\mathcal{M}_g$ has non-negative Kodaira dimension for $g \geqslant 23$ and it is of general type for $g \geqslant 24$. Recently, Farkas has shown that $\mathcal{M}_{23}$ has Kodaira dimension $\geqslant 1$ (see [Far00]). Our starting point has been the following observation: fix a curve D of genus 14 with general moduli. On D there are finitely many line bundles L of degree eight such that $h^0(L) = 2$. For each of them $\omega_D(-L)$ is very ample and defines an embedding $D \subset \mathbf{P}^6$. Now consider the vector space V of quadratic forms vanishing on D: if D is projectively normal then V has dimension five and, hence, $D \subset Q_1 \cap \cdots \cap Q_5$, where $Q_1 \dots Q_5$ are independent quadrics. If $Q_1 \dots Q_5$ define a complete intersection, then $Q_1 \cap \cdots \cap Q_5 = C \cup D$, where C is a curve of degree 14. If C is smooth and connected then its geometric genus is eight. In § 5 we show

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the existence in $\mathbf{P}^6$ of a complete intersection of five quadrics $C_o \cup D_o$ which satisfies all of the previous assumptions. Using the general set up proved in §§ 2, 3 and 4 we are also able to deduce that C_o is a non-specially embedded, projectively normal curve and that the Petri map

$$\mu : H^0(\omega_{D_o}(-1)) \otimes H^0(\mathcal{O}_{D_o}(1)) \to H^0(\omega_{D_o})$$

is injective. Then, in their corresponding Hilbert schemes, C_o and D_o admit irreducible open neighborhoods $\mathcal{C}$ and $\mathcal{D}$ parametrizing curves with the same properties. It follows from the results of § 1 that $\mathcal{C}$ is unirational. On the other hand, the injectivity of μ implies that the natural map $f : \mathcal{D} \to \mathcal{M}_{14}$ is dominant. On $\mathcal{C}$ one can easily construct a Grassmann bundle $\mathcal{G}$ which is locally trivial in the Zariski topology and parametrizes pairs (C, V) such that $C \in \mathcal{C}$ and $V \subset H^0(\mathcal{I}_C(2))$ is a five-dimensional subspace. Since $\mathcal{C}$ is unirational the same holds for $\mathcal{G}$. Finally, the existence of the above complete intersection $C_o \cup D_o$ makes it possible to define a rational map $\phi : \mathcal{G} \to \mathcal{D}$ sending a general pair (C, V) to D, where $C \cup D$ is the scheme defined by V and D is a smooth, irreducible element of $\mathcal{D}$. It turns out that ϕ is birational, hence $f \cdot \phi : \mathcal{G} \to \mathcal{M}_{14}$ is dominant and $\mathcal{M}_{14}$ is unirational. A more elaborated, from the technical point of view, version of this idea works for showing the unirationality of the universal five-symmetric product over $\mathcal{M}_{12}$, of the universal six-symmetric product over $\mathcal{M}_{11}$ and finally of $\mathcal{M}_{13}$. So in this way we are able to obtain the unirationality of $\mathcal{M}_g$, $g = 11, 12, 13$.

To continue with the example of genus 14 we give some more details on the way we prove the unirationality of $\mathcal{C}$. In the general case of a Hilbert scheme of non-special curves of degree d and genus $g \leqslant 10$ the proof of the unirationality is analogous (see § 2). It follows from Mukai's results that a general genus eight canonical curve is a linear section of the Pluecker embedding $G \subset \mathbf{P}^{14}$ of the Grassmannian of lines of $\mathbf{P}^5$ (cf. [Muk88]). Assume $x = (x_1, \dots, x_8) \in G^8$ is general and let P_x be the space spanned by $x_1, \dots, x_8$, then $C_x := P_x \cap G$ is such a general canonical curve. Moreover, C_x is endowed with the line bundle $H_x := \omega_{C_x}(x_1 + \cdots + x_4 - x_5 - \cdots - x_8) \in Pic^{14}(C_x)$. The pair (C_x, H_x) defines a point in the universal Picard variety $Pic_{14,8}$ and a rational map $G^8 \to Pic_{14,8}$. It turns out that the latter is dominant (see § 3), therefore $Pic_{14,8}$ is unirational. Then, with a little bit more effort, the unirationality of $\mathcal{C}$ also follows.

In order to complete this introduction, it is necessary to mention that some constructions produced in the paper can be also effectively achieved via an adequate computer package. As pointed out by the referee, this is true for the projective models of curves of genus 8 and 9 to be considered. The author's choice was for geometric proofs, due to some lack of familiarity with computer packages and due to the beauty of some of the geometry involved.

Some frequently used notation and conventions

- $\mathcal{H}_{d,g,r}$ denotes the *restricted* Hilbert scheme. This is the subscheme, in the Hilbert scheme of curves of degree d and genus g in $\mathbf{P}^r$, parametrizing smoothable, connected, non-degenerate curves.
- $W^r_d(D)$ is the Brill–Noether locus of all line bundles $H \in Pic^d(C)$ such that $h^0(H) \geqslant r+1$. $\mathcal{W}^r_{d,g}$ is the universal Brill–Noether locus over $\mathcal{M}_g$. $Pic_{d,g}$ is the universal Picard variety.
- To simplify notation $X \cap Y$ will be the *scheme-theoretic* intersection of the schemes X and Y, unless differently stated.
- $\mathcal{I}_{X/Y}$ is the ideal sheaf of X in Y. If V is a vector space of sections of a line bundle $|V|$ is the associated linear system. The dual of a vector bundle $\mathcal{E}$ is $\mathcal{E}^*$, $\mathbf{P}(\mathcal{E}) := \mathrm{Proj}\ \mathcal{E}^*$.
- A nodal curve is a curve having ordinary nodes as its only singularities.

1. Auxiliary unirationality results

In this section we show some results of unirationality for the Hilbert schemes $\mathcal{H}_{d,g,r}$ in genus $g \leqslant 9$. More precisely, let

$$\mathcal{H}^{ns}_{d,g,r} := \{C \in \mathcal{H}_{d,g,r} \mid C \text{ is smooth and } \mathcal{O}_C(1) \text{ is non-special}\}, \tag{1.1}$$

where $3 \leqslant r \leqslant d-g$. As is well known, the Zariski closure of $\mathcal{H}^{ns}_{d,g,r}$ is the unique irreducible component of the Hilbert scheme which dominates $\mathcal{M}_g$. We will prove the following.

Theorem 1.2. *We have that $\mathcal{H}^{ns}_{d,g,r}$ is unirational for $g = 7, 8, 9$.*

The extension of the theorem to genus $g \leqslant 6$ is easy: see Remark 1.11. The theorem is an application of the following description, due to Mukai, of a general canonical curve of genus $g = 7, 8, 9$ (cf. [Muk88]).

Theorem 1.3. *For $g = 7, 8, 9$ there exists a rational homogeneous space $P_g \subset \mathbf{P}^{\dim P_g + g - 2}$ whose general curvilinear section is a general canonical curve.*

Let P_g be a homogeneous space as above, we consider the open subset

$$U \subset P_g^g \tag{1.4}$$

of points $x = (x_1, \dots, x_g)$ such that $x_i \neq x_j (i \neq j)$ and moreover:

(i) the linear span $\mathbf{P}_x := \langle x_1 \dots x_g \rangle$ has dimension $g-1$;

(ii) $C_x := \mathbf{P}_x \cap G$ is a smooth, irreducible canonical curve.

On U we have a universal canonical curve $\pi : \mathcal{C} \to U$ with fibre C_x over x. $\mathcal{C}$ contains the divisors $D_i = s_i(U)$, where $s_i : U \to \mathcal{C}$ is the section sending x to x_i. For any $d \geqslant g+3$ we fix *non-zero* integers $n_1, \dots, n_g$ such that $d = n_1 + \dots + n_g + 2 - 2g$. Then we consider the sheaf

$$\mathcal{H} := \omega_\pi(n_1 D_1 + \dots + n_g D_g), \tag{1.5}$$

where ω_π denotes the relative cotangent sheaf of π. For any $x \in U$ we have $\mathcal{H} \otimes \mathcal{O}_{C_x} = \omega_{C_x}(n_1 x_1 + \dots + n_g x_g)$. Let C be a curve of genus g, then the Abel map $a : C^g \to Pic^g(C)$ is surjective. As it is well known this property generalizes as follows.

Lemma 1.6. *Let $n_1, \dots, n_g$ be non-zero integers, then the map $a_{n_1 \dots n_g} : C^g \to Pic^d(C)$ sending $(x_1, \dots, x_g)$ to $\omega_C(n_1 x_1 + \dots + n_g x_g)$ is surjective.*

Lemma 1.7. *Let x be general in U, then $\mathcal{H} \otimes \mathcal{O}_{C_x}$ is non-special and very ample.*

Proof. Let $C = C_x$, by Mukai's result we can assume that C has general moduli. By the previous lemma, each $L \in Pic^d(C)$ is isomorphic to $\omega_C(\Sigma n_i y_i)$, for some $y = (y_1, \dots, y_g) \in C^g$. Therefore, keeping C fixed and possibly replacing x by y, we can assume that $\mathcal{H} \otimes \mathcal{O}_C$ is general in $Pic^d(C)$. On a general curve of genus g a general line bundle of degree $d \geqslant g+3$ is very ample, as follows from [ACGH84, Theorem 1.8, p. 216]. Moreover, such a line bundle is also non-special. $\square$

Finally, up to shrinking the open set U, we can assume that:

(iii) $\mathrm{Proj}\,(\pi_* \mathcal{H}) = U \times \mathbf{P}^{d-g}$;

(iv) $\mathcal{C}$ is embedded in $U \times \mathbf{P}^{d-g}$ by the map associated to the tautological sheaf;

(v) $\mathcal{H} \otimes \mathcal{O}_{C_x}$ is non-special, for all $x \in U$.

Lemma 1.8. *The natural morphism $r : U \to Pic_{d,g}$ is dominant.*

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Proof. Let $x \in U$, by definition $r(x)$ is the isomorphism class of $(C_x, \mathcal{H} \otimes \mathcal{O}_{C_x})$ in the universal Picard variety $Pic_{d,g}$. To prove that r is dominant, it suffices to show that $r(U)$ intersects a general fibre of the forgetful map $f : Pic_{d,g} \to \mathcal{M}_g$ along a dense subset. A general fibre of f is $Pic^d(C)$, where C is general of genus g. By Mukai's theorem C is biregular to a section $C_o = \mathbf{P}_o \cap P_g$, for some point $o = (o_1, \ldots, o_g)$ in $C_o^g \cap U$. Note that $r/C_o^g \cap U : C_o^g \to Pic^d(C)$ associates to $x = (x_1, \ldots, x_g)$ the line bundle $\mathcal{H} \otimes \mathcal{O}_{C_x} = \mathcal{O}_C(n_1 x_1 + \cdots + n_g x_g)$. Then such a map extends to the surjection $a_{n_1 \ldots n_g}$ considered in Lemma 1.6 and, hence, $r(C_o^g \cap U)$ is dense.

