INTEGRALS INVOLVING A MODIFIED BESSEL FUNCTION OF THE SECOND KIND AND AN *E*-FUNCTION

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§ 1. *Introductory.* The first formula to be proved is

$$\int_0^\infty t^{k-1}K_n(t)E(p\,;\alpha_r:q\,;\rho_s:zt)dt=2^{\Sigma\alpha_r-\Sigma\rho_s}(\sqrt{\pi})^{5-p+q}$$

$$\times\left[\begin{aligned}&\frac{2^{k-p+q-2}}{\sin\left(\frac{k+n}{2}\pi\right)\sin\left(\frac{k-n}{2}\pi\right)}E\left(\begin{matrix}\frac{\alpha_1}{2},\frac{\alpha_1+1}{2},\ldots,\frac{\alpha_p+1}{2}:e^{\pm i\pi}4^{q-p+2}z^2\\ \frac{1}{2},1-\frac{k+n}{2},1-\frac{k-n}{2},\frac{\rho_1}{2},\ldots,\frac{\rho_q+1}{2}\end{matrix}\right)\\ &-\frac{2^{k-4}}{\cos\left(\frac{k+n}{2}\pi\right)\cos\left(\frac{k-n}{2}\pi\right)}z^{-1}E\left(\begin{matrix}\frac{\alpha_1+1}{2},\frac{\alpha_1+2}{2},\ldots,\frac{\alpha_p+2}{2}:e^{\pm i\pi}4^{q-p+2}z^2\\ \frac{3}{2},\frac{3-k-n}{2},\frac{3-k+n}{2},\frac{\rho_1+1}{2},\ldots,\frac{\rho_q+2}{2}\end{matrix}\right)\\ &+\sum_{n,-n}\frac{2^{-k-2n-1+(k+n-1)(p-q)}}{\sin n\pi\sin(k+n)\pi}z^{-k-n}\\ &\qquad\times E\left(\begin{matrix}\frac{\alpha_1+k+n}{2},\frac{\alpha_1+k+n+1}{2},\ldots,\frac{\alpha_p+k+n+1}{2}:e^{\pm i\pi}4^{q-p+2}z^2\\ n+1,\frac{k+n+1}{2},\frac{k+n+2}{2},\frac{\rho_1+k+n}{2},\ldots,\frac{\rho_q+k+n+1}{2}\end{matrix}\right)\end{aligned}\right],\;\ldots\ldots(1)$$

where $p \geqq q+1$, $|\operatorname{amp} z|<\pi$, $R(k\pm n+\alpha_r)>0$, $r=1,2,\ldots,p$. For other values of p and q the result is valid if the integral is convergent. A second formula is given in § 3.

The following formulae are required in the proof:

$$\begin{aligned}&\int_0^\infty t^{l-1}K_n(t)K_m(z/t)\,dt\\ &=\sum_{m,-m}2^{l+2m-3}\Gamma(m)\Gamma\left(\frac{l+m+n}{2}\right)\Gamma\left(\frac{l+m-n}{2}\right)z^{-m}\\ &\qquad\times F\left(\,;1-m,1-\frac{l+m+n}{2},1-\frac{l+m-n}{2};\frac{z^2}{16}\right)\\ &+\sum_{n,-n}2^{-l-2n-3}\Gamma(-n)\Gamma\left(\frac{-l+m-n}{2}\right)\Gamma\left(\frac{-l-m-n}{2}\right)z^{l+n}\\ &\qquad\times F\left(\,;1+n,1+\frac{l-m+n}{2},1+\frac{l+m+n}{2};\frac{z^2}{16}\right),\end{aligned}\;\ldots\ldots\ldots\ldots(2)$$

