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Cofiniteness of Generalized Local Cohomology Modules for One-Dimensional Ideals

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Abstract. Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R and M and N two finitely generated R-modules. Our main result asserts that if $\dim R/\mathfrak{a} \leq 1$, then all generalized local cohomology modules $H^i_{\mathfrak{a}}(M, N)$ are $\mathfrak{a}$-cofinite.

1 Introduction

Let R be a commutative Noetherian ring with identity, $\mathfrak{a}$ an ideal of R and M and N two finitely generated R-modules. In 1969, Grothendieck conjectured that the R-module $\operatorname{Hom}_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(N))$ is finitely generated for all i. One year later, Hartshorne [Ha] provided a counter-example to this conjecture. He defined an R-module L to be $\mathfrak{a}$-cofinite if $\operatorname{Supp}_R L \subseteq \mathrm{V}(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, L)$ is finitely generated for all i and he asked when $H^i_{\mathfrak{a}}(N)$ are $\mathfrak{a}$-cofinite for all i. Concerning this question, the following are the most significant results.

(i) If $\mathfrak{a}$ is principal, then $H^i_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-cofinite for all i [K].
(ii) If R is local and $\dim R/\mathfrak{a} = 1$, then $H^i_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-cofinite for all i [DM, Yo].
(iii) If N has finite dimension d, then $H^d_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-cofinite [Me2].
(iv) If $\dim N \leq 2$, then $H^i_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-cofinite for all i [Me2, Proposition 5.1], [MV, Proposition 2.5].).

Yassemi [Ya] asked whether the assertions (i) and (ii) hold for the generalized local cohomology modules $H^i_{\mathfrak{a}}(M, N)$'s. Remember that the i-th generalized local cohomology module of M and N with respect to $\mathfrak{a}$ is defined by

$$H^i_{\mathfrak{a}}(M, N) := \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N);$$

see [He]. Clearly, this notion is a generalization of the usual notion of local cohomology module $H^i_{\mathfrak{a}}(N)$, which corresponds to the case $M = R$.

In this paper, we deal with analogues of the assertions (i)–(iv) in the context of generalized local cohomology modules. Asadollahi and Schenzel, in an unpublished work, proved that if $\mathfrak{a}$ is principal, then $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all i. This provides a complete answer to Yassemi's question in case (i). Under some extra assumptions, the first author of the present paper and Sazeedeh [DS] extended assertion (ii) to

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generalized local cohomology modules. Namely, they showed that if R is local with the unique maximal ideal $\mathfrak{m}$ and it is complete in its $\mathfrak{m}$-adic topology, $\mathfrak{a}$ prime with $\dim R/\mathfrak{a} = 1$ and M has finite projective dimension, then $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all i. Recently, over regular complete local rings, assertion (ii) has been extended to the generalized local cohomology modules $H^i_{\mathfrak{a}}(M, R)$'s, [AD, Lemma 3.5]. See [HV] for more results regarding the extension of (ii) to generalized local cohomology modules. More recently, in a slightly different direction, Bahmanpour and Naghipour removed the assumption on R to be local in (ii), [BN, Corollary 2.7].

In the present paper, we give a complete answer to Yassemi's question in case (ii), even without assuming that R is local. Namely, we prove that if $\dim R/\mathfrak{a} \leq 1$, then $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all i. We do this by adopting the method of the proof of [BN, Theorem 2.6]. For an $\mathfrak{a}$-cofinite R-module L, it easily follows that $\operatorname{Ass}_R L$ is finite and also all the Bass numbers $\mu^i(\mathfrak{p}, L)$ are finite. So, as an immediate application, we generalize the main result of [KK], see Corollary 2.9(i) below. Next we try to extend the assertions (iii) and (iv) to generalized local cohomology modules. Assume that $d := \dim(M \otimes_R N) + \sup\{i \in \mathbb{N}_0 : \operatorname{Ext}^i_R(M, N) \neq 0\}$ is finite. Then we show that $H^d_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite. In particular, in the case $d \leq 2$, this implies that $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all i.