Let

$$p : \mathcal{H}^{ns}_{d,g,d-g} \to Pic_{d,g} \tag{1.9}$$

be the morphism sending C to the isomorphism class of the pair $(C, \mathcal{O}_C(1))$ and let

$$q : U \times PGL(d-g+1) \to \mathcal{H}^{ns}_{d,g,d-g} \tag{1.10}$$

be the morphism sending (x, α) to $\alpha(C_x)$. We observe that $r(U)$ is the image of $U \times \{id\}$ under the map $p \cdot q$. Hence, by Lemma 1.8, $p \cdot q$ is dominant. $\square$

Proof of Theorem 1.2. We first show the case $r = d - g$. We know from Lemma 1.8 and (1.10) that both $p(q(U \times PGL(r+1)))$ and $p(\mathcal{H}^{ns}_{d,g,d-g})$ contain a non-empty open set of $Pic_{d,g}$. Therefore,

$$A := p^{-1}\{p(q(U \times PGL(d-g+1))) \cap p(\mathcal{H}^{ns}_{d,g,d-g})\}$$

contains a non-empty open set. Then, since $\mathcal{H}^{ns}_{d,g,r}$ is irreducible, it suffices to show that $A \subset q(U \times PGL(r+1))$. Indeed this implies that q is dominant and, hence, that $\mathcal{H}^{ns}_{d,g,r}$ is unirational. To prove the above inclusion we observe that: $C \in A \Longrightarrow p(C) \in p(q(U \times PGL(d-g+1))) \Longrightarrow C$ is smooth and $\mathcal{O}_C(1)$ is non-special $\Longrightarrow C$ is linearly normal $\Longrightarrow p^{-1}(p(C)) = P_C$, where P_C is the $PGL(d-g+1)$-orbit of C. However, then there exists $\beta \in PGL(r+1)$ such that $\beta(C) = q(x, \alpha)$, that is $C = q(x, \beta^{-1}\alpha)$. Let us complete the proof with the case $r < d - g$: consider the space M_r of all linear projections $\mathbf{P}^{d-g} \to \mathbf{P}^r$ and the map $s : Y \times M_r \to \mathcal{H}^{ns}_{d,g,r}$ sending (C, α) to $\alpha(C)$. It is easy to see that s is dominant, therefore $\mathcal{H}^{ns}_{d,g,r}$ is also unirational. $\square$

Remark 1.11. Let $4 \leqslant g \leqslant 6$ and let $P_g \subset \mathbf{P}^{g+1}$ be a fixed, general Fano threefold of index one and genus g. A parameters count shows that a general canonical curve of genus g is a curvilinear section of P_g. Using this property one can show, with exactly the same proof as above, that $\mathcal{H}^{ns}_{d,g,r}$ is unirational if $4 \leqslant g \leqslant 6$. We leave to the reader the extension of the result to the case $g \leqslant 3$.

2. General set-up

In this section we build up a somehow general strategy to apply the previous unirationality results. The basic idea is to consider families of irreducible curves D of genus g' which are linked to a general $C \in \mathcal{H}^{ns}_{d,g,r}$ by a complete intersection of fixed type $(f_1, \ldots, f_{r-1})$. These families are unirational. In some cases they dominate $\mathcal{M}_{g'}$. We start with the following.

Definition 2.1. Let $3 \leqslant r \leqslant d - g$ and let $g \leqslant 9$, then

$$\mathcal{C}_{d,g,r} \ := \ \{C \in \mathcal{H}^{ns}_{d,g,r} \mid \rho_f \text{ has maximal rank for each } f\}$$

where $\rho_f : H^0(\mathcal{I}_{C/\mathbf{P}^r}(f)) \to H^0(\mathcal{O}_C(f))$ is the restriction map.

By semicontinuity $\mathcal{C}_{d,g,r}$ is open, it is non-empty because the maximal rank condition is generically satisfied in the cases we are considering [BE87]. Due to the results of the previous section, $\mathcal{C}_{d,g,r}$ is irreducible and unirational. We fix a sequence of integers

$$\sigma = (f_1, \ldots, f_s, k_1, \ldots, k_s) \tag{2.2}$$

satisfying

$$1 \leqslant k_i \leqslant n_i \quad \text{and} \quad k_1 + \cdots + k_s = r - 1$$

where n_i is the constant value of $h^0(\mathcal{I}_{C/\mathbf{P}^r}(f_i))$ when C moves in $\mathcal{C}_{d,g,r}$. Then we consider the ideal sheaf $\mathcal{J}$ of the universal curve $\mathcal{C} \subset \mathcal{C}^{ns}_{d,g,r} \times \mathbf{P}^r$ and the natural projections $p_1 : \mathcal{C} \to \mathcal{C}_{d,g,r}$ and $p_2 : \mathcal{C} \to \mathbf{P}^r$. By Grauert's theorem the sheaf

$$\mathcal{F}_i := p_{1*}(\mathcal{J} \otimes p_2^* \mathcal{O}_{\mathbf{P}^r}(f_i)) \tag{2.3}$$

is a vector bundle on $\mathcal{C}_{d,g,r}$, the fibre at C being the space $H^0(\mathcal{I}_{C/\mathbf{P}^r}(f_i))$. For each $k \geqslant 1$ we can also consider the Grassmann bundle

$$u_k : G(k, \mathcal{F}_i) \to \mathcal{C}^{ns}_{d,g,r} \tag{2.4}$$

defined by k and $\mathcal{F}_i$.

DEFINITION 2.5. $\mathcal{G}^\sigma_{d,g,r}$ is the fibre product $G(k_1, \mathcal{F}_1) \times_{u_{k_1}} \cdots \times_{u_{k_s}} G(k_s, \mathcal{F}_s)$ over $\mathcal{C}_{d,g,r}$.

We have that $\mathcal{G}^\sigma_{d,g,r}$ is birational to the product $\mathcal{C}_{d,g,r} \times G(k_1, n_1) \times \cdots \times G(k_s, n_s)$, therefore the next statement is immediate.

PROPOSITION 2.6. *We have that $\mathcal{G}^\sigma_{d,g,r}$ is irreducible and unirational.*

A point of $\mathcal{G}^\sigma_{d,g,r}$ is a sequence $(C, V_1, \ldots, V_s)$, where $C \in \mathcal{C}^{ns}_{d,g,r}$ and V_i is a k_i-dimensional subspace of $H^0(\mathcal{I}_{C/\mathbf{P}^r}(f_i))$. Let $B \subset \mathbf{P}^r$ be the scheme defined by the set of homogeneous forms $V_1 \cup \cdots \cup V_s$. Since $\dim V_1 + \cdots + \dim V_s = r - 1$ it is possible that B is a curve: in this case B is a complete intersection.

DEFINITION 2.7.

(1) A point $(C, V_1, \ldots, V_s) \in \mathcal{G}^\sigma_{d,g,r}$ is a key-point if the above scheme B is a nodal curve $C \cup D$ and D is smooth, irreducible and non-degenerate.

(2) $\mathcal{G}^\sigma_{d,g,r}$ satisfies the key condition if the set of its key-points is non-empty.

Clearly the set of the key-points of $\mathcal{G}^\sigma_{d,g,r}$ is open. Let $C \cup D$ be a nodal complete intersection as above, from now on we will keep the following notation:

$$d' = \deg(D), \quad g' = p_a(D), \quad n = \text{cardinality of Sing } B. \tag{2.8}$$

The numbers d', g' and n can be readily computed from (d, g, σ), we have

$$\begin{gathered} d + d' = f_1^{k_1} \cdots f_s^{k_s}, \\ (g - g') = \tfrac{1}{2}(k_1 f_1 + \cdots + k_s f_s - r - 1)(d - d') \\ n = (k_1 f_1 + \cdots + k_s f_s - r - 1)d + 2 - 2g, \end{gathered}$$

see [Ful84, Example 9.1.12, p. 159]. Finally, we fix the notation for some natural maps.

DEFINITION 2.9. Assume that $\mathcal{G}^\sigma_{d,g,r}$ satisfies the key condition, then $\gamma_{d,g,r} : \mathcal{G}^\sigma_{d,g,r} \to \mathcal{H}_{d',g',r}$ is the map sending a point $(C, V_1, \ldots, V_s)$ as above to D.

The natural map from $\mathcal{H}_{d,g,r}$ to the universal Brill–Noether locus will be denoted as

$$\beta_{d,g,r} : \mathcal{H}_{d,g,r} \to \mathcal{W}^r_{d,g} \tag{2.10}$$

and the forgetful map from $\mathcal{W}^r_{d,g}$ to the moduli space will be denoted as

$$\alpha_{d,g,r} : \mathcal{W}^r_{d,g} \to \mathcal{M}_g. \tag{2.11}$$

The next statement only summarizes our program for showing the unirationality of some moduli spaces, the proof is immediate.

Proposition 2.12. *Assume that $\mathcal{G}^{\sigma}_{d,g,r}$ satisfies the key condition and that the image of $\gamma_{d,g,r}$: $\mathcal{G}^{\sigma}_{d,g,r} \to \mathcal{H}_{d',g',r}$ dominates $\mathcal{M}_{g'}$, then $\mathcal{M}_{g'}$ is unirational.*

3. Some useful criteria I

In this section we prove sufficient conditions to ensure that the key condition holds for $\mathcal{G}^{\sigma}_{d,g,r}$. This criterion is an elementary version of more general known properties and we show it for completeness and the lack of references. We recall that a subvariety $Y \subset \mathbf{P}^r$ is a scheme-theoretic intersection of hypersurfaces of degree f if and only if $\mathcal{I}_{Y/\mathbf{P}^r}(f)$ is globally generated. Throughout this section we will assume that

$$Y =: Y^c \cup Z \tag{3.1}$$

where:

- Y^c is an equidimensional variety of codimension c, which is locally complete intersection with at most finitely many singular points;
- Z is disjoint from Y^c and it is either smooth zero-dimensional or empty.

