Proceedings of the Edinburgh Mathematical Society (2001) **44**, 661–666 ©

BOOK REVIEWS

CARTER, M. AND VAN BRUNT, B. *The Lebesgue–Stieltjes integral: a practical introduction* (Undergraduate Texts in Mathematics, Springer, 2000), ix+228 pp., 0 387 95012 5 (hardcover), £30.50.

I was attracted to this book when I read the following in the publisher's publicity leaflet: '... those whose interests lie more in the direction of applied mathematics will in all probability find themselves needing to use the Lebesgue or Lebesgue–Stieltjes integral without having the necessary theoretical background. It is to such readers that this book is addressed. The authors aim to introduce the Lebesgue–Stieltjes integral on the real line in a natural way as an extension of the Riemann integral. They have tried to make the treatment as practical as possible. The evaluation of Lebesgue–Stieltjes integrals is discussed in detail, as are the key theorems of integral calculus as well as the standard convergence theorems.' Armed with an easily accessible version of the Lebesgue–Stieltjes integral, statisticians could unify the treatment of discrete, continuous and mixed distributions and physicists would have a mathematically sound representation of the Dirac δ-function—in general, we could deal simultaneously with certain discrete and continuous phenomena which are traditionally regarded as being quite distinct. I do not think we have to be convinced of the need for a book of the type which the authors have tried to provide.

They certainly touch upon an impressive range of topics for such a small volume. Chapter 1 reviews some ideas of the real numbers, including countable and uncountable sets and sups and infs. Sequences and series, limits, monotonicity, step functions, bounded variation, and absolute continuity are discussed in Chapter 2. The Riemann integral is defined in Chapter 3 along with its extension by means of improper integrals. Chapter 4 introduces the Lebesgue–Stieltjes integral: the measure of an interval with respect to an increasing function α is defined and extended to the union of a finite collection of disjoint intervals; step functions are integrated and an upper integral $L_\alpha(f)$ is defined for non-negative functions by means of sequences of step functions whose sums dominate the function; finally, an integrable function is a function f for which there is a sequence of α-summable step functions (θ_n) such that $L_\alpha(|f-\theta_n|) \to 0$. The basic properties of the integral, including the monotone and dominated convergence theorems, are discussed in Chapter 5, which also deals with null sets and functions and the extension to integrators of bounded variation. Chapter 6, which is entitled *Integral Calculus*, deals with techniques of integration and includes change of variable, integration by parts, the Fundamental Theorem of Calculus, and differentiation under the integral sign. Double and repeated integrals appear with Fubini's theorem in Chapter 7. There is a brief introduction to normed spaces followed by L^p and H^p spaces and a mention of the Sobolev spaces $W^{k,p}$ in Chapter 8. Hilbert spaces form the subject matter of Chapter 9 with orthogonal sets of functions and the Sturm–Liouville problem to the fore. Finally, in Chapter 10, the *Epilogue*, there is some discussion of historical development and other types of integrals. Most results and definitions are supported by well-chosen illustrative examples.

The setting-up of the integral is done in some detail, but thereafter very little is proved, so that the reader ends up with a mass of facts but not a lot of real substance. In earlier sections

https://doi.org/10.1017/S0013091501214436 Published online by Cambridge University Press

too I think we could benefit from a little more detail in some places. For example, on p. 46 we are presented with

$$\int_{\pi}^{\infty} \frac{\sin x}{x}\,dx$$

as an example of a conditionally convergent integral; convergence is proved but we are just told that it can be shown that the integral is not absolutely convergent. I feel that the whole point of this example is lost by not at least giving a sketch of the proof. I found several typographical errors, whose corrections are mostly obvious. A few examples refer to probability distributions, but otherwise I see no genuine applications in the text—it is arguable, of course, that those who want to apply the theory already know their applications and their need is to understand the tools they have to use.

I am sure that the authors have produced an interesting and useful account of their topic. The overview presented in the book and the illustrative examples should be of value both to those whose interests are in applications and those who are concerned with the theory. However, I feel that the book can only be a prelude to further study if a proper understanding of the applications is to be secured.

I. TWEDDLE

NARKIEWICZ, W. *The development of prime number theory from Euclid to Hardy and Littlewood* (Monographs in Mathematics, Springer, 2000), xii+448 pp., 3 540 66289 8 (hardback), £58.50.

This is an extraordinary book both in the breadth of the subject matter addressed and its incredible attention to detail. At times, it has the flavour of a specialized and updated version of Dickson's *History of the Theory of Numbers*: there are twelve proofs of the infinitude of primes within the first ten pages and twenty pages on various prime number formulae. Yet very shortly after, we are browsing through Dirichlet's Theorem on primes in arithmetic progressions (with complete and up to date references and proofs) where one notes the influence of Davenport's *Multiplicative Number Theory*.

After a quick foray into Chebychev's Theorem and its applications, we are presented with a chapter on the Riemann zeta-function and Dirichlet series. Here there is a wealth of historical insight in the form of quotations and letters as well as, of course, concrete mathematical proofs of the more familiar results as well as one or two off the beaten path, such as Stieltjes's representation of the zeta function as a Laurent series around 1 and an interesting exposition on the work of Cahen on general Dirichlet series. We also have a discussion and analysis of Riemann's original memoir, especially the part relevant to analytic prime number theory. What is particularly striking from this part of the book is the extraordinary statement (backed up with a certain amount of evidence) that 'the only correct and fully proved result dealing with the growth of $\pi(x)$ and obtained in the period between Riemann's memoir (1860) and Hadamard's novel approach to the zeta-function, ..., was a theorem of Phragmen (1891)'!

We are now in Chapter 5, which deals with the new era of analytic prime number theory initiated by Hadamard's first paper on the zeta function (1893) and its far reaching implications on the distribution of primes. Included here are the results of von Mangoldt on the Riemann assertions, both Hadamard's and de la Vallée–Poussin's proofs of the Prime Number Theorem and a short assortment of other proofs of the non-vanishing of the zeta-function and L-functions on the line $s = 1$. There is also a discussion on the size of the error term in the standard asymptotic formulae. It is difficult to overstate the extent of sheer detail of references and the scholarship displayed here and elsewhere throughout the book. It suffices to note the almost 80 pages of references to published articles.

https://doi.org/10.1017/S0013091501214436 Published online by Cambridge University Press