2 The Results

To prove our main result, we need the following lemmas. The statement of the first lemma involves the notion of attached prime ideals. For the convenience of the reader, we review this notion briefly below. Let $M = S_1 + \cdots + S_n$ be a minimal secondary representation of a representable R-module M. Then $\mathfrak{p}_i := \operatorname{Rad}(\operatorname{Ann}_R S_i)$ is a prime ideal for all $i = 1, \dots, n$ and it is known that the set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ is independent of the chosen minimal secondary representation of M. This set is denoted by $\operatorname{Att}_R M$ and each of its elements is said to be an attached prime ideal of M. It is known that a representable R-module M is zero if and only if $\operatorname{Att}_R M = \varnothing$ and that if $0 \to N \to M \to L \to 0$ is an exact sequence of representable R-modules and R-homomorphisms, then $\operatorname{Att}_R L \subseteq \operatorname{Att}_R M \subseteq \operatorname{Att}_R N \cup \operatorname{Att}_R L$. Also, it is known that any Artinian R-module is representable. Let $f\colon R \to T$ be a ring homomorphism and L a T-module. As for any T-module K, we have $\operatorname{Ann}_R K = f^{-1}(\operatorname{Ann}_T K)$; it follows that if L is representable as a T-module, then it is also representable as an R-module and $\operatorname{Att}_R L = \{f^{-1}(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Att}_T L\}$. For the basic theory concerning attached prime ideals, we refer the reader to [Ma, §6, Appendix].

In what follows, we denote the set of maximal ideals of R by $\operatorname{Max} R$. The following slightly extends [BN, Lemma 2.4].

Lemma 2.1 *Let A be an Artinian R-module. Then A has finite length if and only if $\operatorname{Att}_R A$ consists of maximal ideals. In particular, if for a finitely generated R-module N, the set $\operatorname{Supp}_R N \cap \operatorname{Att}_R A$ consists of maximal ideals, then $A \otimes_R N$ has finite length.*

Proof Assume that A has finite length. This, in particular, implies that $\operatorname{V}(\operatorname{Ann}_R A) = \operatorname{Supp}_R A \subseteq \operatorname{Max} R$. It is clear from the definition that each element of $\operatorname{Att}_R A$ contains

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$\operatorname{Ann}_R A$. Thus $\operatorname{Att}_R A \subseteq \operatorname{Max} R$. Conversely, assume that

$$\operatorname{Att}_R A = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\} \subseteq \operatorname{Max} R.$$

Then there exists $t > 0$ such that $(\mathfrak{m}_1 \cdots \mathfrak{m}_n)^t A = 0$. So A has finite length.

Let N be a finitely generated R-module. Then one has

$$\operatorname{Att}_R(A \underset{R}{\otimes} N) = \operatorname{Supp}_R N \cap \operatorname{Att}_R A,$$

see [Zo] or [D, Theorem 2.2]. So, the last assertion of the lemma follows immediately by its first assertion. ■

Lemma 2.2 *Let $\mathfrak{a}$ be an ideal of R and M and N two finitely generated R-modules. Assume that $\dim R/\mathfrak{a} = 0$. Then $H^i_{\mathfrak{a}}(M, N)$ is Artinian for all i, and so*

$$\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M, N))$$

has finite length for all i and j.

Proof By [Za, Theorem 2.2], for each integer i, the generalized local cohomology module $H^i_{\mathfrak{a}}(M, N)$ is Artinian. Since $\dim R/\mathfrak{a} = 0$, it becomes clear that $\operatorname{V}(\mathfrak{a})$ consists of finitely many maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$, say. Thus for any two integers i and j, $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M, N))$ is an Artinian R-module which is annihilated by some power of the ideal $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n$, and so it has finite length. ■

For reference we summarize the following easy facts in a lemma.

Lemma 2.3 *Let $\mathfrak{a}$ be an ideal of R.*

(i) *Let $M \to N \to P \to Q$ be an exact sequence of R-modules and R-homomorphisms. Assume that M and Q are finitely generated and N and P are supported in $\operatorname{V}(\mathfrak{a})$. Then N is $\mathfrak{a}$-cofinite if and only if P is $\mathfrak{a}$-cofinite.*
(ii) *Let Y be an R-module such that $\operatorname{Hom}_R(R/\mathfrak{a}, Y)$ is finitely generated. Then for any finitely generated submodule Z of Y, the R-module $\operatorname{Hom}_R(R/\mathfrak{a}, Y/Z)$ is finitely generated.*

Proof The proof is easy and we leave it to the reader. ■

In the next result, we are concerned with minimax modules. For its proof, we apply part of the argument given in the proof of [BN, Theorem 2.6]. Recall that an R-module M is said to be minimax if there exists a finitely generated submodule N of M such that M/N is Artinian [Zo].