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where $R(z)>0$, (1);

$$K_{\frac{1}{2}}(z)=\sqrt{\left(\frac{\pi}{2z}\right)}\,e^{-z}=\sqrt{\left(\frac{\pi}{2z}\right)}\,E\left(\,::\frac{1}{z}\right); \qquad \ldots\ldots(3)$$

$$\int_0^\infty e^{-\lambda}\lambda^{\alpha-1}E(p\,;\alpha_r:q\,;\rho_s:z/\lambda^2)d\lambda=2^{\alpha-1}\pi^{-\frac{1}{2}}E\{\alpha_1,\ldots,\alpha_p,\alpha/2,(\alpha+1)/2:q\,;\rho_s:\tfrac{1}{4}z\}, \qquad \ldots\ldots(4)$$

where $R(\alpha)>0$, $|\operatorname{amp} z|<\pi$, (2);

$$\frac{1}{2\pi i}\int e^{\zeta}\zeta^{-\rho}E(p\,;\alpha_r:q\,;\rho_s:z\zeta^2)d\zeta=2^{1-\rho}\pi^{\frac{1}{2}}E(p\,;\alpha_r:\rho_1,\ldots,\rho_q,\rho/2,(\rho+1)/2:4z), \qquad \ldots\ldots(5)$$

where the contour starts from $-\infty$ on the ξ-axis, passes round the origin in the positive direction, and ends at $-\infty$ on the ξ-axis, the initial value of amp ζ being $-\pi$, (3).

§ 2. *Proof of the Formula.* In (2) put $m=\frac{1}{2}$, and apply (3), replacing l by $k-\frac{1}{2}$ and z by $1/z$; then, on multiplying by $\sqrt{\{2/(\pi z)\}}$, it is found that, if $R(z)>0$,

$$\int_0^1 t^{k-1}K_n(t)E(::zt)\,dt$$

$$=\frac{2^{k-2}\pi^{5/2}}{\sin\left(\frac{k+n}{2}\pi\right)\sin\left(\frac{k-n}{2}\pi\right)}E\left(\,:\tfrac{1}{2},1-\frac{k+n}{2},1-\frac{k-n}{2}:e^{\pm i\pi}16z^2\right)$$

$$-\frac{2^{k-4}\pi^{5/2}}{\cos\left(\frac{k+n}{2}\pi\right)\cos\left(\frac{k-n}{2}\pi\right)}z^{-1}E\left(\,:\frac{3}{2},\frac{3-k-n}{2},\frac{3-k+n}{2}:e^{\pm i\pi}16z^2\right)$$

$$+\sum_{n,-n}\frac{2^{-k-2n-1}\pi^{5/2}}{\sin n\pi\sin(k+n)\pi}z^{-k-n}E\left(\,:n+1,\frac{k+n+1}{2},\frac{k+n+2}{2}:e^{\pm i\pi}16z^2\right).$$

On generalising the E-function on the left in the usual way, and applying formulae (4) and (5) to the E-functions on the right, formula (1) is obtained. When it is necessary to avoid values of $z\zeta$ such that $R(z\zeta)\not>0$, the contour in (5) may be replaced by a line to the right of the origin parallel to the η-axis, z being taken real and positive. This restriction on z and any restrictions on the ρ's required for convergence may subsequently be removed by analytical continuation.

§ 3. *A Second Integral.* The formula to be proved is

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_n(\lambda)E(p\,;\alpha_r:q\,;\rho_s:z\lambda)\,d\lambda$$

$$=\frac{\pi^{3/2}2^{-k}\cos k\pi}{\sin(k+n)\pi\sin(k-n)\pi}E\begin{pmatrix}\frac{1}{2}-k,\alpha_1,\ldots,\alpha_p:\frac{1}{2}e^{\pm i\pi}z\\ 1-k+n,1-k-n,\rho_1,\ldots,\rho_q\end{pmatrix}$$

$$+\sum_{n,-n}\frac{\pi^{3/2}2^{n-1}z^{-k-n}}{\sin(k+n)\pi\sin n\pi}E\begin{pmatrix}\frac{1}{2}+n,\alpha_1+k+n,\ldots,\alpha_p+k+n:\frac{1}{2}e^{\pm i\pi}z\\ 1+2n,1+k+n,\rho_1+k+n,\ldots,\rho_q+k+n\end{pmatrix}, \qquad \ldots\ldots(6)$$

where $p\geqq q+1$, $R(k\pm n+\alpha_r)>0$, $r=1,2,\ldots,p$, $|\operatorname{amp} z|<\pi$. For other values of p and q the result is valid if the integral is convergent.