Proposition 3.2. *Assume that $\mathcal{I}_{Y/\mathbf{P}^r}(f)$ is globally generated and that $\dim Y \leqslant 3$. Then there exists a complete intersection of c hypersurfaces $Q_1 \dots Q_c$ of degree f such that either:*

- *$Q_1 \cap \dots \cap Q_c = Y$; or*
- *$Q_1 \cap \dots \cap Q_c = X \cup Y$ and, moreover, (1) X is smooth and contains Z, (2) $X \cap Y^c$ is smooth and equidimensional of codimension $c+1$.*

Proof. Let $I =: H^0(\mathcal{I}_{Y/\mathbf{P}^r}(f))$, we denote by G^c the Grassmannian of codimension c subspaces of I and by G_c the Grassmannian of c-dimensional subspaces. We assume that Y is not a complete intersection of c hypersurfaces of degree f: otherwise there is nothing else to show. Let $\sigma : P \to \mathbf{P}^r$ be the blowing up of Y^c. Then the strict transform of $|I|$ by σ is $|fH - E|$, where E is the exceptional divisor of σ and H is the pull-back of a hyperplane. Since $\mathcal{I}_{Y/\mathbf{P}^r}(f)$ is globally generated and $Y = Y^c \cup Z$, the base locus of $|fH - E|$ is $\sigma^{-1}(Z)$. Let $V \in G_c$ be general and let B_V be the base locus of the strict transform of $|V|$ on P. By Bertini's theorem we can assume that B_V is smooth: this follows because $\sigma^{-1}(Z)$ is smooth and finite. Moreover, we can assume that B_V intersects transversally the exceptional divisor E. Since Y^c is locally complete intersection, E is a projective bundle with fibre of dimension $c-1$. Then, since V is general of codimension c and Sing Y^c is finite, we can also assume that $B_V \cap \sigma^{-1}(\text{Sing } Y^c)$ is empty. We claim that $\sigma / B_V : B_V \to \mathbf{P}^r$ is an embedding: this is obvious on $B_V - (B_V \cap E)$. To complete the proof consider $p \in B_V \cap E$ and $F = \sigma^{-1}(\sigma(p))$. B_V is the complete intersection of c independent divisors $D_1, \dots, D_c$ of $|fH - E|$. Since $\mathcal{O}_F(fH - E) \cong \mathcal{O}_{\mathbf{P}^{c-1}}(1)$ the intersection scheme $B_V \cap F$ is a linear space. This must be zero-dimensional because B_V is transversal to E, hence, σ / B_V is an embedding at p. Let $X_V = \sigma(B_V)$ then X_V is smooth and, moreover, $X_V \cup Y$ is the complete intersection of the hypersurfaces $\sigma(D_1), \dots, \sigma(D_c) \in |\mathcal{I}_{Y/\mathbf{P}^r}(f)|$. This proves all of our statements except that $X_V \cap Y^c$ is smooth. To prove this latter property we consider, for any $y \in Y^c - \text{Sing } Y$, the vector space $I_y = \{q \in I \mid q \in m_y^2\}$. Since Y is a scheme-theoretic intersection of hypersurfaces of degree f, I_y has codimension c in I. This defines a morphism $\phi : Y^c - \text{Sing } Y \to G^c$ sending y to I_y. For any $V \in G_c$ we can consider the Schubert cycle $\sigma_V = \{L \in G^c \mid \dim (L \cap V) \geqslant 1\}$. It is well known that Sing $\sigma_V = \{L \in G^c \mid \dim (L \cap V) \geqslant 2\}$ and, moreover, that Sing σ_V has codimension 4 in G_c. Since $\dim Y \leqslant 3$ we have Sing $\sigma_V \cap \phi(Y^c - \text{Sing } Y) = \emptyset$ for a general V. Then, by the transversality of a general σ_V, we can assume that $\phi^{-1}(\sigma_V)$ is smooth of codimension $c+1$. On the other hand, it turns out that $\phi^{-1}(\sigma_V) = Y^c \cap X_V$. Indeed we have $y \in Y^c \cap X_V \Leftrightarrow y \in Y^c - \text{Sing } Y$ and B_V is singular at $y \Leftrightarrow y \in Y^c - \text{Sing } Y$ and $\dim I_y \cap V = 1$. This completes the proof. $\square$

In particular we will apply the lemma when Y^c is a nodal curve. So we point out the following.

Proposition 3.3. *Let $Y = C \cup Z$, where C is a nodal curve and Z is a smooth, zero-dimensional scheme disjoint from C. Assume that $\mathcal{I}_{Y/\mathbf{P}^r}(f)$ is globally generated, then:*

(1) *Y lies in a smooth surface S which is complete intersection of $r-2$ hypersurfaces of degree f;*

(2) *$|fH - C|$ is base-point-free, so that a general $D \in |fH - C|$ is transversal to C; D is connected if $D^2 > 0$ (H = hyperplane section of S).*

Proof. By Proposition 3.2 there exists a nodal complete intersection $C \cup D$, of $r-1$ hypersurfaces of degree f, such that D is smooth and contains Z. Let $V = H^0(\mathcal{I}_{C\cup D/\mathbf{P}^r}(f))$ and let $x \in C \cup D$: if x is smooth no $Q \in |V|$ is singular at x. If x is singular then x is a node and there exists exactly one Q singular at x. Since Sing $C \cup D$ is finite, the base locus of a general hyperplane in $|V|$ is a smooth surface S as required. This shows part (1). Part (2) follows immediately. $\square$

Proposition 3.4. *Let $Y = C \cup Z$ be as in Proposition 3.3 and let $\sigma = (f, r-1)$. Assume that:*

(1) *$\mathcal{I}_{Y/\mathbf{P}^r}(f)$ is globally generated and $C \in \mathcal{C}_{d,g,r}$;*

(2) *$d \leqslant f^{r-1} - f^{r-2}$ and $d'(f^{r-2} - r - 1) < 2g' - 2$.*

Then $\mathcal{G}^\sigma_{d,g,r}$ satisfies the key condition.

Proof. By Proposition 3.3 $Y \subset S$, where S is a smooth complete intersection of $r-2$ hypersurfaces of degree f. Moreover, there exists a smooth $D \in |fH - C|$ transversal to C. Conditions in part (2) are equivalent to $(H - D)H \leqslant 0$ and $D^2 > 0$. Hence, D is connected, not degenerate and the statement follows. $\square$

4. Some useful criteria II

Now we want to give some sufficient conditions, on some of the unirational Grassmann bundles $\mathcal{G}^\sigma_{d,g,r}$, so that $\mathcal{G}^\sigma_{d,g,r}$ dominates the moduli space $\mathcal{M}_{g'}$.

Lemma 4.1. *Let $C \cup D \subset \mathbf{P}^r$ be a nodal complete intersection of $r-1$ hypersurfaces of degree $f = (r+2)/(r-2)$. Then:*

(1) *C is f-normal if and only if D is linearly normal and D is f-normal if and only if C is linearly normal;*

(2) *$\mathcal{O}_C(1)$ is non-special if and only if $h^0(\mathcal{I}_{D/\mathbf{P}^r}(f)) = r-1$;*

(3) *C is non-degenerate if and only if $\mathcal{O}_D(f)$ is non-special and D is non-degenerate if and only if $\mathcal{O}_C(f)$ is non-special.*

Proof. As in Proposition 3.3 $C \cup D$ is contained in a smooth complete intersection S of $r-2$ hypersurfaces of degree f.

(1) Let H be a hyperplane section of S, the assumption $f = (r+2)/(r-2)$ simply means that H is a canonical divisor. It follows from the standard exact sequence

$$0 \to \mathcal{O}_S(H - D) \to \mathcal{O}_S(H) \to \mathcal{O}_D(H) \to 0$$

that D is linearly normal if and only if $h^1(\mathcal{O}_S(H - D)) = 0$. Since H is canonical we have $h^1(\mathcal{O}_S(H - D)) = h^1(\mathcal{O}_S(D)) = h^1(\mathcal{O}_S(fH - C))$. Hence, D is linearly normal if and only if C is f-normal. Since $C + D \sim fH$, the second equivalence follows exchanging D with C.

(2) At first we remark that $h^0(\mathcal{I}_{D/\mathbf{P}^r}(f)) = h^0(\mathcal{O}_S(fH - D)) + r - 2 = h^0(\mathcal{O}_S(C)) + r - 2$. Secondly the standard exact sequence

$$0 \to \mathcal{O}_S(H - C) \to \mathcal{O}_S(H) \to \mathcal{O}_C(1) \to 0$$

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yields $h^1(\mathcal{O}_C(1)) = h^2(\mathcal{O}_S(H-C)) - h^2(\mathcal{O}_S(H)) = h^0(\mathcal{O}_S(C)) - 1$. Hence, $h^0(\mathcal{I}_{D/\mathbf{P}^r}(2)) = r-1 \Leftrightarrow h^0(\mathcal{O}_S(C)) = 1 \Leftrightarrow h^1(\mathcal{O}_C(1)) = 0$.

(3) Consider the standard exact sequence

$$0 \to \mathcal{O}_S(C) \to \mathcal{O}_S(fH) \to \mathcal{O}_D(f) \to 0$$

and its associated long exact sequence. Since $f \geqslant 1$, we have $h^1(\mathcal{O}_S(fH)) = h^2(\mathcal{O}_S(fH)) = 0$ and, hence, $h^1(\mathcal{O}_D(f)) = h^2(\mathcal{O}_S(C)) = h^0(\mathcal{O}_S(H-C))$. This implies the first equivalence. Again the second follows exchanging C and D. $\square$

Proposition 4.2. *Let $\sigma = (f, r-1)$, with $f = (r+2)/(r-2)$ and $r = d-g$. Assume that the map*

$$\beta_{d',g',r} \cdot \gamma_{d,g,r} \colon \mathcal{G}^{\sigma}_{d,g,r} \longrightarrow \mathcal{W}^r_{d',g'}$$

exists, i.e. that $\mathcal{G}^{\sigma}_{d,g,r}$ satisfies the key assumption. Then the image of such a map is open.

Proof. Under the assumption a general $C \in \mathcal{C}_{d,g,r}$ is linked to a smooth, irreducible, non-degenerate curve D by a complete intersection of $r-1$ hypersurfaces of degree f. Note that C is both f-normal and 1-normal: this follows because the restriction map $\rho_m : H^0(\mathcal{O}_{\mathbf{P}^r}(m)) \to H^0(\mathcal{O}_C(m))$ has maximal rank and $h^0(\mathcal{I}_{C/\mathbf{P}^4}(f)) > 0$. Hence, $\rho(f)$ is surjective and C is f-normal. On the other hand, $r = d-g$ implies $h^0(\mathcal{O}_C(1)) = r+1$, because $\mathcal{O}_C(1)$ is non-special. Then $\rho(1)$ is an isomorphism and C is linearly normal. D has the following properties: (i) D is linearly normal; (ii) D is f-normal; (iii) $h^0(\mathcal{I}_{D/\mathbf{P}^r}(f)) = r-1$, (iv) $\mathcal{O}_D(f)$ is non-special. This follows from Lemma 4.1. Using the same lemma it is easy to see that the set

$$U' = \{D' \in \mathcal{H}_{d',g',r} \mid D' \text{ is smooth, irreducible, non-degenerate and satisfies (i)}, \dots, \text{(iv)}\},$$

is open. Let $D' \in U'$, assume that: (v) the scheme defined by $V' =: H^0(\mathcal{I}_{D/\mathbf{P}^r}(f))$ is a complete intersection $C' \cup D'$, where C' is smooth and irreducible. Then, by Lemma 4.1, $C' \in \mathcal{C}_{d,g,r}$ and, hence, $D' = \gamma_{d,g,r}(C', V')$. Conversely, if $D' = \gamma_{d,g,r}(C', V')$, then D' satisfies condition (v). Thus, condition (v) defines an open set $U \subset U'$ and U is the image of $\gamma_{d,g,r}$. U is invariant under the action of $PGL(r+1)$, moreover each $D \in U$ is linearly normal. This implies that $\beta_{d',g',r}(U) = U/PGL(r+1)$ and that $\beta_{d',g',r}(U)$ is open in $\mathcal{W}^r_{d',g'}$. $\square$

However, we recall that $\mathcal{W}^r_{d',g'}$ can be reducible, even if the Brill–Noether number $\rho(d',g',r)$ is greater than or equal to zero. So it could happen that $\gamma_{d,g,r} \cdot \beta_{d',g',r}$ is not dominant and that its image does not dominate $\mathcal{M}_{g'}$.

Definition 4.3. Let $x \in \mathcal{W}^r_{d',g'}$ be the moduli point of the pair (D, L), $L \in Pic^{d'}(D)$. We will say that x is Petri general if the Petri map $\mu : H^0(\omega_D(-L)) \otimes H^0(L) \to H^0(\omega_D)$ is injective. We define as the main universal Brill–Noether locus the open set

$$\mathcal{U}^r_{d',g'} := \{x \in \mathcal{W}^r_{d',g'} \mid x \text{ is Petri general}\}.$$

Let $\rho(d',g',r) \geqslant 0$, by the main theorems of the Brill–Noether theory $\mathcal{U}^r_{d',g'}$ is irreducible and dominates $\mathcal{M}_g$ via the natural map. This motivates the previous definition.