Lemma 2.4 *Let $\mathfrak{a}$ be an ideal of R with $\dim R/\mathfrak{a} = 1$. Let U be an $\mathfrak{a}$-torsion R-module such that $\operatorname{Hom}_R(R/\mathfrak{a}, U)$ is finitely generated. Assume that for any one-dimensional associated prime ideal $\mathfrak{p}$ of $\mathfrak{a}$, the $R_{\mathfrak{p}}$-module $U_{\mathfrak{p}}$ has finite length. Then U is minimax and $\mathfrak{a}$-cofinite.*

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Proof Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ be the set of all one-dimensional associated prime ideals of $\mathfrak{a}$. Let $1 \leq i \leq n$. Since $U_{\mathfrak{p}_i}$ has finite length, there exists a finitely generated submodule U_i of U such that $U_{\mathfrak{p}_i} = (U_i)_{\mathfrak{p}_i}$. Set $V := U_1 + U_2 + \cdots + U_n$. Then V is a finitely generated submodule of U and

$$\operatorname{Supp}_R(U/V) \subseteq \mathrm{V}(\mathfrak{a}) - \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} \subseteq \operatorname{Max} R.$$

Now Lemma 2.3(ii) yields that $\operatorname{Hom}_R(R/\mathfrak{a}, U/V)$ is finitely generated. It is also Artinian, because it is supported only at maximal ideals. Hence, by [Me1, Theorem 1.3], the R-module U/V is Artinian, and so U is minimax. Next, since $\operatorname{Hom}_R(R/\mathfrak{a}, U)$ is finitely generated, [Me2, Proposition 4.3] implies that U is $\mathfrak{a}$-cofinite. ■

Now it is time to prove our main result.

Theorem 2.5 *Let $\mathfrak{a}$ be an ideal of R with $\dim R/\mathfrak{a} \leq 1$ and M and N two finitely generated R-modules. Then $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all i.*

Proof In view of Lemma 2.2, it remains to consider the case $\dim R/\mathfrak{a} = 1$. Hence, in what follows, we assume that $\dim R/\mathfrak{a} = 1$.

The claim will be proved if for any non-negative integer t, we show that $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all $i < t$ and $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M, N))$ is finitely generated. To do this, we apply induction on t. The claim clearly holds for $t = 0$. Now assume that $t > 0$ and that the claim holds for $t - 1$. It remains to prove that $H^{t-1}_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite and $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M, N))$ is finitely generated. [DH, Corollary 2.8 i)] yields that $H^i_{\mathfrak{a}}(M, \Gamma_{\mathfrak{a}}(N)) \cong \operatorname{Ext}^i_R(M, \Gamma_{\mathfrak{a}}(N))$ for all i. Hence, from the short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(N) \longrightarrow N \longrightarrow N/\Gamma_{\mathfrak{a}}(N) \longrightarrow 0,$$

one can deduce the exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i_R(M, \Gamma_{\mathfrak{a}}(N)) \longrightarrow H^i_{\mathfrak{a}}(M, N) \longrightarrow H^i_{\mathfrak{a}}(M, N/\Gamma_{\mathfrak{a}}(N)) \longrightarrow \operatorname{Ext}^{i+1}_R(M, \Gamma_{\mathfrak{a}}(N)) \longrightarrow \cdots,$$

so by Lemma 2.3(i), we may assume that N is $\mathfrak{a}$-torsion free. Let $T := \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ be the set of all one-dimensional ideals in $\operatorname{Ass}_R(R/\mathfrak{a})$. By Lemma 2.2,

$$H^i_{\mathfrak{a}R_{\mathfrak{p}_k}}(M_{\mathfrak{p}_k}, N_{\mathfrak{p}_k}) \cong H^i_{\mathfrak{a}}(M, N)_{\mathfrak{p}_k}$$