This result can be derived from Ragab's formula (4)

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_n(\lambda)K_m(z/\lambda)\,d\lambda$$

$$= \sum_{m,-m} \frac{\pi^3\cos(k+m)\pi\, 2^{-3/2}(2/z)^m}{\sin(k+m+n)\pi\sin(k+m-n)\pi\sin m\pi}$$

$$\times E\begin{pmatrix}\frac{1}{4}-\frac{1}{2}k-\frac{1}{2}m,\ \frac{3}{4}-\frac{1}{2}k-\frac{1}{2}m : e^{\pm i\pi}4/z^2\\ 1-m,\ 1-\frac{1}{2}k-\frac{1}{2}m-\frac{1}{2}n,\ 1-\frac{1}{2}k-\frac{1}{2}m+\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}k-\frac{1}{2}m-\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}k-\frac{1}{2}m+\frac{1}{2}n\end{pmatrix}$$

$$+\sum_{n,-n}\left\{\begin{array}{l} -\dfrac{\pi^3 2^{-7/2}(z/2)^{k+n}}{\sin\left(\dfrac{k-m+n}{2}\pi\right)\sin\left(\dfrac{k+m+n}{2}\pi\right)\sin n\pi} \\ \qquad\times E\begin{pmatrix}\frac{1}{4}+\frac{1}{2}n,\ \frac{3}{4}+\frac{1}{2}n : e^{\pm i\pi}4/z^2\\ 1+\frac{1}{2}k+\frac{1}{2}m+\frac{1}{2}n,\ 1+\frac{1}{2}k-\frac{1}{2}m+\frac{1}{2}n,\ \frac{1}{2},\ \frac{1}{2}+n,\ 1+n\end{pmatrix} \\ +\dfrac{\pi^3 2^{-7/2}(z/2)^{k+n+1}}{\cos\left(\dfrac{k-m+n}{2}\pi\right)\cos\left(\dfrac{k+m+n}{2}\pi\right)\sin n\pi} \\ \qquad\times E\begin{pmatrix}\frac{3}{4}+\frac{1}{2}n,\ \frac{5}{4}+\frac{1}{2}n : e^{\pm i\pi}4/z^2\\ \frac{3}{2}+\frac{1}{2}k+\frac{1}{2}m+\frac{1}{2}n,\ \frac{3}{2}+\frac{1}{2}k-\frac{1}{2}m+\frac{1}{2}n,\ \frac{3}{2},\ 1+n,\ \frac{3}{2}+n\end{pmatrix}\end{array}\right\} \quad \ldots\ldots\ldots(7)$$

where $R(z)>0$.

The formula

$$E(p\,;\alpha_r:q\,;\rho_s:e^{\pm i\pi}z) = 2^{\Sigma\alpha_r-\Sigma\rho_s}(\sqrt{\pi})^{q-p+1}$$

$$\times\left\{\begin{array}{l} 2^{q-p}E\left(\dfrac{\alpha_1}{2},\dfrac{\alpha_1+1}{2},\ldots,\dfrac{\alpha_p+1}{2}:\dfrac{1}{2},\dfrac{\rho_1}{2},\dfrac{\rho_1+1}{2},\ldots,\dfrac{\rho_q+1}{2}:e^{\pm i\pi}4^{q-p+1}z^2\right) \\ +\dfrac{1}{2z}E\left(\dfrac{\alpha_1+1}{2},\dfrac{\alpha_1'+2}{2},\ldots,\dfrac{\alpha_p+2}{2}:\dfrac{3}{2},\dfrac{\rho_1+1}{2},\dfrac{\rho_1+2}{2},\ldots,\dfrac{\rho_q+2}{2}:e^{\pm i\pi}4^{q-p+1}z^2\right)\end{array}\right\} \quad \ldots(8)$$

will be required. It can be derived by generalising the formula

$$e^{1/z} = E(\ ::\ -z) = \Gamma(\tfrac{1}{2})\,E(\ :\tfrac{1}{2}:\ -4z^2) + \Gamma(\tfrac{3}{2})\frac{1}{z}E(\ :\tfrac{3}{2}:\ -4z^2).$$

Now in (7) replace z by $1/z$, put $m=\frac{1}{2}$, so that, from (3), $K_{\frac{1}{2}}\{1/(\lambda z)\}=\sqrt{(\frac{1}{2}\pi z\lambda)}\,E(\ ::z\lambda)$, replace k by $k-\frac{1}{2}$ and multiply by $\sqrt{\{2/(\pi z)\}}$: then

