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ON THE AFFINE DIAMETER OF CLOSED CONVEX HYPERSURFACES

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In this paper we prove that the affine diameter of any closed uniformly convex hypersurface in Euclidean space enclosing finite volume is bounded from above.

1. INTRODUCTION

In this paper we establish an upper bound for the affine diameter of a closed convex hypersurface in Euclidean space. Let Γ be a closed, smooth, uniformly convex hypersurface in R^{n+1}, $n \geqslant 1$, and Σ the domain in R^{n+1} enveloped by Γ. Here we call a hypersurface Γ in R^{n+1} uniformly convex if the principal curvatures at each point of Γ with respect to its inner normal vector are positive. Suppose the volume of Σ is equal to 1. If g is a Riemannian metric defined on Γ, then the diameter of Γ is defined as

$$\operatorname{diam}(\Gamma) := \sup_{p,q\in\Gamma} d(p,q),$$

where $d(\cdot,\cdot)$ denotes the distance function of the metric g, that is

$$d(p,q) = \inf\{L(\gamma) \mid \gamma \text{ is any curve on } \Gamma \text{ which connects } p \text{ and } q\}.$$

Here $L(\gamma)$ is the arc-length of γ with respect to the metric g. In affine geometry we consider the affine metric (or Berwald-Blaschke metric) on Γ, given by

$$g = K^{-1/(n+2)}II,$$

where K is the Gauss curvature and II is the second fundamental form of Γ (see [**1**, **5**, **6**]). The hypersurface Γ becomes a Riemannian manifold under this metric. In this paper we prove the following theorem.

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THEOREM 1. *Let Γ be a closed, smooth, uniformly convex hypersurface in R^{n+1}, $n \geqslant 1$, and let Σ be the domain in R^{n+1} enveloped by Γ. Suppose* $\mathrm{Vol}(\Sigma) = 1$. *Then there exists a constant $C(n)$ depending only on n such that the affine diameter of Γ,* $\mathrm{diam}(\Gamma)$, *satisfies* $\mathrm{diam}(\Gamma) \leqslant C(n)$.

In Euclidean geometry it is well known that the diameter of Γ, $\mathrm{diam}(\Gamma)$, under the Euclidean metric is bounded from below by a constant $C(n)$ which depends only on n, and $\mathrm{diam}(\Gamma) = C(n)$ holds if and only if Γ is an Euclidean sphere S^n in R^{n+1}.

REMARKS.

(i) We have not got the best upper bound of the affine diameter in this paper. It is reasonable to believe that the best upper bound $C(n)$ exists and $\mathrm{diam}(\Gamma) = C(n)$ holds if and only if Γ is an ellipsoid. In fact, if $n = 1$, the affine isoperimetric inequality (see [**1, 5**]) implies $\mathrm{diam}(\Gamma) \leqslant \pi^{2/3}$ and the equality holds if and only if Γ is an ellipsoid.

(ii) If Γ is a closed locally uniformly convex hypersurface in Euclidean space, it must be uniformly convex [**4, 7, 9**]. Thus the same statement is true in this case.

2. PROOF OF THEOREM 1

At first we make use of the fact ([**3**]) that for any bounded convex domain $\Sigma \subset R^{n+1}$, there exists a unique ellipsoid E, called the minimum ellipsoid of Σ, which attains the minimum volume among all ellipsoids concentric with and containing Σ, and a positive constant α_n, depending only on n, such that

$$\alpha_n E \subset \Sigma \subset E$$

where $\alpha_n E$ is the α_n dilation of E with respect to its centre. Let T be an equi-affine transformation that normalises Σ (see [**2**]), $T(E) = B(0, r_n)$, where r_n is a constant depending only on n and $\mathrm{Vol}(E) = \mathrm{Vol}\big(B(0, r_n)\big)$. Then

$$B(0, \alpha_n r_n) \subset T(\Sigma) \subset B(0, r_n).$$

Here $B(x, t)$ denotes the Euclidean ball with centre x and radius t. Since the affine distance is affine invariant, we only need to estimate the affine diameter of $T(\Gamma)$.