Lemma 4.4. *Let $C \cup D \subset \mathbf{P}^r$ be a nodal complete intersection of $r-1$ hypersurfaces of degree $f = (r+2)/(r-2)$. Then the following conditions are equivalent.*

(1) *The multiplication $\mu_C : H^0(\mathcal{O}_{\mathbf{P}^r}(1)) \otimes H^0(\mathcal{I}_{C/\mathbf{P}^r}(f)) \to H^0(\mathcal{I}_{C/\mathbf{P}^r}(f+1))$ is surjective.*

(2) *The Petri map $\mu : H^0(\omega_D(-1)) \otimes H^0(\mathcal{O}_D(1)) \to H^0(\omega_D)$ is surjective.*

In particular, μ is surjective if the homogeneous ideal of C is generated by forms of degree f.

Proof. We already know that $C \cup D \subset S$, where S is a smooth canonical surface which is a complete intersection of $r-2$ hypersurfaces of degree f. Then it holds $\omega_D(-1) \cong \mathcal{O}_D(D)$. Moreover, the standard exact sequence

$$0 \to \mathcal{O}_S \otimes H^0(\mathcal{O}_S(H)) \to \mathcal{O}_S(D) \otimes H^0(\mathcal{O}_S(H)) \to \mathcal{O}_D(D) \otimes H^0(\mathcal{O}_S(H)) \to 0$$

induces the following exact commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{O}_S(H)) \arrow[r] \arrow[d, "id"] & H^0(\mathcal{O}_S(D)) \otimes H^0(\mathcal{O}_S(H)) \arrow[r] \arrow[d, "\mu_s"] & H^0(\omega_D(-1)) \otimes H^0(\mathcal{O}_D(1)) \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & H^0(\mathcal{O}_S(H)) \arrow[r] & H^0(\mathcal{O}_S(H+D)) \arrow[r] & H^0(\omega_D) \arrow[r] & 0
\end{tikzcd}$$

Since the left vertical arrow is an isomorphism, μ_S is surjective if and only if μ is surjective. Now observe that $\mathcal{I}_{C/S}(f) \cong \mathcal{O}_S(D)$ and consider the exact sequence

$$0 \to V \otimes \mathcal{I}_{S/\mathbf{P}^r}(f) \to V \otimes \mathcal{I}_{C/\mathbf{P}^r}(f) \to V \otimes \mathcal{O}_S(D) \to 0,$$

where $V = H^0(\mathcal{O}_{\mathbf{P}^r}(1))$. The sequence induces the following exact diagram.

$$\begin{tikzcd}
0 \arrow[r] & V \otimes H^0(\mathcal{I}_{S/\mathbf{P}^r}(f)) \arrow[r] \arrow[d] & V \otimes H^0(\mathcal{I}_{C/\mathbf{P}^r}(f)) \arrow[r] \arrow[d, "\mu_C"] & V \otimes H^0(\mathcal{O}_S(D)) \arrow[r] \arrow[d, "\mu_S"] & 0 \\
0 \arrow[r] & H^0(\mathcal{I}_{S/\mathbf{P}^r}(f+1)) \arrow[r] & H^0(\mathcal{I}_{C/\mathbf{P}^r}(f+1)) \arrow[r] & H^0(\mathcal{O}_S(H+D)) \arrow[r] & 0
\end{tikzcd}$$

Since S is a complete intersection of hypersurfaces of degree f, the left vertical arrow is surjective. Therefore, μ_C is surjective if and only if μ_S is surjective if and only if μ is surjective. □

THEOREM 4.5. *Let $\sigma = (f, r-1)$, $f = (r+2)/(r-2)$ and $r = d-g$. Assume $\rho(d', g', r) \geqslant 0$ and that one of the following conditions hold.*

(1) *The ideal of a general $C \in \mathcal{C}_{d,g,r}$ is generated by forms of degree f and $2g'-2 > d' > f^{r-2}$.*
(2) *$\mathcal{G}^{\sigma}_{d,g,r}$ satisfies the key condition and $\mathcal{W}^r_{d',g'}$ is irreducible.*

Then both $\mathcal{U}^r_{d',g',r}$ and $\mathcal{M}_{g'}$ are unirational.

Proof. Assume that condition (1) holds and consider a general $(C, V) \in \mathcal{G}^{\sigma}_{d,g,r}$. Then the scheme defined by V is a nodal complete intersection $C \cup D$, where D is a smooth curve of arithmetic genus g' and degree d'. It is easy to show that, under the assumption $2g'-2 > d' > f^{r-2}$, D is irreducible and non-degenerate. Hence, $\mathcal{G}^{\sigma}_{d,f,r}$ satisfies the key condition. Note also that condition (1) implies the surjectivity of the multiplication map

$$\mu_C : H^0(\mathcal{O}_{\mathbf{P}^r}(1)) \otimes H^0(\mathcal{I}_{C/\mathbf{P}^r}(f)) \to H^0(\mathcal{I}_{C/\mathbf{P}^r}(f+1)).$$

Then, by Lemma 4.4, $\mu : H^0(\omega_D(-1)) \otimes H^0(\mathcal{O}_D(1)) \to H^0(\omega_D)$ is surjective. This implies that μ is also injective. Indeed the assumption $\rho(d', g', r) \geqslant 0$ implies that $\dim H^0(\omega_D(-1)) \otimes H^0(\mathcal{O}_D(1)) \leqslant \dim H^0(\omega_D)$, otherwise the Petri map would never be injective for the values d', g', r. However, then $\beta_{d',g',r}(D)$ is a point of the main universal Brill–Noether locus $\mathcal{U}^r_{d',g'}$ and the image of

$$\beta_{d',g',r} \cdot \gamma_{d,g,r} : \mathcal{G}^{\sigma}_{d,g,r} \to \mathcal{W}^r_{d',g'}$$

is contained in $\mathcal{U}^r_{d',g'}$. On the other hand, it follows from Proposition 4.2 that such a image is open. Since $\mathcal{U}^r_{d',g'}$ is irreducible and dominates $\mathcal{M}_{g'}$ the statement follows. Finally, assume that (2) holds. Then the image of the above map is open and also dense in $\mathcal{W}^r_{d',g'}$. Moreover, $\mathcal{W}^r_{d',g'}$ dominates $\mathcal{M}_{g'}$ because $\rho(d', g', r) \geqslant 0$. Hence, the statement follows again. □

5. Curves of degree 14 and genus 8 in $\mathbf{P}^6$

Now we want to prove that the homogeneous ideal of a general curve $C \in \mathcal{C}_{14,8,6}$ is generated by quadrics. We start with a smooth, non-degenerate Del Pezzo surface

$$Y \subset \mathbf{P}^6 \tag{5.1}$$

of degree 6. It is well known that $\mathcal{I}_{Y/\mathbf{P}^6}(2)$ is globally generated. Then, by Proposition 3.2, there exists a reducible, nodal complete intersection of four quadrics

$$X \cup Y \tag{5.2}$$

where X is a smooth, irreducible surface of degree 10 and

$$F = X \cap Y \tag{5.3}$$

is a smooth curve. We have $\mathcal{O}_{X\cup Y}(1) \cong \omega_{X\cup Y}$ for the dualizing sheaf of $X \cup Y$, moreover it holds

$$\omega_X(F) \cong \omega_{X\cup Y} \otimes \mathcal{O}_X, \quad \omega_Y(F) \cong \omega_{X\cup Y} \otimes \mathcal{O}_Y. \tag{5.4}$$

Since Y is a Del Pezzo, it follows that $F \in |\mathcal{O}_Y(2)|$ is a quadratic section of Y. In particular, F is a smooth canonical curve of genus 7 in $\mathbf{P}^6$. We point out that F is non-trigonal. This follows because a trigonal canonical curve has infinitely many trisecant lines. Since Y is the intersection of quadrics, these lines would be contained in Y: a contradiction. Let $H_X \cup H_Y$ be a general hyperplane section of $X \cup Y$, with $H_X \in |\mathcal{O}_X(1)|$ and $H_Y \in |\mathcal{O}_Y(1)|$. Since F has degree 12, it follows $17 = p_a(H_X \cup H_Y) = p_a(H_X) + p_a(H_Y) + 11$ and, hence, $p_a(H_X) = 5$.

Proposition 5.5. *X is rational and projectively normal. Moreover, the homogeneous ideal of X is generated by quadrics.*

Proof. To prove that a surface is projectively normal it suffices to show the same property for a smooth, hyperplane section. Now H_X is a smooth, non-degenerate curve of genus p and degree $2p$ in $\mathbf{P}^p$, with $p = 5$. The projective normality of such a model of a genus p curve is proved in [GL86]. To prove that X is rational observe that $K_X \sim H_X - F$ by (5.4), hence, $mH_XK_X = -2m$ and $P_m(X) = 0$ for $m \geqslant 1$. Moreover, the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(H_X) \to \mathcal{O}_{H_X}(H_X) \to 0$$

implies that X is regular. Indeed the restriction $H^0(\mathcal{O}_X(H_X)) \to H^0(\mathcal{O}_{H_X}(H_X))$ is surjective because $h^0(\mathcal{O}_{H_X}(1)) = 6$ and X, H_X are not degenerate. Then, passing to the long exact sequence, $H^1(\mathcal{O}_X)$ injects in $H^1(\mathcal{O}_X(H_X))$. On the other hand, $h^2(\mathcal{O}_X(H_X)) = h^0(\mathcal{O}_X(K_X - H_X)) = h^0(\mathcal{O}_X(-F)) = 0$. Then Riemann–Roch yields $h^1(\mathcal{O}_X(H_X)) = 0$. Hence, $h^1(\mathcal{O}_X) = 0$ and X is regular. To show that the ideal of X is generated by quadrics we consider the exact diagram

$$\begin{tikzcd}
0 \arrow[r] & V \otimes H^0(\mathcal{I}_{X\cup Y}(2)) \arrow[r] \arrow[d] & V \otimes (H^0(\mathcal{I}_X(2)) \oplus H^0(\mathcal{I}_Y(2))) \arrow[r] \arrow[d, "u_X(2)\oplus u_Y(2)"] & V \otimes H^0(\mathcal{I}_F(2)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(\mathcal{I}_{X\cup Y}(3)) \arrow[r] & H^0(\mathcal{I}_X(3)) \oplus H^0(\mathcal{I}_Y(2)) \arrow[r] & H^0(\mathcal{I}_F(3)) \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are the natural multiplication maps and $V = H^0(\mathcal{O}_{\mathbf{P}^6}(1))$. The construction of the diagram is easy: the starting point is tensoring by V the Mayer–Vietoris sequence

$$0 \to \mathcal{I}_{X\cup Y}(2) \to \mathcal{I}_X(2) \oplus \mathcal{I}_Y(2) \to \mathcal{I}_F(2) \to 0.$$

