A RELATIONSHIP BETWEEN ARBITRARY POSITIVE MATRICES AND STOCHASTIC MATRICES

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1. Introduction. The author **(2)** has shown that corresponding to each positive square matrix A (i.e. every $a_{ij} > 0$) is a unique doubly stochastic matrix of the form $D_1 A D_2$, where the D_i are diagonal matrices with positive diagonals. This doubly stochastic matrix can be obtained as the limit of the iteration defined by alternately normalizing the rows and columns of A.

In this paper, it is shown that with a sacrifice of one diagonal D it is still possible to obtain a stochastic matrix. Of course, it is necessary to modify the iteration somewhat. More precisely, it is shown that corresponding to each positive square matrix A is a unique stochastic matrix of the form DAD where D is a diagonal matrix with a positive diagonal. It is shown further how this stochastic matrix can be obtained as a limit to an iteration on A.

Immediate corollaries to this result are a theorem of Marcus and Newman **(1)**, which states that if A is a positive symmetric matrix, then there exists a diagonal matrix D with a positive main diagonal such that DAD is doubly stochastic, and its generalization, which states that if A is positive $N \times N$ and if $p_1, \ldots, p_N$ are positive real numbers, then there exists a unique matrix of the form DAD with row sums $p_1, \ldots, p_N$ where D is a diagonal matrix with a positive diagonal.

2. Stochastic matrices and positive matrices. The main result is:

THEOREM. *Corresponding to each positive matrix A there exists a unique stochastic matrix of the form DAD where D is a diagonal matrix with a positive diagonal.*

The existence part of the proof is absorbed into three lemmas which follow.

LEMMA 1. *Let $V \subseteq E^N \times E^N$ consist of vector pairs (x, y) with positive components that satisfy*

$$\sum_{j=1}^{N} y_i a_{ij} x_j = 1, \qquad i = 1, \ldots, N,$$

with $||x|| = \max |x_i| \leqslant a^{-\frac{1}{2}}$ and $||y|| = \max |y_i| \leqslant a^{-\frac{1}{2}}$ where a is the minimal element of the positive matrix $A = (a_{ij})$. Then the function

$$\phi(x, y) = \max_i \sum_{j=1}^{N} x_i a_{ij} y_i - \min_i \sum_{j=1}^{N} x_i a_{ij} y_j$$

achieves a minimum of zero on V.

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Proof. Certainly V is not empty since it contains (x^0, y^0) where

$$x_{i,0} = a^{-\frac{1}{2}}, \qquad y_{i,0} = (\sum_j a_{ij})^{-1}a^{\frac{1}{2}}, \qquad \text{for } i = 1, \ldots, N.$$

Note that

$$|y_{i,0}| \leqslant a^{\frac{1}{2}}/a_{ij} \leqslant a^{\frac{1}{2}}/a = a^{-\frac{1}{2}}$$

for any i, j.

Construct a sequence $(x^n, y^n) \in V$ as follows. Let (x^0, y^0) be as above and set

$$x_{i,n+1} = M_n^{-1}a^{-\frac{1}{2}}\rho_{i,n}^{-1}x_{i,n}, \qquad y_{j,n+1} = M_n a^{\frac{1}{2}}\delta_{j,n}^{-1}y_{j,n},$$

where

$$\rho_{i,n} = \sum_j x_{i,n}a_{ij}y_{j,n}, \qquad \delta_{j,n} = \sum_i \rho_{i,n}^{-1}x_{i,n}a_{ji}y_{j,n},$$

$$M_n = \max_i \rho_{i,n}^{-1}x_{i,n}.$$

It is easy to see that each (x^n, y^n) lies in V, for certainly $\sum_j y_{i,n}a_{ij}x_{j,n} = 1$ for all i. Since for all i, j, n,