Now since $T(\Gamma)$ is closed and uniformly convex, the Gauss map $G : T(\Gamma) \to S^n$ is a diffeomorphism from $T(\Gamma)$ onto S^n. We divide S^n into several pieces

$$U_i^+ = \big\{(x_1, \dots, x_i, \dots, x_{n+1}) \in R^{n+1} \mid x_i > (1/4n)\big\} \cap S^n,$$
$$U_i^- = \big\{(x_1, \dots, x_i, \dots, x_{n+1}) \in R^{n+1} \mid x_i < -(1/4n)\big\} \cap S^n,$$

$i = 1, 2, \ldots, n+1$. Then $\{U_1^{\pm}, \ldots, U_{n+1}^{\pm}\}$ is an open covering of S^n which is an Euclidean sphere of radius 1 and $\{G^{-1}(U_i^{\pm}) \mid i = 1, \ldots, n+1\}$ is an open covering of $T(\Gamma)$. Each $G^{-1}(U_i^+)$ or $G^{-1}(U_i^-)$ can be expressed as a graph

$$x_i = u_i^+(x_1, \ldots, \widehat{x}_i, \ldots, x_{n+1}) \text{ on } \Omega_i^+ \subset B^n(r_n) \subset R^n$$

or

$$x_i = u_i^-(x_1, \ldots, \widehat{x}_i, \ldots, x_{n+1}) \text{ on } \Omega_i^- \subset B^n(r_n) \subset R^n,$$

and we set $\Omega_i^{+*} = Du_i^+(\Omega_i^+)$ and $\Omega_i^{-*} = Du_i^-(\Omega_i^-)$. We only need to prove that on each open set $G^{-1}(U_i^{\pm})$, the affine distance of any two points p, $q \in G^{-1}(U_i^{\pm})$ is bounded. Then by the triangle inequality we can prove the affine distance of any two points on $T(\Gamma)$ is bounded.

Now we denote $U := U_{n+1}^-$, $\Omega := \Omega_{n+1}^-$ and consider $G^{-1}(U) \subset T(\Gamma)$ as a graph defined on Ω, given by

$$x_{n+1} = u(x_1, \ldots, x_n), \quad x = (x_1, \ldots, x_n) \in \Omega. \tag{1}$$

The affine metric on the graph can be written as $\rho u_{x_i x_j} dx_i dx_j$, where

$$\rho = \left(\det D^2 u\right)^{-(1/(n+2))}.$$

Then the affine arc-length of a curve γ on the graph is given by

$$L(\gamma) = \int_l (\rho u_{\xi\xi})^{1/2}\, ds \tag{2}$$

where l is the projection of γ on $\{x_{n+1} = 0\}$, s is the (Euclidean) arc-length parameter on l, $\xi = (\xi_1, \ldots, \xi_n)$ is the unit tangent vector on l and $u_{\xi\xi} = \Sigma \xi_i \xi_j u_{x_i x_j}$.

Let $p \in G^{-1}(U)$ and $G(p) = -e_{n+1} = (0, \ldots, 0, -1)$. For each point $q \in G^{-1}(U)$, there is a unique geodesic γ from p to q such that $G(\gamma)$ is a geodesic line on the south hemisphere. Let l be the projection of γ on $\{x_{n+1} = 0\}$. Then $Du(l)$ is a line segment in Ω^*. Ω^* is the spherical projection on $\{x_{n+1} = -1\}$ of U, which is a ball in R^n.

Let u^* be the Legendre transformation of u, given by

$$u^*(y) = x \cdot y - u(x), \quad y \in \Omega^*, \tag{3}$$

where $x \in \Omega$ is chosen such that $Du(x) = y$ (x is unique in this way). Then u^* is a convex function and u is the Legendre transformation of u^* such that $x = Du^*(y)$ and

$$\det D^2 u(x) \cdot \det D^2 u^*(y) = 1. \tag{4}$$

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By the Legendre transformation, a curve γ on the graph of u corresponds to a curve γ^* of the graph of u^* such that a point $(x, u(x)) \in \gamma$ corresponds to a point $(y, u^*(y)) \in \gamma^*$, where $y = Du(x)$, and vice versa. The projection of γ in Ω, l, then corresponds to the projection of γ^* in Ω^*, l^*, with $l = Du^*(l^*)$. Then the affine arc-length of the curve γ can be expressed as (see [8])

$$L = \int_{l^*} \left(\rho^* u^*_{\eta\eta}\right)^{1/2} ds \tag{5}$$

where s is the arc-length parameter of l^* and $\rho^* = [\det D^2 u^*]^{1/(n+2)}$.

