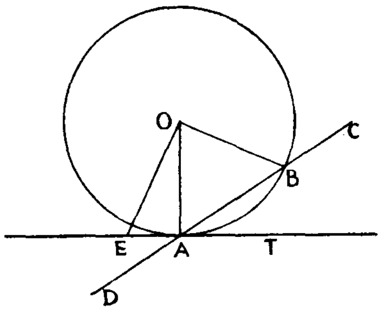
Sixth Meeting, 13th May 1904.

Mr CHARLES TWEEDIE, President, in the Chair.

Note on the Treatment of Tangents in recent Textbooks of Elementary Geometry.

By Professor GEORGE A. GIBSON.

Several textbooks of Elementary Geometry have recently been put on the market, and in nearly all that I have examined (and I have gone carefully through many of them) the treatment of tangents is based on what the writers call the Method of Limits. The usual form given to the proof that the tangent at any point of a circle is at right angles to the radius to the point of contact is somewhat as follows.



The radii OA, OB are equal and therefore

$$\angle OBA = \angle OAB \qquad (1)$$

This equation is true however near B is to A; when B coincides with A the angle AOB is zero, and the angle OAB is a right angle. But when B coincides with A the secant ABC is the tangent, and therefore the tangent at A is at right angles to OA.

Sometimes, instead of the angles OAB, OBA the supplements OAD, OBC are taken and instead of (1) we have

$$\angle OAD = \angle OBC \quad - \quad - \quad - \quad - \quad (2)$$

but this makes no real difference in the proof; neither C nor D can be found except through the *two* points A and B.

It is more important, however, to note that some writers explicitly state an assumption which all who adopt this mode of

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proof actually make, either explicitly or implicitly: namely, after the words "this equation is true however near B is to A" they add "therefore it is true when B coincides with A." This assumption is of course identical with that implied in the venerable dictum that "what is true up to the limit is true in the limit."

Now, it is surely not hypercritical to call in question the logic of this proof. So far as the reasoning is concerned, in what respect does it differ from the following? On the line through A at right angles to OA take any point E distinct from A. The angle OAE is greater than the angle OEA and therefore

$$OE > OA \quad - \quad - \quad - \quad - \quad - \quad - \quad (3)$$

This inequality is true however near E is to A and therefore it is true when E coincides with A; that is, OA is greater than itself.

As a mere matter of reasoning, the conclusion is as sound in the one case as in the other, on the assumption of the dictum quoted.

Of course the whole difficulty lies in the failure to grasp, or at least to state and apply, the proper definition of a limit. It is rather disheartening to find the absurdities, so clearly pointed out by Berkeley nearly two hundred years ago, still flourishing and apparently endowed with a new lease of life. It is all the more regrettable to find these in English textbooks, when one considers that we owe to one Englishman the explicit statement, and to another a thoroughly satisfactory exposition, of the Method of Limits. (See the notice of the *Analyst* Controversy, *Proc. Edin. Math. Soc.*, Vol. XVII., pages 9–32).

The radical error of all such proofs as that sketched above lies, it seems to me, in a wrong conception of a limit; a limit seems to be considered *as a particular case.* Thus the straight line OA (or the two *coincident* straight lines OA, OB which still make but one line) is considered to be a particular case of the triangle OAB. But, however *convenient* it may be to use the language of coincident points and lines, there is absolutely no cogency in the reasoning that is often based on the conception of coincident points and lines. Equation (1) above is established on the express understanding that A and B are distinct points and cannot be established unless they are distinct; equation (1) (like the inequality (3)) is true only so long as B is distinct from A (or E distinct from A). When B coincides

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with A, OAB is no longer a triangle. It is surely not going to be accepted as an axiom of the modern geometry that, when a theorem has been established on the express understanding that certain conditions hold, we are at liberty to maintain that the theorem is true when one or more of these conditions are violated. The theorem may be true when one or more of the conditions are violated, but that is a matter for proof and is not a legitimate assumption.

It is a mere commonplace of careful writers on mathematics that the limit to which a function $f(x)$ converges when x converges to a limit, a say, has by its definition nothing whatever to do with *the particular case* of the function when *x is equal to a.* In fact the reason for the introduction of the notion of a limit is, that the usual definition of the function ceases to give a definite meaning for the particular value a of x; though of course the definition of a limit holds equally well whether $f(a)$ has or has not a definite meaning when evaluated by the ordinary rules of algebra. It is not easy to say how much of the erroneous conception of a limit is due simply to defective language and notation; the phrase "when x is equal to a" in the clause "the limit of $f(x)$ when x is equal to a" has, I fear, led many astray. It cannot be too emphatically insisted upon that, in finding the limit of $f(x)$ when x converges to a as its limit, the value a must not be assigned to x; the limit depends, not on the value of $f(x)$ when x has the value a but on the values of $f(x)$ when x is *all but* equal to a. So far as the limit is concerned, it does not matter in the least whether $f(x)$ has or has not a definite value when x is equal to a; cases are quite common in which $f(x)$ has a definite value when x is equal to a and also a definite limit when x converges to a, and yet the value and the limit are not equal.

If the method of limits is to be used with absolute beginners in geometry (personally, I am inclined to hold that it is not suitable as a method of reasoning for absolute beginners) there should be greater care taken to show the *reasonableness* of the definition, and the proofs should be genuine and not merely plausible. For the beginner the process by which a secant through a fixed point outside a circle is gradually rotated till it becomes a tangent, is very valuable *by way of suggestion*, and a teacher who does not frequently use the process in order to gain theorems on tangents loses a great oppor-

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tunity. But when the process has suggested a theorem, that theorem should be demonstrated by a method which implies that the tangent has been actually drawn. Thus, I think Euclid was wise in proving III 36 as well as III 35.

When the notion of a limit is first introduced, it should, I think, be strictly confined to the case of a tangent; the general definition is too abstract. I would suggest some such definition as the following :—the tangent at a point A on a curve is a line AT such that the angle TAB between AT and the secant AB, through A and any other point B on the curve near to A, is small when B is near to A, and can be made as small as we please simply by taking B near enough to A.

The definition is rather long-winded, but it merely states in other words what, I think, is the ordinary conception of a tangent; namely, the tangent at A is a line (i) that meets the curve in only one point near A but (ii) that, if rotated about A as a pivot through any small angle (no matter how small that angle may be) will again cut the curve near A.

Now, to prove that the tangent to a circle is the line at right angles to the radius to the point of contact, first draw AT perpendicular to OA. Then, since the angle TAB is half the angle AOB, that angle is small when B is near A and can obviously be made as small as we please by taking B near enough to A. Hence AT is the tangent at A.

I hesitated for some time about asking the Society to accept this Note, but I finally felt myself justified in making the request on considering that we are now at the beginning of a series of great changes in the teaching of mathematics, and that there is almost a consensus of opinion among recent writers of textbooks as to the treatment of tangents by the method of limits. The exposition actually given of the method seems to me to be so radically faulty and so well fitted to make it difficult for a pupil to gain a sound knowledge of the method in his later studies, that I have ventured to take up the time of the Society with matters that are certainly well understood and properly expounded in various works.

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