Since $X \cup Y$ is a complete intersection, $h^1(\mathcal{I}_{X\cup Y}(m)) = 0,\ m \geqslant 1$, hence, the diagram is exact. We already know from the proof of the previous proposition that F is a not trigonal canonical curve. Hence, the right vertical arrow is surjective. The same is true for the left arrow, because $X \cup Y$ is complete intersection. Thus, the central arrow is surjective. However, this is the direct sum of the multiplication maps $u_X(2)$ and $u_Y(2)$,therefore both $u_X(2)$ and $u_Y(2)$ are surjective. To complete the proof it suffices to show that the multiplication

$$u_X(m) : V \otimes H^0(\mathcal{I}_X(m)) \to H^0(\mathcal{I}_X(m+1))$$

is surjective for $m \geqslant 3$. By the Castelnuovo–Mumford theorem $u_X(m)$ is surjective for $m \geqslant 3$ if $h^i(\mathcal{I}(3-i)) = 0$, $i > 0$. This follows from the standard exact sequence

$$0 \to \mathcal{I}_X(3-i) \to \mathcal{O}_{\mathbf{P}^6}(3-i) \to \mathcal{O}_X(3-i) \to 0,$$

using the projective normality of X and the vanishing of $h^1(\mathcal{O}_X(2))$ and of $h^2(\mathcal{O}_X(1))$. □

Let

$$f : X \to \mathbf{P}^4 \tag{5.6}$$

be the adjoint map defined by the linear system $|K_X + H_X| = |2H_X - F|$. Since X is projectively normal, the linear system defining f is cut on X by the quadrics containing F. We know from Noether's theorem that the ideal of F is generated by quadrics, since F is not trigonal. Hence, $|2H_X - F|$ is base-point-free and f is a morphism. Let

$$S = f(X), \tag{5.7}$$

it is easy to compute $(K_X + H_X)^2 = 4$ and $p_a(H_X + K_X) = 1$. This implies that f is birational onto S and that S is a quartic Del Pezzo surface. Applying Reider's theorem to f it follows that f contracts an integral curve E in the following cases: (1) $1 \leqslant HE \leqslant 2$ and $E^2 = 0$; (2) E is a line and $E^2 = -1$ (see [Rei88]). It is well known that f is not a morphism in case (1), hence, only case (2) is possible. It is easy to compute that exactly six lines are contracted by f to the distinct points

$$b_1 \dots b_6 \in S. \tag{5.8}$$

Lemma 5.9. *One can assume that $b_1, \dots, b_6$ are general points on S and that S is a general quartic Del Pezzo surface.*

Proof. In the Hilbert scheme of quartic Del Pezzo surfaces consider the open set U parametrizing integral surfaces. Let $u : \mathcal{S} \to U$ be the universal surface and let $u_6 : \mathcal{S}^6 \to U$ be the six times fibre product of u. U is irreducible and the fibre of u_6 at S is S^6, therefore $\mathcal{S}^6$ is irreducible. Let $b = (b_1, \dots, b_6)$, then (b, S) is a point of $\mathcal{S}^6$. Note that (b, S) defines the six-dimensional linear system $|\mathcal{I}_{Z_b/S}(2)|$ where $Z_b =: \{b_1 \dots b_6\}$. The associated map $f_b : S \to \mathbf{P}^6$ is just f^{-1} and the image X_b of f_b is just X. Thus, for the point (b, S), the surface X_b is smooth, projectively normal and its ideal is generated by quadrics. All of these properties are preserved in a neighbourhood of (b, S), hence, they are satisfied by $X_{b'}$ for a general $(b', S') \in \mathcal{S}^6$. This implies the statement. □

Next we recall that there exists a blowing down $\sigma : S \to \mathbf{P}^2$ of five disjoint lines $L_1, \dots, L_5$ of S. By the lemma we can assume that $L_i \cap \{b_1 \dots b_6\}$ is empty. Thus, $\tau =: \sigma \cdot f$ is the blowing up of 11 distinct points of $\mathbf{P}^2$, that is

$$l_i = \sigma(L_i), \quad i = 1 \dots 5 \quad \text{and} \quad e_j = \tau(E_j), \quad j = 1 \dots 6, \tag{5.10}$$

where E_j is the exceptional line contracted by f to b_j. Let $P \in |\sigma^* \mathcal{O}_{\mathbf{P}^2}(1)|$, note that

$$Pic\ X = \mathbf{Z}[P] \oplus \mathbf{Z}[L_1] \oplus \cdots \oplus \mathbf{Z}[L_5] \oplus \mathbf{Z}[E_1] \oplus \cdots \oplus \mathbf{Z}[E_6]. \tag{5.11}$$

It is easy to compute that

$$|H_X| = |6P - 2(L_1 + \cdots + L_5) - (E_1 + \cdots + E_6)|. \tag{5.12}$$

By the lemma we can assume that $l_1, \dots, l_5, e_1, \dots, e_6$ are sufficiently general, in particular that l_1, l_2, e_1, e_2 are the base points of an irreducible pencil of conics. The strict transform on X of a

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conic of this pencil will be denoted by R. It is clear that $|R|$ is irreducible and base-point-free, a general R is a smooth rational curve in $\mathbf{P}^6$ of degree $6 = H_X R$. We assume from now on that $l_1 \dots l_5,\ e_1 \dots e_6$ are in general position in $\mathbf{P}^2$.

Lemma 5.13.

(1) *R is non-degenerate.*

(2) *Let $R' \subset R$ be a proper irreducible component, R' is linearly normal.*

Proof. (1) Note that $H_X - R \sim 4P - 2(L_3+L_4+L_5) - (L_1+L_2+E_3+E_4+E_5+E_6)$. Therefore, $|H_X - R|$ is non-empty if and only if there exists a quartic curve $Q \subset \mathbf{P}^2$ passing through $l_1, \dots, l_5, e_3, e_4, e_5, e_6$ and singular at l_3, l_4, l_5. This does not happen if these points are sufficiently general in $\mathbf{P}^2$.

(2) R' is either the strict transform of a line joining two of the points l_1, l_2, e_1, e_2 or the strict transform of a smooth conic through l_1, l_2, e_1, e_2 and $o \in \{l_3, l_4, l_5, e_3, e_4, e_5, e_6\}$. Let $o = e_i$, $i = 3 \dots 6$. Then R' is a smooth rational quintic and it suffices to show that $\dim |H_X - R + E_6| = 0$. This is equivalent to saying that there exists a unique plane quartic passing through $l_1, \dots, l_5, e_3, e_4, e_5$ and singular at l_3, l_4, l_5: since $l_1, \dots, l_5, e_3, e_4, e_5$ are general this is true. We omit further details. $\square$

Finally, a curve $C \in |2H_X - R|$ has arithmetic genus 8 and degree 14. C is exactly the curve we are looking for.

Theorem 5.14. *A smooth $C \in |2H_X - R|$ belongs to $\mathcal{C}_{14,8,6}$, moreover its homogeneous ideal is generated by quadrics.*

Proof. A general R as above is a smooth, non-degenerate rational sextic curve in $\mathbf{P}^6$. Hence, the homogeneous ideal of R is generated by quadrics. From this and the projective normality of X it follows that $|2H_X - R|$ is base-point-free. Then, by Bertini's theorem, a general $C \in |2H - R|$ is smooth, and it is connected because $C^2 > 0$. Tensoring the standard exact sequence

$$0 \to \mathcal{I}_{X/\mathbf{P}^6}(2) \to \mathcal{I}_{C/\mathbf{P}^6}(2) \to \mathcal{I}_{C/X}(2) \to 0$$

by $V = H^0(\mathcal{O}_{\mathbf{P}^6}(1))$ and passing to the long exact sequence we obtain

$$0 \to V \otimes H^0(\mathcal{I}_{X/\mathbf{P}^6}(2)) \to V \otimes H^0(\mathcal{I}_{C/\mathbf{P}^6}(2)) \to V \otimes H^0(\mathcal{I}_{C/X}(2)) \to 0.$$

The multiplication $l : V \otimes H^0(\mathcal{I}_{X/\mathbf{P}^6}(2)) \to H^0(\mathcal{I}_{X/\mathbf{P}^6}(3))$ is surjective because the ideal of X is generated by quadrics. On the other hand, we have $\mathcal{I}_{C/X}(2) \cong \mathcal{O}_X(R)$. Thus, if the multiplication

$$r : V \otimes H^0(\mathcal{O}_X(R)) \to H^0(\mathcal{O}_X(H_X + R))$$

has maximal rank, then the same is true for $\mu_C : V \otimes H^0(\mathcal{I}_{C/\mathbf{P}^6}(2)) \to H^0(\mathcal{I}_{C/\mathbf{P}^6}(3))$. From the exact sequence

$$0 \to \mathcal{O}_X(H_X) \to \mathcal{O}_X(H_X + R) \to \mathcal{O}_R(H_X + R) \to 0$$

we obtain $h^0(\mathcal{O}_X(H_X + R)) = 14$. Hence, r has maximal rank if and only if r is an isomorphism. Now $|R|$ is a base-point-free pencil, in particular it has no fixed divisor. Then, applying the base-point-free pencil trick as proved for curves in [ACGH84, p. 126], it follows that $\operatorname{Ker}\ r = H^0(\mathcal{O}_X(H_X - R))$. Since R is non-degenerate, $\operatorname{Ker}\ r = (0)$. Hence, r is an isomorphism and μ_C is surjective. Using this fact, and the Castelnuovo–Mumford theorem as in Proposition 5.5, one deduces that the ideal of C is generated by quadrics if $h^i(\mathcal{I}_{C/\mathbf{P}^6}(3-i)) = 0$ for $i > 0$. This follows from the long exact sequence of

$$0 \to \mathcal{I}_{C/\mathbf{P}^6}(3-i) \to \mathcal{O}_{\mathbf{P}^6}(3-i) \to \mathcal{O}_C(3-i) \to 0$$

if $h^1(\mathcal{I}_{C/\mathbf{P}^6}(2)) = h^1(\mathcal{O}_C(1)) = 0$. Since $h^1(\mathcal{O}_X(R)) = h^1(\mathcal{I}_{X/\mathbf{P}^6}(2)) = 0$, we already have $h^1(\mathcal{I}_{C/\mathbf{P}^6}(2)) = 0$. To show that $\mathcal{O}_C(1)$ is non-special, consider the long exact sequence of

$$0 \to \mathcal{O}_X(R - H_X) \to \mathcal{O}_X(H_X) \to \mathcal{O}_C(1) \to 0.$$

We have $h^1(\mathcal{O}_X(H_X)) = 0$ then it suffices to show $h^2(\mathcal{O}_X(R - H_X)) = 0$. By Serre duality this is $h^0(\mathcal{O}_X(K_X + H_X - R))$. In $Pic(X)$ we have $K_X + H_X - R \sim P - (L_3 + L_4 + L_5) + E_1 + E_2$. Since l_3, l_4, l_5 are not collinear points the latter divisor is not linearly equivalent to an effective one. Hence, $h^2(\mathcal{O}_X(R - H_X)) = 0$ and $\mathcal{O}_C(1)$ is non-special. Finally, we show that C is projectively normal: C is non-degenerate because $H_X(H_X - C) < 0$, moreover the non-speciality of $\mathcal{O}_C(1)$ implies that C is linearly normal. Since C is also 2-normal the projective normality of C follows [ACGH84, D-5, p. 140]. In particular, we have also shown that C is in $\mathcal{C}_{14,8,6}$. $\square$

6. The unirationality of $\mathcal{M}_{14}$

In order to show the unirationality of $\mathcal{M}_{14}$ we consider our usual unirational Grassmann bundle $u : \mathcal{G}^{\sigma}_{14,8,6} \to \mathcal{C}_{14,8,6}$, where we put $\sigma = (2, 5)$. Then, from the formulae given in Proposition 2.12, we compute that

$$(d, g, r) = (14, 8, 6) \;\Leftrightarrow\; (d', g', r) = (18, 14, 6).$$

By Theorem 5.14 the homogeneous ideal of a general $C \in \mathcal{C}_{14,8,6}$ is generated by quadrics. Moreover, the condition $2g' - 2 > d' > f^{r-2}$ is satisfied and the Brill–Noether number $\rho(d', g', r)$ is 0. Then, applying Theorem 4.5, $\mathcal{G}^{\sigma}_{14,8,6}$ dominates $\mathcal{U}^6_{18,14}$. This shows the following.