Let $S_\theta = S^{n-1} \cap \{x_1 > \cos\theta\}$ and $S_\theta(r) = rS_\theta$, where $\theta \in (0, \pi/2)$, $r > 0$. For any point $y \in S_\theta(r)$, let $l^* = l^*_y$ be the (open) line segment joining the origin to y and $\mathcal{C}_\theta(r)$ the union of the line segments l^*_y for all $y \in S_\theta(r)$. $\mathcal{C}_\theta$ is a cone with vertex at the origin, radius r, aperture θ and axial direction $e_1 = (1, 0, \dots, 0)$. $\widetilde{\mathcal{C}}_\theta = \mathcal{C}_\theta \cap \{x_1 < \cos\theta\}$. Then for each $y \in S_\theta(r)$, there exists a unique point $\widetilde{y} \in P = \{x \in R^n \mid x_1 = \cos\theta\}$ such that $\widetilde{y}$ is on the line segment l^*_y. More generally, for $z \in R^n$, $r > 0$, $\xi \in S^n$, $\theta \in (0, \pi/2)$, we let $\mathcal{C}_\theta = \mathcal{C}(z, r, \xi)$ denote the congruent cone with vertex at z, radius r, aperture θ and axial direction ξ, and $S_\theta = S_\theta(z, r, \theta) = \overline{\mathcal{C}}_\theta \cap \{|x - z| = r\}$. We also denote $\widetilde{\mathcal{C}}_\theta = \widetilde{\mathcal{C}}_\theta(z, r, \xi) = \mathcal{C}_\theta(z, r, \xi) \cap \{x \mid (x - z - r\cos\theta\xi)\xi < 0\}$, $P_\theta(z, r, \xi) = \{x \in R^n \mid (x - z - r\cos\theta\xi)\xi = 0\}$ and $\widetilde{P}_\theta(z, r, \xi) = P_\theta(z, r, \xi) \cap \mathcal{C}_\theta$.

Lemma. *Suppose $\mathcal{C}_\theta = \mathcal{C}_\theta(z, r, \xi) \subset \Omega^{-*}_{n+1}$. Then there exists a constant $C_0(n)$ depending only on n such that for any fixed $k > 0$, the measure of the set*

$$Q = \left\{\alpha \mid L_{\widetilde{y}} = \int_{l^*_{\widetilde{y}}} \left(\rho^* u^*_{\eta\eta}\right)^{1/2} ds > (C_0(n)k)/(2\theta), \widetilde{y} \in \widetilde{P}_\theta(z, r, \xi), \frac{(\widetilde{y} - z)\xi}{|\widetilde{y} - z|} = \cos\alpha, -\theta < \alpha < \theta\right\}$$

satisfies $|Q| < (2\theta)/k$.

Proof: For any $\widetilde{y} \in \widetilde{P}_\theta(z, r, \xi)$ satisfying $(\widetilde{y} - z)/\xi|\widetilde{y} - z| = \cos\alpha$ and $-\theta < \alpha < \theta$, there exists a unique point $y \in S_\theta$, such that $\widetilde{y} \in l^*_y$ and $l^*_{\widetilde{y}} \subset l^*_y$. Then we have by (5)

$$L_{\widetilde{y}} \leqslant L_y \leqslant \left(\int_0^r \rho^* \, ds\right)^{1/2} \left(\int_0^r u^*_{\eta\eta} \, ds\right)^{1/2}$$

where $\eta = (y - z)/|y - z|$. The second integral is less than $u^*_\eta(y) - u^*_\eta(z) \leqslant C$. Here the constant C depends only on n because $T(\Gamma)$ is located in a bounded domain of R^{n+1}. Since

$$\int_0^r \rho^* \, ds \leqslant C\left(\int_0^r s^{n-1}(\rho^*)^{n+2} \, ds\right)^{1/(n+2)},$$