is an Artinian $R_{\mathfrak{p}_k}$-module for all i and all $k = 1, \dots, n$. Set

$$\mathfrak{U} = \bigcup_{i=0}^{t-1} \bigcup_{k=1}^{n} \operatorname{Att}_R(H^i_{\mathfrak{a}R_{\mathfrak{p}_k}}(M_{\mathfrak{p}_k}, N_{\mathfrak{p}_k})),$$

and take

$$x \in \mathfrak{a} - \Big(\bigcup_{\mathfrak{p} \in (\mathfrak{U} - \mathrm{V}(\mathfrak{a})) \cup \operatorname{Ass}_R N} \mathfrak{p} \Big).$$

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Then for any $j \geq 0$, the short exact sequence $0 \to N \xrightarrow{x} N \to N/xN \to 0$ induces the exact sequence

$$(*) \quad 0 \longrightarrow H^j_{\mathfrak{a}}(M,N)/xH^j_{\mathfrak{a}}(M,N) \xrightarrow{\rho_j} H^j_{\mathfrak{a}}(M,N/xN) \longrightarrow (0 :_{H^{j+1}_{\mathfrak{a}}(M,N)} x) \longrightarrow 0.$$

Now the induction hypothesis yields that $H^j_{\mathfrak{a}}(M,N/xN)$ is $\mathfrak{a}$-cofinite for all $j < t-1$ and $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{\mathfrak{a}}(M,N/xN))$ is finitely generated. Set

$$U_j := H^j_{\mathfrak{a}}(M,N)/xH^j_{\mathfrak{a}}(M,N) \quad \text{and} \quad V_j := H^j_{\mathfrak{a}}(M,N/xN)$$

for $j \leq t-1$. For any $\mathfrak{p} \in T$, from the choice of x, it follows that

$$\mathrm{V}(xR_{\mathfrak{p}}) \cap \operatorname{Att}_{R_{\mathfrak{p}}}(H^j_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})) \subseteq \{\mathfrak{p}R_{\mathfrak{p}}\},$$

and so Lemma 2.1 implies that $(U_j)_{\mathfrak{p}} = H^j_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})/xH^j_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ has finite length for all $j \leq t-1$. On the other hand, it follows from $(*)$ that the R-modules $\operatorname{Hom}_R(R/\mathfrak{a}, U_{t-2})$ and $\operatorname{Hom}_R(R/\mathfrak{a}, U_{t-1})$ are finitely generated. Therefore, Lemma 2.4 implies that U_{t-2} and U_{t-1} are minimax and $\mathfrak{a}$-cofinite. Now by using the exact sequence

$$0 \longrightarrow U_{t-2} \xrightarrow{\rho_{t-2}} V_{t-2} \longrightarrow (0 :_{H^{t-1}_{\mathfrak{a}}(M,N)} x) \longrightarrow 0,$$

we deduce that $(0 :_{H^{t-1}_{\mathfrak{a}}(M,N)} x)$ is $\mathfrak{a}$-cofinite. Next, since both R-modules

$$(0 :_{H^{t-1}_{\mathfrak{a}}(M,N)} x) \quad \text{and} \quad H^{t-1}_{\mathfrak{a}}(M,N)/xH^{t-1}_{\mathfrak{a}}(M,N)$$

are $\mathfrak{a}$-cofinite, [Me2, Corollary 3.4] implies that $H^{t-1}_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite. Finally, the long Ext sequence induced by the exact sequence

$$0 \longrightarrow U_{t-1} \xrightarrow{\rho_{t-1}} V_{t-1} \longrightarrow (0 :_{H^t_{\mathfrak{a}}(M,N)} x) \longrightarrow 0$$

implies that $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M,N)) \cong \operatorname{Hom}_R(R/\mathfrak{a}, 0 :_{H^t_{\mathfrak{a}}(M,N)} x)$ is finitely generated. Hence the proof is complete by induction. ■

In order to present our next result, we need to include the following definition from [DH].