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_n(\lambda)E(\ ::z\lambda)\,d\lambda = \frac{\pi^{5/2}\cos k\pi\,2^{-\frac{1}{2}}}{\sin(k+n)\pi\sin(k-n)\pi}$$

$$\times\left\{E\begin{pmatrix}\frac{1}{4}-\frac{1}{2}k,\ \frac{3}{4}-\frac{1}{2}k : e^{\pm i\pi}4z^2\\ \frac{1}{2},\ 1-\frac{1}{2}k-\frac{1}{2}n,\ 1-\frac{1}{2}k+\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}k-\frac{1}{2}n,\ \frac{1}{2}-\frac{1}{2}k+\frac{1}{2}n\end{pmatrix}\right.$$

$$\left.+\frac{1}{2z}E\begin{pmatrix}\frac{3}{4}-\frac{1}{2}k,\ \frac{5}{4}-\frac{1}{2}k : e^{\pm i\pi}4z^2\\ \frac{3}{2},\ \frac{3}{2}-\frac{1}{2}k-\frac{1}{2}n,\ \frac{3}{2}-\frac{1}{2}k+\frac{1}{2}n,\ 1-\frac{1}{2}k-\frac{1}{2}n,\ 1-\frac{1}{2}k+\frac{1}{2}n\end{pmatrix}\right\}$$

$$+\sum_{n,-n}\frac{\pi^{5/2}\,2^{-3/2}(2z)^{-k-n}}{\sin(k+n)\pi\sin n\pi}\left\{E\begin{pmatrix}\frac{1}{4}+\frac{1}{2}n,\ \frac{3}{4}+\frac{1}{2}n : e^{\pm i\pi}4z^2\\ \frac{1}{2}+\frac{1}{2}k+\frac{1}{2}n,\ 1+\frac{1}{2}k+\frac{1}{2}n,\ \frac{1}{2},\ \frac{1}{2}+n,\ 1+n\end{pmatrix}\right.$$

$$\left.+\frac{1}{2z}E\begin{pmatrix}\frac{3}{4}+\frac{1}{2}n,\ \frac{5}{4}+\frac{1}{2}n : e^{\pm i\pi}4z^2\\ 1+\frac{1}{2}k+\frac{1}{2}n,\ \frac{3}{2}+\frac{1}{2}k+\frac{1}{2}n,\ \frac{3}{2},\ 1+n,\ \frac{3}{2}+n\end{pmatrix}\right\},$$

where $R(z)>0$.

Here apply (8), and formula (6) with $p=q=0$ is obtained; formula (6) can then be derived in the usual way.

In particular, on putting $p=2$, $q=0$, $\alpha_1=\frac{1}{2}+m$, $\alpha_2=\frac{1}{2}-m$, replacing z by $2z$ and k by $k-\frac{1}{2}$, and applying the formula

$$\cos m\pi\, E(\tfrac{1}{2}+m,\ \tfrac{1}{2}-m : : 2z)=\surd(2\pi z)e^z K_m(z), \qquad \text{.........................(9)}$$

it is found that

$$\int_0^\infty e^{-(1-z)\lambda}K_n(\lambda)K_m(z\lambda)\lambda^{k-1}\,d\lambda$$

$$=\frac{\pi 2^{-k}\sin k\pi\, z^{-\frac{1}{2}}}{\cos m\pi \cos(k+n)\pi\cos(k-n)\pi}\,E\left(\begin{matrix}1-k,\ \frac{1}{2}+m,\ \frac{1}{2}-m : e^{\pm i\pi}z\\ \frac{3}{2}-k+n,\ \frac{3}{2}-k-n\end{matrix}\right)$$

$$-\sum_{n,-n}\frac{\pi 2^{-k-1}z^{-k-n}}{\sin n\pi\cos m\pi\cos(k+n)\pi}\,E\left(\begin{matrix}\frac{1}{2}+n,\ k+m+n,\ k-m+n : e^{\pm i\pi}z\\ 1+2n,\ \frac{1}{2}+k+n\end{matrix}\right), \qquad \text{.........(10)}$$

where $R(k\pm m\pm n)>0$.

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