we have following estimate, using the spherical polar coordinates,

$$\begin{aligned}\int_{\widetilde{P}_\theta} L_{\widetilde{y}} d\widetilde{y} &\leqslant \int_{S_\theta} L_y dy \leqslant C \int_{S_\theta} \left(\int_0^r \rho^*\right)^{1/2} dy \\ &\leqslant C \int_{S_\theta} \left(\int_0^r s^{n-1}(\rho^*)^{n+2}\, ds\right)^{1/2(n+2)} dy \\ &= C \int_{S_\theta} \left(\int_0^r s^{n-1} \det D^2 u^*\, ds\right)^{1/2(n+2)} dy \\ &\leqslant C\left(\int_{S_\theta}\int_0^r s^{n-1} \det D^2 u^*\, ds dy\right)^{1/2(n+2)} \\ &\leqslant C\left(\int_{C_\theta} \det D^2 u^*\right)^{1/2(n+2)} \\ &= C\left(\left|Du^*(C_\theta)\right|\right)^{1/2(n+2)}.\end{aligned}$$

Since $C_\theta \subset \Omega$ and $Du^*(\Omega^*) = \Omega \subset B^n(r_n)$, we have

$$\begin{aligned}\int_{\widetilde{P}_\theta} L_{\widetilde{y}} d\widetilde{y} &\leqslant C\left|Du^*(\Omega^*)\right|^{1/2(n+2)} \\ &= C|\Omega|^{1/2(n+2)} \\ &\leqslant C\left|B^n(r_n)\right|^{1/2(n+2)} =: C_0(n). \qquad (6)\end{aligned}$$

Then for any $k > 0$,

$$|Q| = \left|\left\{\alpha \mid L_{\widetilde{y}} > \frac{C_0(n)k}{2\theta}, \widetilde{y} \in \widetilde{P}_\theta, \frac{(\widetilde{y}-z)\xi}{|\widetilde{y}-z|} = \cos\alpha\right\}\right| < \frac{2\theta}{k}. \qquad (7)$$

This proves the Lemma. ∎

By the above Lemma, we have

COROLLARY. *For any fixed $\theta \in (0, \pi/2)$, there exists a constant $C(n,\theta)$ depending only on n and θ, such that the measure of the set*

$$\widetilde{Q} = \left\{\alpha \mid L_{\widetilde{y}} \leqslant C(n,\theta), \widetilde{y} \in \widetilde{P}_\theta(z,r,\xi), \frac{(\widetilde{y}-z)\xi}{|\widetilde{y}-z|} = \cos\alpha, -\theta < \alpha < \theta\right\}$$

satisfies $|\widetilde{Q}| \geqslant (4/3)\theta$.

PROOF: In fact, we can take $k = 3$ and $C(n,\theta) = (3C_0(n))/\theta$. Then by (7), we have $|\widetilde{Q}| \geqslant 2\theta - |Q| \geqslant (4\theta)/3$. ∎

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Now we take a fixed $\theta \in (0, \pi/2)$. For any two points p and q in $G^{-1}(U) \subset T(\Gamma)$, $p = \big(x_p, u(x_p)\big)$, $q = \big(x_q, u(x_q)\big)$, x_p, $x_q \in \Omega$, let $x_p^* = Du(x_p)$, $x_q^* = Du(x_q) \in \Omega^*$. We denote

$$P(p,q) = \big\{x \in R^n \;\big|\; |x - x_p^*| = |x - x_q^*|\big\}$$

which is a $(n-1)$-plane in R^n. Then we get two cones $\widetilde{C}_\theta\big(x_p^*, r_p, \xi_p\big)$ and $\widetilde{C}_\theta\big(x_q^*, r_q, \xi_q\big)$ in R^n with the same base on plane $P(p,q)$, and $r_p = r_q$, $\xi_p = \overrightarrow{x_p^* x_q^*}/|\overrightarrow{x_p^* x_q^*}| = -\xi_q$. We consider the parts of these two cones in Ω^*, $\widetilde{C}'_\theta(p) = \widetilde{C}_\theta\big(x_p^*, r_p, \xi_p\big) \cap \Omega^*$ and $\widetilde{C}'_\theta(q) = \widetilde{C}_\theta\big(x_q^*, r_q, \xi_q\big) \cap \Omega^*$. We also denote $\widetilde{P}'_\theta = P(p,q) \cap \Omega^*$. Since Ω^* is a ball, the measure of the set