Definition 2.6 Let M and N be two finitely generated R-modules. The *Gorenstein projective dimension of M relative to N* is defined by

$$\operatorname{Gpd}_N M := \sup\{i \in \mathbb{N}_0 : \operatorname{Ext}^i_R(M,N) \neq 0\}.$$

Let $\mathfrak{a}$ be an ideal of R and N a finitely generated R-module with finite dimension d. It is known that $H^d_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-cofinite and Artinian [Me2, Proposition 5.1]. In the spirit of our results in [DH, §2], part (i) of our next result might be considered as a natural extension of this fact to generalized local cohomology modules. It is worth mentioning that it also improves [HV, Proposition 3.1] considerably. Also, it is known and easy to see that if $d \leq 2$, then $H^i_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-cofinite for all i. Part (ii) of the following result is a generalization of this fact to generalized local cohomology modules. (Mafi and Saremi [MS, Theorem 2.9] have established another generalization of this fact. Note that their generalization and ours cannot be deduced from each other.)

Theorem 2.7 *Let $\mathfrak{a}$ be an ideal of R and M and N two finitely generated R-modules. Let $d := \dim(M \otimes_R N) + \operatorname{Gpd}_N M$.*

(i) *If d is finite, then $H^d_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite and Artinian.*
(ii) *If $d \leq 2$, then $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all i.*

Proof (i) Let $p := \operatorname{Gpd}_N M$. From Grothendieck's Vanishing Theorem (see [BS]), one has $\dim(M \otimes_R N) \geq \operatorname{cd}_{\mathfrak{a}}(M \otimes_R N)$. If this inequality is strict, then $H^d_{\mathfrak{a}}(M, N) = 0$ by [DH, Theorem 2.5]. So we may assume that $\dim(M \otimes_R N) = \operatorname{cd}_{\mathfrak{a}}(M \otimes_R N)$, and then [DH, Theorem 2.5] implies the isomorphism $H^d_{\mathfrak{a}}(M, N) \cong H^{d-p}_{\mathfrak{a}}(\operatorname{Ext}^p_R(M, N))$. Since $\operatorname{Ext}^p_R(M, N)$ is supported in $\operatorname{Supp}_R M \cap \operatorname{Supp}_R N = \operatorname{Supp}_R(M \otimes_R N)$, it follows that $\dim_R(\operatorname{Ext}^p_R(M, N)) \leq d - p$. In the case $\dim_R(\operatorname{Ext}^p_R(M, N)) = d - p$, the claim follows by [Me2, Proposition 5.1] and otherwise it follows by Grothendieck's Vanishing Theorem.

(ii) The module $H^0_{\mathfrak{a}}(M, N)$ is finitely generated and it is supported in $V(\mathfrak{a})$. Thus, it is clearly $\mathfrak{a}$-cofinite. Since $\operatorname{Hom}_R(M, N)$ is supported in $\operatorname{Supp}_R(M \otimes_R N)$, our assumption yields that $\dim_R(\operatorname{Hom}_R(M, N)) \leq 2$. Hence, by [Me2, Proposition 5.1] and [MV, Proposition 2.5], it follows that $H^i_{\mathfrak{a}}(\operatorname{Hom}_R(M, N))$ is $\mathfrak{a}$-cofinite for all i. On the other hand, by [MS, Lemma 2.7], $H^1_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite if and only if $H^1_{\mathfrak{a}}(\operatorname{Hom}_R(M, N))$ is $\mathfrak{a}$-cofinite. Hence $H^1_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite. Next, [DH, Theorem 2.5] implies that $H^i_{\mathfrak{a}}(M, N) = 0$ for all $i > \dim(M \otimes_R N) + \operatorname{Gpd}_N M$. Therefore, the assertion follows by (i). ■

Part (i) of the following corollary improves the main result of [KK].

Corollary 2.8 *Let $\mathfrak{a}$ be an ideal of R and M and N two finitely generated R-modules. Assume that one of the following conditions is satisfied.*

(i) $\dim R/\mathfrak{a} \leq 1$.
(ii) $\dim(M \otimes_R N) + \operatorname{Gpd}_N M \leq 2$.

Then $\operatorname{Ass}_R(H^i_{\mathfrak{a}}(M, N))$ is finite for all i. Also, the Bass numbers $\mu^j(\mathfrak{p}, H^i_{\mathfrak{a}}(M, N))$ are finite for all i and j and all prime ideals $\mathfrak{p}$ of R.