THEOREM 6.1. *Both $\mathcal{M}_{14}$ and $\mathcal{U}^6_{18,14}$ are unirational.*

Note that, via Serre duality, the main Brill–Noether locus $\mathcal{U}^6_{18,14}$ is biregular to $\mathcal{U}^1_{8,14}$.

7. The unirationality of $\mathcal{M}_{12}$

We put $\sigma = (2, 5)$ again and apply a very similar argument.

CLAIM 7.1. We have that $\mathcal{G}^{\sigma}_{15,9,6}$ satisfies the key assumption.

We note that $(d, g, r) = (15, 9, 6) \Leftrightarrow (d', g', r) = (17, 12, 6)$. Under the claim the map

$$\beta_{17,12,6} \cdot \gamma_{15,9,6} : \mathcal{G}^{\sigma}_{15,9,6} \to \mathcal{W}_{17,12,6}$$

exists, by Proposition 4.2 its image is open. Again the condition $2g' - 2 > d' > f^{r-2}$ holds and $\rho(d', g', r)$ is greater than or equal to zero. Via Serre duality $\mathcal{W}^6_{17,12,6}$ is biregular to the universal 5-symmetric product $\mathcal{W}^0_{5,12}$, hence, it is irreducible. Then, applying Theorem 4.5, $\mathcal{G}^{\sigma}_{15,9,6}$ dominates $\mathcal{W}^6_{17,12}$. This shows the following.

THEOREM 7.2. *Both $\mathcal{W}^0_{5,12}$ and $\mathcal{M}_{12}$ are unirational.*

Proof of Claim 7.1. Let $X \cup Y$ be the reducible complete intersection of four quadrics considered in § 4. Keeping the assumptions and notation used there, we consider on X the irreducible, base-point-free pencil of rational normal sextics $|R|$. This pencil contains the curve

$$D_1 + E_6 \in |R|, \tag{7.3}$$

where D_1 is the strict transform on X of the irreducible conic passing through the points l_1, l_2, e_1, e_2, e_6. We know from Lemma 5.13(2) that D_1 is a smooth rational normal quintic spanning a hyperplane in $\mathbf{P}^6$. Now we consider the linear system $|2H_X - D_1|$ of curves of genus 9 and degree 15. Since D_1 is a non-degenerate rational normal quintic, the sheaf $\mathcal{I}_{D_1/\mathbf{P}^6}(2)$ is globally generated. Hence, the image of the natural restriction map

$$\rho : |\mathcal{I}_{D_1/\mathbf{P}^6}(2)| \to |2H_X - D_1|$$

is base-point-free. Since $(2H_X - D_1)^2 > 0$ it follows that a general $C \in |2H_X - D_1|$ is a smooth, irreducible curve. Moreover, we have the following. $\square$

Theorem 7.4. *The curve C is projectively normal and $\mathcal{O}_C(1)$ is non-special, so that $C \in \mathcal{C}_{17,12,6}$.*

Proof. Note that $C \sim C' + E_6$, where $C' \in |2H_X - R|$. We have already studied $|2H_X - R|$: we know from Theorem 5.14 and its proof that $|2H_X - R|$ is base-point-free and that a general $C' \in |2H_X - R|$ is a projectively normal element of $\mathcal{C}_{14,8,6}$. In particular, we can assume that such a general C' is transversal to E_6. We also recall that E_6 is a line and that $Z =: E_6 \cap C'$ is supported on two points. The non-speciality of $\mathcal{O}_{C' \cup E_6}(1)$ follows from the long exact sequence of

$$0 \to \mathcal{O}_{C' \cup E_6}(1) \to \mathcal{O}_{C'}(1) \oplus \mathcal{O}_{E_6}(1) \to \mathcal{O}_Z(1) \to 0.$$

The vanishing of $h^1(\mathcal{I}_{C' \cup E_6}(m))$, $m \geqslant 1$, follows in a similar way from the long exact sequence of

$$0 \to \mathcal{I}_{C' \cup E_6/\mathbf{P}^6}(m) \to \mathcal{I}_{C'}(m) \oplus \mathcal{I}_{E_6/\mathbf{P}^6}(m) \to \mathcal{I}_{Z/\mathbf{P}^6}(m) \to 0.$$

Then, by semicontinuity the same properties hold for a general $C \in |C' + E_6|$. □

Now we fix a general, smooth $C \in |2H_X - D_1|$, then C is transversal to D_1 and $C \cup D_1$ is a nodal quadratic section of X. X is a scheme-theoretic intersection of quadrics, hence, the same property holds for $C \cup D_1$. Then, by Proposition 3.2, there exists a nodal complete intersection of five quadrics

$$C \cup D_1 \cup D_2 = Q_1 \cap \cdots \cap Q_5. \tag{7.5}$$

From the formulae in (2.8) it follows that $D_1 \cup D_2$ is a nodal curve of degree 17 and arithmetic genus 12. On the other hand, the surface X is linked to a smooth sextic Del Pezzo Y by a complete intersection of four quadrics, so it is not restrictive to assume $Q_1 \cap \cdots \cap Q_4 = X \cup Y$. However, then D_2 is a quadratic section of Y, hence, it is smoothable to a canonical curve of genus 7. Observe that $\operatorname{Sing} D_2 \cap (C \cup D_1)$ is empty because $C \cup D_1 \cup D_2$ is nodal. Moreover, $b := D_1 \cap D_2$ is a smooth divisor of degree six in the rational normal quintic D_1. Then, keeping D_1 fixed and moving D_2 in $|\mathcal{I}_{B/Y}(D_1)|$, one can smooth D_2. This shows that there exists a flat family $\{D_1 \cup D_{2,t}, \quad t \in T\}$, such that $D_{2,t}$ is a smooth canonical curve for $t \in T - o$ and $D_{2,o} = D_2$.

Lemma 7.6. *It holds that $h^1(\mathcal{I}_{D_1 \cup D_2/\mathbf{P}^6}(2)) = 0$ and $h^0(\mathcal{I}_{D_1 \cup D_2/\mathbf{P}^6}(2)) = 5$. Therefore, up to shrinking T, one can assume $h^0(\mathcal{I}_{D_1 \cup D_{2,t}}(2)) = 5$ for each t.*

Proof. D_2 is a quadratic section of Y, so we have the standard exact sequence of ideal sheaves

$$0 \to \mathcal{I}_{Y/\mathbf{P}^6}(2) \to \mathcal{I}_{D_2/\mathbf{P}^6}(2) \to \mathcal{O}_Y(D_1) \to 0.$$

The associated long exact sequence yields $h^1(\mathcal{I}_{D_2/\mathbf{P}^6}(2)) = 0$. Then the long exact sequence of

$$0 \to \mathcal{I}_{D_1 \cup D_2/\mathbf{P}^6}(2) \to \mathcal{I}_{D_1/\mathbf{P}^6}(2) \oplus \mathcal{I}_{D_2/\mathbf{P}^6}(2) \to \mathcal{I}_{D_1 \cap D_2/\mathbf{P}^6}(2) \to 0$$

yields $h^0(\mathcal{I}_{D_1 \cup D_2/\mathbf{P}^6}(2)) = 5$. The final part of the statement immediately follows by semicontinuity.

By the lemma, $C \cup D_1 \cup D_2$ deforms in a flat family of complete intersections of five quadrics:

$$\{C_t \cup D_1 \cup D_{2,t} = Q_{1,t} \cap \cdots \cap Q_{5,t}, \quad t \in T\}.$$

Since $C \in \mathcal{C}_{15,9,6}$, a general C_t belongs to $\mathcal{C}_{15,9,6}$. Since a general D_t is smooth, we conclude that it is not restrictive to assume that D_2 is a smooth canonical curve of genus 7. □

Lemma 7.7. *We have $h^1(T_{\mathbf{P}^6} \otimes \mathcal{O}_{D_1 \cup D_2}) = 0$ so that $D_1 \cup D_2$ is smoothable.*

Proof. Consider the Mayer–Vietoris sequence

$$0 \to T_{\mathbf{P}^6} \otimes \mathcal{O}_{D_1 \cup D_2} \to T_{\mathbf{P}^6} \otimes (\mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2}) \to T_{\mathbf{P}^6} \otimes \mathcal{O}_{D_1 \cap D_2} \to 0. \tag{7.8}$$

As is well known, $h^1(T_{\mathbf{P}^6} \otimes \mathcal{O}_{D_i}) = 0$ for the curves we are considering. On the other hand, $D_1 \cap D_2$ is a set of six linearly independent points and D_1 is a non-degenerate rational normal curve.

Hence, the restriction map $H^0(T_{\mathbf{P}^6} \otimes \mathcal{O}_{D_1}) \to T_{\mathbf{P}^6} \otimes \mathcal{O}_{D_1 \cap D_2}$ is surjective. These remarks, and the long exact sequence of (7.8), imply $h^1(T_{\mathbf{P}^6} \otimes \mathcal{O}_{D_1 \cup D_2}) = 0$. Then $D_1 \cup D_2$ is smoothable (cf. [HH85, 2.1]).

We conclude the proof of our claim: let $\{D_t, t \in T\}$ be a flat family of curves in $\mathbf{P}^6$ such that D_t is smooth for $t \in T - o$ and $D_o = D_1 \cup D_2$. Let $V_t = H^0(\mathcal{I}_{D_t/\mathbf{P}^6}(2))$. By Lemma 7.6 we can assume $\dim V_t = 5$ for each $t \in T$. The scheme defined by V_o is $C \cup D_1 \cup D_2$. Hence, V_t defines a complete intersection of quadrics $C_t \cup D_t$, with $C_t \in \mathcal{C}_{15,9,6}$ and the claim follows. $\square$

8. The unirationality of $\mathcal{M}_{11}$

In this case we shift to curves in $\mathbf{P}^4$, but the arguments are the same. We definitely assume $\sigma = (3,3)$ and consider the unirational Grassmann bundle $u : \mathcal{G}^{\sigma}_{13,9,4} \to \mathcal{C}_{13,9,4}$. For $\sigma = (3,3)$ it turns out that $(d, g, r) = (13, 9, 4) \Longleftrightarrow (d', g', r) = (14, 11, 4)$.

CLAIM 8.1. We claim that $\mathcal{G}^{\sigma}_{13,9,4}$ satisfies the key assumption.