$$Q_1(p) = \{\alpha \mid \quad l_\alpha \text{ is a line segment from } x_p^* \text{ to some point of } \widetilde{P}'_\theta \text{ and the angle between } l_\alpha \text{ and } \xi_p \text{ equals to } \alpha\}$$

satisfies $\theta \leqslant |Q_1(p)| \leqslant 2\theta$. On $\widetilde{C}'_\theta(p)$, by the same argument as (6) and (7), we have

$$\int_{\widetilde{P}'_\theta} L_{\widetilde{y}}\, d\widetilde{y} \leqslant C_0(n)$$

and

$$|Q| = \left|\left\{\alpha \mid L_{\widetilde{y}} > \frac{3C_0(n)}{|Q_1|}, \widetilde{y} \in \widetilde{P}'_\theta, \frac{(\widetilde{y} - x_p^*)\xi_p}{|\widetilde{y} - x_p^*|} = \cos\alpha, -\theta < \alpha < \theta\right\}\right| < \frac{|Q_1|}{3}.$$

The fact $\theta \leqslant |Q_1(p)| \leqslant 2\theta$ implies

$$\begin{aligned}\widetilde{Q} &= \{\alpha \mid L_{\widetilde{y}} \leqslant C(n,\theta), \widetilde{y} \in \widetilde{P}'_\theta, \frac{(\widetilde{y} - x_p^*)\xi_p}{|\widetilde{y} - x_p^*|} = \cos\alpha, -\theta < \alpha < \theta\} \\ &\supset \{\alpha \mid L_{\widetilde{y}} \leqslant \frac{3C_0(n)}{|Q_1|}, \widetilde{y} \in \widetilde{P}'_\theta, \frac{(\widetilde{y} - x_p^*)\xi_p}{|\widetilde{y} - x_p^*|} = \cos\alpha, -\theta < \alpha < \theta\}.\end{aligned}$$

Here we recall $C(n,\theta) = 3C_0(n)/\theta$. Therefore

$$(8) \qquad |\widetilde{Q}| \geqslant |Q_1| - \frac{|Q_1|}{3} = \frac{2|Q_1|}{3}.$$

In the same way, we can prove that (8) holds on $\widetilde{C}'_\theta(q)$. Then there exists at least one point $z \in \widetilde{P}'_\theta$ such that $L\big(x_p^*, z\big) \leqslant C(n,\theta)$ and $L\big(x_q^*, z\big) \leqslant C(n,\theta)$, where $L(\cdot,\cdot)$ is the affine distance on $T(\Gamma)$ in the sense of (5). Then by the triangle inequality, we have $d(p,q) \leqslant 2C(n,\theta)$.

The same conclusion holds on any $G^{-1}(U_i^{\pm})(i = 1, \dots, n+1)$ for any two points on $G^{-1}(U_i^{\pm})$. If p and q belong to two neighbouring sets $G^{-1}(U_i)$ and $G^{-1}(U_j)$, and $G^{-1}(U_i) \cap G^{-1}(U_j) \neq \varnothing$, we can pick any point $w \in G^{-1}(U_i) \cap G^{-1}(U_j)$, then $d(p,q) \leqslant d(p,w) + d(w,q) \leqslant 4C(n,\theta)$. In general, for any two points p and q in $T(\Gamma)$, since $\{G^{-1}(U_i^{\pm}), i = 1, \dots, n+1\}$ is a covering of $T(\Gamma)$, there exist finite sets $\{G^{-1}(U_{i_k}), k = 1, \dots, m\}(m \leqslant 2(n+1))$ such that the intersection of any two neighbouring sets $G^{-1}(U_{i_k})$ and $G^{-1}(U_{i_{k+1}})$ is nonempty, and $G^{-1}(U_{i_1}) = G^{-1}(U_i)$, $G^{-1}(U_{i_m}) = G^{-1}(U_j)$. Then $d(p,q) \leqslant 4(n+1)C(n,\theta)$. Now we take $\theta = \pi/4$ and $C(n) = 4(n+1)C(n,\pi/4)$, then Theorem 1 follows immediately from the above discussion. This finishes the proof of Theorem 1. ∎

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