Proof Theorem 2.5 and Theorem 2.7(ii) yield that $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all i. It is well known that modules cofinite with respect to an ideal have finitely many associated prime ideals and their Bass numbers are finite. ■

References

[AD] M. Asgharzadeh and K. Divaani-Aazar, *Finiteness properties of formal local cohomology modules and Cohen-Macaulayness.* Comm. Algebra, to appear. arXiv:0807.5042

[BN] K. Bahmanpour and R. Naghipour, *Cofiniteness of local cohomology modules for ideals of small dimension.* J. Algebra **321**(2009), no. 7, 1997–2011. doi:10.1016/j.jalgebra.2008.12.020

[BS] M. Brodmann and R. Y. Sharp, *Local cohomology: An algebraic introduction with geometric applications.* Cambridge Studies in Advanced Mathematics, 60, Cambridge University Press, Cambridge, 1998.

[DM] D. Delfino and T. Marley, *Cofinite modules and local cohomology.* J. Pure. Appl. Algebra, **121**(1997), no. 1, 45–52. doi:10.1016/S0022-4049(96)00044-8

[D] K. Divaani-Aazar, *On associated and attached prime ideals of certain modules*. Colloq. Math. **89**(2001), no. 1, 147–157. doi:10.4064/cm89-1-11

[DH] K. Divaani-Aazar and A. Hajikarimi, *Generalized local cohomology modules and homological Gorenstein dimensions.* Comm. Algebra, to appear. arXiv:0803.0107.

[DS] K. Divaani-Aazar and R. Sazeedeh, *Cofiniteness of generalized local cohomology modules*. Colloq. Math. **99**(2004), no. 2, 283–290. doi:10.4064/cm99-2-12

[Ha] R. Hartshorne, *Affine duality and cofiniteness.* Invent. Math. **9**(1969/1970), 145–164. doi:10.1007/BF01404554

[HV] S. H. Hassanzadeh and A. Vahidi, *On vanishing and cofiniteness of generalized local cohomology modules.* Comm. Algebra **37**(2009), no. 7, 2290–2299. doi:10.1080/00927870802622718

[He] J. Herzog, *Komplex Auflösungen und Dualität in der lokalen Algebra*, Habilitationsschrift, Universität Regensburg, 1970.

[KK] S. Kawakami and K.-I. Kawasaki, *On the finiteness of Bass numbers of generalized local cohomology modules.* Toyama Math. J. **29**(2006), 59–64.

[K] K.-I. Kawasaki, *Cofiniteness of local cohomology modules for principle ideals*. Bull. London. Math. Soc. **30**(1998), no. 3, 241–246. doi:10.1112/S0024609397004347

[MS] A. Mafi and H. Saremi, *Cofinite modules and generalized local cohomology.* Houston J. Math. **35**(2009), no. 4, 1013–1018.

[MV] T. Marley and J. C. Vassilev, *Cofiniteness and associated primes of local cohomology modules.* J. Algebra **256**(2002), no. 1, 180–193. doi:10.1016/S0021-8693(02)00151-5

[Ma] H. Matsumura, *Commutative Ring Theory.* Second edition. Cambridge Studies in Advanced Mathematics 8, Cambridge University Press, Cambridge, 1989.

[Me1] L. Melkersson, *On asymptotic stability for sets of prime ideals connected with the powers of an ideal.* Math. Proc. Cambridge Philos. Soc. **107**(1990), no. 2, 267–271. doi:10.1017/S0305004100068535

[Me2] ———, *Modules cofinite with respect to an ideal.* J. Algebra **285**(2005), no. 2, 649–668. doi:10.1016/j.jalgebra.2004.08.037

[Ya] S. Yassemi, *Cofinite modules.* Comm. Algebra **29**(2001), no. 6, 2333–2340. doi:10.1081/AGB-100002392

[Yo] K.-I. Yoshida, *Cofiniteness of local cohomology modules for ideals of dimension one.* Nagoya Math. J. **147**(1997), 179–191.

[Za] N. Zamani, *On graded generalized local cohomology.* Arch. Math. (Basel) **86**(2006), no. 4, 321–330.

[Zo] H. Zöschinger, *Minimax-moduln.* J. Algebra,**102**(1986), no. 1, 1–32. doi:10.1016/0021-8693(86)90125-0

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