This will be shown in Theorem 9.10. Under the claim the map $\gamma_{13,9,4} : \mathcal{G}^{\sigma}_{14,11,4} \to \mathcal{W}^4_{14,11}$ exists. By Proposition 4.2 the image of $\beta_{14,11,4} \cdot \gamma_{13,9,4} : \mathcal{G}^{\sigma}_{13,9,4} \to \mathcal{W}^2_{14,11}$ is open. Serre duality yields a biregular map between $\mathcal{W}^4_{14,11}$ and the universal 6-symmetric product $\mathcal{W}^0_{6,11}$. Hence, $\mathcal{W}^4_{14,11}$ irreducible. Applying, as usual, Theorem 4.5, we have the following.

THEOREM 8.2. *Both $\mathcal{W}^0_{6,11}$ and $\mathcal{M}_{11}$ are unirational.*

9. Curves of degree 12 and genus 8 in $\mathbf{P}^4$

In this section we prove our previous Claim 8.1. Preliminarily, we also show that a general curve of degree 12 and genus 8 in $\mathbf{P}^4$ is a scheme-theoretic intersection of cubics. This will be used in the next section. Instead of our favourite rational surface of degree 10 in $\mathbf{P}^6$, we now use a smooth septic $X \subset \mathbf{P}^4$ having sectional genus 5 and birational to a K3 surface. This surface is very well known and its properties are described in [DES93]. We preliminarily recall some of them.

PROPOSITION 9.1. *The surface X is projectively normal and its ideal is generated by three cubic forms.*

The geometric construction of X is also well known (cf. [Bau95]).

PROPOSITION 9.2. *Let $X' \subset \mathbf{P}^5$ be any smooth complete intersection of three quadrics and let $e \in X'$ be a point not on a line of X'. Then the image of X' under the linear projection of center e is a smooth surface X as above.*

Thus, X is defined by the blowing up $\sigma : X \to X'$ at e and $K_X = \sigma^{-1}(e)$, we have $\mathit{Pic}\ X = \sigma^* \mathit{Pic}\ X' \oplus \mathbf{Z}[K_X]$. To construct some curves of genus 8 and 9 we choose a suitable X' with Picard number two.

PROPOSITION 9.3. *There exists a smooth complete intersection of three quadrics $X' \subset \mathbf{P}^5$ such that $\mathit{Pic}\ X' = \mathbf{Z}[L'] \oplus \mathbf{Z}[H']$, where H' is a hyperplane section of X' and L' is a very ample curve of degree 10 and genus 3. Moreover, any effective divisor on X' is very ample.*

Proof. The existence of a K3 surface X' with Picard lattice as above is a standard consequence of the surjectivity of the periods map for K3 surfaces. Such a lattice does not contain non-zero vectors v such that $v^2 = 0, -2, 2$. Indeed, let $v = xH' + yL'$, then $v^2 = 4(2x^2 + y^2 + 5xy) \neq 2, -2, 0$. Let D be any effective divisor, then $D^2 \geqslant 4$ and $\dim |D| \geqslant 3$. Let F be a fixed irreducible component of $|D|$, then $\dim |F| = 0$ and, hence, $F^2 < 0$: a contradiction. Since X' is a K3 surface, then $|D|$ is

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base-point-free and irreducible, moreover $|D|$ is very ample unless $D^2 = 2$ or there exists a curve F such that $DF \leqslant 2$ and $F^2 \in \{0, -2\}$. Hence, D is very ample. Up to changing their sign, we can assume that both the generators H' and L' of $Pic\ X'$ are effective. In particular, we can assume that X' is embedded in $\mathbf{P}^5$ by H'. Then either X' is a complete intersection of three quadrics or contains a pencil $|F|$ of plane cubics. Since $F^2 = 0$, the latter case is excluded. □

From now on we assume that X' is a K3 surface as in the previous statement. On X' we have the very ample linear system

$$|3H' - L'| \tag{9.4}$$

of curves of degree 14 and genus 9. Let $e \in X'$ be a general point, due to the very ampleness of the linear systems we are considering we can assume that:

(1) there exists $A'_e \in |3H' - L'|$ having an ordinary node at e and no other singular point;

(2) there exists $B'_e \in |L'|$ having an ordinary node at e and no other singular point;

(3) the linear systems $|L' - e|$ and $|3H' - L' - e|$ have a unique, simple base point at e.

Finally, let

$$\pi : X' \to X \subset \mathbf{P}^4 \tag{9.5}$$

be the projection of center e, π is the inverse of the blow up $\sigma : X \to X'$ at e. Let

$$A \subset X \subset \mathbf{P}^4 \tag{9.6}$$

be the strict transform of A'_e by σ. Then A is a smooth, irreducible curve of genus 8 and degree 12. Let $H =: \sigma^* H'$, $L =: \sigma^* L'$ then $A \in |3H - L - 2K_X|$. Unfortunately, $\mathcal{O}_A(1)$ is special: this happens to every curve in X, since X is regular and $h^1(\mathcal{O}_X(1)) = 1$. Nevertheless, we can use A to show the following.

Theorem 9.7. *A general $C \in \mathcal{C}_{12,4,8}$ is a scheme-theoretic intersection of cubics.*

Proof. It is easy to see that the Hilbert scheme $\mathcal{H}_{12,8,4}$ is irreducible, in particular $\mathcal{C}_{12,8,4}$ is dense in $\mathcal{H}_{12,8,4}$. Hence, there exists a flat family $\{C_t, t \in T\}$ such that $C_t \in \mathcal{C}_{12,8,4}$ if $t \neq o$ and $A = C_o$. Then, to prove the theorem, it suffices to show that: (1) $h^0(\mathcal{I}_{C_t/\mathbf{P}^4}(3))$ is constant on T; (2) A is a scheme-theoretic intersection of cubics. To show part (1) it suffices to show that A is 3-normal. This follows from the long exact sequence of

$$0 \to \mathcal{I}_{X/\mathbf{P}^4}(3) \to \mathcal{I}_{A/\mathbf{P}^4}(3) \to \mathcal{O}_X(3H - A) \to 0 \tag{9.8}$$

observing that $h^1(\mathcal{O}_X(3H - A)) = 0$ and that X is 3-normal. To show part (2), recall that the ideal of X is generated by cubics. Then, to prove that A is a scheme-theoretic intersection of cubics, it suffices to show that $|3H - A|$ is base-point-free. This follows because $|3H - A|$ is the strict transform on X of $|3H' - A' - e|$, whose unique base point is e. □

Now we turn to curves of degree 13 and genus 9: the linear system $|3H - L - K_X|$ is just the strict transform by σ of $|3H' - L' - e|$, hence, a general

$$C_o \in |3H - L - K_X| \tag{9.9}$$

is smooth, irreducible of degree 13 and genus 9. Let $B \subset X \subset \mathbf{P}^4$ be the strict transform of the curve B'_e considered above, B is a smooth octic of genus 2. We can assume that $C_o \cup B$ is nodal, note also that $C_o + B$ is a cubic section of X. Since the ideal of X is generated by three cubics, the ideal of $C_o \cup B$ is generated by four. Then, by Proposition 3.2, there exists a nodal complete intersection $F_1 \cap F_2 \cap F_3 = C_o \cup B \cup B_1$, where F_1, F_2, F_3 are cubics. By Proposition 3.3 we can also choose F_1, F_2 so that $F_1 \cap F_2 = X \cup Y$ where Y is smooth. Then Y is a quadric and B_1 is a smooth curve of type $(3, 3)$ on it. Using C_o and the previous remarks we can finally show the following.

THEOREM 9.10. *A general $C \in \mathcal{C}_{13,9,4}$ is linked to a smooth, irreducible curve by a complete intersection of three cubics. In particular, $\mathcal{G}^{\sigma}_{13,9,4}$ satisfies the key condition.*

Proof. It is easy to see that $\mathcal{H}_{13,9,4}$ is irreducible. Therefore, there exists an irreducible flat family $\{C_t, t \in T\}$ such that $C_t \in \mathcal{C}_{13,9,4}$ if $t \neq o$ and $C_t = C_o$ if $t = o$. C_o is 3-normal: the proof is exactly the same used for the curve A in the proof of Theorem 9.7. Let $W_t = H^0(\mathcal{I}_{C_t/\mathbf{P}^4}(3))$, then W_t has constant dimension 4. Let $V_o \subset W_o$ be the subspace defining the complete intersection $C_o \cup B \cup B_1$, then we can move V_o in a family $\{V_t, t \in T\}$ of three-dimensional subspaces $V_t \subset W_t$. We can assume that V_t defines a nodal complete intersection of three cubics $C_t \cup D_t$ and that D_t is nodal, non-degenerate, of degree 14 and arithmetic genus 11. Let $\mathcal{H}$ be the complete Hilbert scheme of D_t, clearly there exists a rational map $\gamma : \mathcal{G}^{\sigma}_{13,9,4} \to \mathcal{H}$ sending a general $(C, V) \in \mathcal{G}^{\sigma}_{12,8,4}$ to D, $C \cup D$ being the complete intersection defined by V. However, we only know that $D_o = \gamma(C_o, V_o)$ is the nodal union of two smooth curves, so any D in the image of γ could be singular. To complete the proof we show that this does not happen.

We recall that, by Lemma 4.1, each D in the image of γ is a linearly normal curve such that $h^0(\mathcal{I}_D(3)) = 3$. Let $D = \gamma(C, V)$, the latter property implies that $V = H^0(\mathcal{I}_{D/\mathbf{P}^4}(3))$ and, hence, that γ is birational onto its image. So we can compare dimensions.

Let D be general in the image of γ. D is a flat deformation of D_o. D_o has six nodes, moreover $D_o = B_o \cup B_1$, where B_o, B_1 are smooth, irreducible and B_o is not degenerate. Assume that D is not smooth. Then, since D is general, we have the following cases: (1) D is the nodal union of two smooth, irreducible curves, one of them not degenerate; (2) D is nodal, irreducible with at most 6 nodes. We discuss the two cases separately.

(1) Let $f : \mathcal{D} \to T$ be a flat family such that $\mathcal{D}_t$ is general in the image of γ and $\mathcal{D}_o = D_o$. We can assume that T is smooth, irreducible and that each $\mathcal{D}_t$ satisfies the condition in (1). Let $\mathcal{D}$ be irreducible, then the two irreducible components of a general $\mathcal{D}_t$ must have the same degree and genus. This implies that the degree is 7 and the genus is 3. Let $\mathcal{F} \subset \mathcal{H}$ be the family of all 6-nodal curves $D_1 \cup D_2$ such that D_i is a smooth septic of genus 3: we have $\dim \mathcal{F} = 48 < \dim \mathcal{G}^{\sigma} = 60$. Hence, the image of γ is not in $\mathcal{F}$. Assume now that $\mathcal{D}$ is reducible, it is easy to deduce that then a general $\mathcal{D}_t$ is like D_o i.e. it has six nodes and it is the union of a smooth canonical curve of genus 4 and of a smooth octic of genus 3. Again this family of reducible curves has dimension less than $\dim \mathcal{G}^{\sigma}_{13,9,4}$.

(2) We can consider an analogous family $f : \mathcal{D} \to T$. In this case $\mathcal{D}_t$ is irreducible, nodal with $\nu \leqslant 6$ nodes if $t \neq o$ and $\mathcal{D}_o = D_o$. Moreover, $\mathcal{D}_t$ is non-degenerate and linearly normal. Let $\mathcal{F}_\nu$ be the corresponding family of irreducible, nodal curves of genus $11 - \nu$ and degree 14. It suffices to compute that $\dim \mathcal{F}_\nu < 60$. This is a not difficult exercise. □

10. The unirationality of $\mathcal{M}_{13}$

Let $\sigma = (3, 3)$; continuing in the same vein we first point out that $\mathcal{G}^{\sigma}_{12,14,4}$ satisfies the key condition as follows from Theorem 9.7. Since $(d, g, r) = (12, 8, 4) \iff (d', g', r') = (15, 14, 4)$, a general $C \in \mathcal{C}_{12,8,4}$ is linked to a smooth, irreducible, non-degenerate D of genus 14 by a complete intersection of three cubics. D has genus 14 and not 13, however we will turn very soon to curves D having exactly one node. We already know that

$$C \cup D \subset S, \tag{10.1}$$

where S is a smooth complete intersection of two cubics. Let H be a hyperplane section of S: we know as well that, since C is a scheme-theoretic intersection of cubics, then the linear system $|3H - C|$ is base-point-free. From $h^0(\mathcal{I}_{C/\mathbf{P}^4}(3)) = 6$ it follows that $\dim |3H - C| = 3$. Note also that $(3H - C)^2 = 11$. This implies that, for every $D \in |3H - C|$, the linear series $|\mathcal{O}_D(D)|$ is a complete g^2_{11} with no base points. Since the degree is 11, the associated morphism $f_D : D \to \mathbf{P}^2$ is birational onto its image. We will say that a curve is *uninodal* if it is nodal with a unique node. From now on we assume that $D \in |3H - C|$ is uninodal.

Definition 10.2. The family of all uninodal $D \in \mathcal{H}_{15,14,4}$ such that there exists a complete intersection $C \cup D$ as above will be denoted as $\mathcal{D}$.

Let $\nu : D' \to D$ be the normalization map and let $L_{D'} := \nu^* \mathcal{O}_D(D)$, $L_{D'}$ defines the morphism $f_D \cdot \nu$. Hence, $|L_{D'}|$ is base-point-free of degree 11 and $f_D(D)$ is an element of the quasi-projective variety of all reduced, irreducible linearly normal plane curves of degree 11 and genus 13. It follows from the general results on families of plane nodal curves that such a variety is irreducible and that a non-empty open set of it parametrizes nodal curves (cf. [HM91, p. 40]). Moreover, its image

$$U^2_{11,13} \subset \mathcal{W}^2_{11,13} \tag{10.3}$$

dominates the moduli space $\mathcal{M}_{13}$ via the natural map. This follows since the Brill–Noether number $\rho(11, 13, 2)$ is greater than or equal to zero. The image of the element $f_D(D)$ is just the moduli point of the pair $(D', L_{D'})$ in $\mathcal{W}^2_{11,13}$. Since $L_{D'} = \omega_{D'}(n) \otimes \nu^* \mathcal{O}_D(-1)$, with $n := \nu^* \operatorname{Sing} D$, we can define a morphism

$$\phi : \mathcal{D} \to U^2_{11,13} \tag{10.4}$$

sending a uninodal $D \in \mathcal{D}$ to the moduli point of $(D', L_{D'})$.

Lemma 10.5. *We have that ϕ is dominant.*

Proof. We fix an irreducible flat family of pairs $\{(D_t, L_t), t \in T\}$ such that D_t is a smooth, irreducible curve of genus 13 and $L_t \in Pic^{11}(D_t)$ is globally generated with $h^0(L_t) = 3$. Moreover, we choose a T which dominates $\mathcal{U}^2_{11,13}$ via the natural map and such that $(D_o, L_o) = (D', L_{D'})$. Up to a base change we can assume that: (1) there exists a rationally determined divisor $n_t \in Div^2(D_t)$ which is contracted to a point by the morphism $f_t : D_t \to \mathbf{P}^2$ defined by L_t, (2) $n_o = n$.

Then we consider the family of pairs $(D_t, \omega_{D_t}(n_t) \otimes L_t^{-1})$ and the corresponding family of associated maps $h_t : D_t \to \mathbf{P}^4$. To show that ϕ is dominant it suffices to show that $h_t(D_t) \in \mathcal{D}$ for t general. Since $D = h_o(D_o)$ is uninodal, the general $h_t(D_t)$ is uninodal. It is immediately computed that $h^0(\mathcal{I}_{h_t(D_t)/\mathbf{P}^4}(3)) \geqslant 3$. On the other hand, we have $h^0(\mathcal{I}_{D/\mathbf{P}^4}(3)) = 3$, as follows by applying Lemma 4.1 to $C \cup D$. Thus, by semicontinuity, the same property holds for a general D_t. Finally, let $V_t = H^0(\mathcal{I}_{h_t(D_t)/\mathbf{P}^4}(3))$: the scheme defined by V_o is the curve $C \cup D$. Hence, the scheme defined by a general V_t is a nodal curve $C_t \cup D_t$, with $C_t \in \mathcal{C}_{12,8,4}$. Hence, $h_t(D_t) \in \mathcal{D}$ and ϕ is dominant. $\square$

Lemma 10.6. *A point $(C, V) \in \mathcal{G}^\sigma_{12,8,4}$ is uninodal if $D = \gamma_{12,8,4}(C, V)$ is uninodal. The family of all uninodal points (C, V) will be denoted by $\mathcal{N}$.*

Lemma 10.7. *The families $\mathcal{N}$ and $\mathcal{D}$ are unirational.*

Proof. It is clear that $\gamma_{12,8,4}(\mathcal{N}) = \mathcal{D}$. Hence, it is sufficient to show that $\mathcal{N}$ is unirational. Fix a general pair $(C, x) \in \mathcal{C}_{12,8,4} \times \mathbf{P}^4$, then consider the family $F(C, x)$ of all uninodal points (C, V) such that $x = \operatorname{Sing} D$ and $\gamma_{12,8,4}(C, V) = D$. It is easy to see that $F(C, x)$ is biregular to an open subset of the Grassmannian $G(2, I_x/I_{2,x})$, where $I_x =: \{f \in I \mid f \in m_x\}$, $I_{2,x} =: \{f \in I_x \mid f \in m_x^2\}$ and $I =: H^0(\mathcal{I}_{C/\mathbf{P}^4}(3))$. Moreover, $F(C, x)$ is the fibre at the point (C, x) of the morphism $\pi : \mathcal{N} \to \mathcal{C}_{12,8,4} \times \mathbf{P}^4$ sending (C, V) to (C, x), with $x = \operatorname{Sing} D$ and $D = \gamma_{12,8,4}(C, V)$. We first prove that $\pi(\mathcal{N})$ is dense: let $\pi(C, V) = (C, x)$, then there exists a nodal complete intersection of three cubics $C \cup D$ such that $\operatorname{Sing} D = x$. This condition is open on (C, V) and x, hence, it holds on open neighbourhoods U_C of C and U_x of x. Therefore, $U_C \times U_x \subset \pi(\mathcal{N})$ and $\pi(\mathcal{N})$ is dense. $\pi(\mathcal{N})$ contains a dense open set A on which $I_x/I_{2,x}$ has constant dimension. It is standard to construct on A a vector bundle $\mathcal{Q}$ having fibre $I_x/I_{2,x}$ at the point (C,x). On the other hand, the fibre $F(C,x)$ of π at (C,x) is open in $G(2,I_x/I_{2,x})$. Hence, $\mathcal{N}$ is biregular to an open set of the Grassmann bundle $G(2,\mathcal{Q})$. This is birational to $\mathcal{C}_{12,8,4} \times \mathbf{P}^4 \times G(2,4)$, therefore it is unirational. $\square$

As a straightforward consequence of the lemma we have the following.

Theorem 10.8. *The moduli space $\mathcal{M}_{13}$ is unirational as well as the Severi variety of nodal plane curves of degree 11 and genus 13.*

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References

ACGH84 E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, *Geometry of algebraic curves I* (Springer, Berlin, 1984), 1–386.

Bau95 I. Bauer, *Inner projections of algebraic surfaces: a finiteness result*, J. reine angew. Math. **460** (1995), 1–13.

BE87 E. Ballico and Ph. Ellia, *The maximal rank conjecture for non special curves in* $\mathbf{P}^n$, Math. Z. **196** (1987), 355–367.

CR84 M. C. Chang and Z. Ran, *Unirationality of the moduli space of curves of genus 11, 13 (and 12)*, Invent. Math. **76** (1984), 41–54.

CR91 M. C. Chang and Z. Ran, *On the slope and Kodaira dimension of* $\overline{M}_g$ *for small* g, J. Diff. Geom. **34** (1991), 267–274.

DES93 W. Decker, L. Ein and F. O. Schreyer, *Construction of surfaces in* $\mathbf{P}^4$, J. Algebraic Geom. **2** (1993), 185–237.

EH83 D. Eisenbud and J. Harris, *The Kodaira dimension of the moduli space of curves of genus* $\leqslant 23$, Invent. Math. **74** (1983), 359–387.

Far00 G. Farkas, *The geometry of the moduli space of curves of genus 23*, Math. Ann. **318** (2000), 43–65.

FP02 G. Farkas and M. Popa, *Effective divisors on* M_g *and a counterexample to the slope conjecture*, Preprint (2002).

Ful84 W. Fulton, *Intersection theory* (Springer, Berlin, 1984), 470.

GL86 M. Green and R. Lazarsfeld, *On the projective normality of complete linear series on an algebraic curve*, Invent. Math. **83** (1986), 73–90.

HH85 R. Hartshorne and A. Hirschowitz, *Smoothing algebraic space curves*, in *Algebraic geometry* (Sitges, 1983), eds E. Casas-Alvero, G. E. Welters, S. Xambo-Descamps, Lecture Notes in Mathematics, vol. 1124 (Springer, Berlin, 1985), 98–131.

HM82 J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), 23–86.

HM90 J. Harris and I. Morrison, *Slopes of effective divisors on the moduli space of curves*, Invent. Math. **99** (1990), 321–335.

HM91 J. Harris and I. Morrison, *Moduli of curves* (Springer, Berlin, 1991), 1–366.

MM83 S. Mori and S. Mukai, *The uniruledness of the moduli space of curves of genus 11*, in *Algebraic Geometry* (Tokyo/Kyoto 1982), Lecture Notes in Mathematics, vol. 1016, eds M. Raynaud and T. Shioda (Springer, Berlin, 1983), 334–353.

Muk88 S. Mukai, *Curves, K3 surfaces and Fano manifolds of genus* $\leqslant 10$, in *Algebraic Geometry in honor of Masayoshi Nagata*, eds H. Hijikata and H. Hironaka (Kikokuniya, Tokyo, 1988), 367–377.

Rei88 I. Reider, *Vector bundles of rank two and linear systems on algebraic surfaces*, Ann. of Math. (2) **127** (1988), 309–316.

Ser81 E. Sernesi, *L'unirazionalitá della varietá dei moduli delle curve di genere 12*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **8** (1981), 405–439.

Sev21 F. Severi, *Vorlesungen über Algebraische Geometrie* (translated by E. Loeffler) (Teubner, Leipzig, 1921).

ST02 F. O. Schreyer and F. Tonoli, *Needles in a haystack: special varieties via small fields*, in *Mathematical computations with Macaulay 2* (Springer, Berlin, 2002).

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