$$\delta_{j,n}^{-1}y_{j,n} = (\sum_i \rho_{i,n}^{-1}x_{i,n}a_{ji})^{-1} \leqslant (\rho_{i,n}^{-1}x_{i,n}a_{ji})^{-1} \leqslant a^{-1}(\rho_{i,n}^{-1}x_{i,n})^{-1},$$

in particular

$$\delta_{j,n}^{-1}y_{j,n} \leqslant a^{-1}M_n^{-1}$$

for all j and n. Thus

$$y_{j,n+1} \leqslant M_n a^{\frac{1}{2}}a^{-1}M_n^{-1} = a^{-\frac{1}{2}};$$

also

$$x_{i,n+1} \leqslant M_n^{-1}a^{-\frac{1}{2}}M_n = a^{-\frac{1}{2}},$$

and hence

$$||x^n|| \leqslant a^{-\frac{1}{2}} \quad \text{and} \quad ||y^n|| \leqslant a^{-\frac{1}{2}} \qquad \text{for all } n.$$

Then from $x_{i,n}\sum_j a_{ij}y_{j,n} = \rho_{i,n}$, it follows that

$$\rho_{i,n}^{-1}x_{i,n} = (\sum_j a_{ij}y_{j,n})^{-1} \geqslant a^{\frac{1}{2}}(\sum_j a_{ij})^{-1} \geqslant Ra^{\frac{1}{2}},$$

where $R^{-1} = \max_i \sum_j a_{ij}$. Also

$$y_{j,n} = (\sum_i x_{i,n}a_{ji})^{-1} \geqslant a^{\frac{1}{2}}(\sum_i a_{ji})^{-1} \geqslant Ra^{\frac{1}{2}},$$

and therefore, in particular,

$$d_n = \min_{i,j} \rho_{i,n}^{-1}x_{i,n}a_{ij}y_{j,n} \geqslant Ra^{\frac{1}{2}}aRa^{\frac{1}{2}} = R^2a^2 = \mu > 0 \qquad \text{for all } n.$$

Let

$$\rho_{i_1,n+1} = \min_i \rho_{i,n+1}, \qquad \rho_{i_2,n+1} = \max_i \rho_{i,n+1},$$

$$\delta_{j_1,n} = \min_j \delta_{j,n}, \qquad \delta_{j_2,n} = \max_j \delta_{j,n}.$$

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Then

$$\begin{aligned}\phi(x^{n+1}, y^{n+1}) &= \rho_{i_2,n+1} - \rho_{i_1,n+1}\\ &\leqslant [x_{i_2,n}\,\rho_{i_2,n}^{-1}\,a_{i_2,j_2}\,\delta_{j_2,n}^{-1}\,y_{j_2,n} + \delta_{j_1,n}^{-1}(1 - x_{i_2,n}\,\rho_{i_2,n}^{-1}\,a_{i_2j_2}\,y_{j_2,n})]\\ &\quad - [x_{i_1,n}\,\rho_{i_1,n}^{-1}\,a_{i_1,j_1}\,\delta_{j_1,n}^{-1}\,y_{j_1,n} + \delta_{j_2,n}^{-1}(1 - x_{i_1,n}\,\rho_{i_1,n}^{-1}\,a_{i_1,j_1}\,y_{j_1,n})]\\ &\leqslant [\delta_{j_2,n}^{-1}\,d_n + \delta_{j_1,n}^{-1}(1 - d_n)] - [\delta_{j_1,n}^{-1}\,d_n + \delta_{j_2,n}^{-1}(1 - d_n)]\\ &= (1 - 2d_n)(\delta_{j_1,n}^{-1} - \delta_{j_2,n}^{-1}) \leqslant (1 - 2d_n)(\max_i \rho_{i,n} - \min_i \rho_{i,n})\\ &= (1 - 2d_n)\phi(x^n, y^n) \leqslant (1 - 2\mu)\phi(x^n, y^n).\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \phi(x^n, y^n) = 0.$$

Since V is bounded, the sequence $\{(x^n, y^n)\}$ has a limit point $(\tilde{x}, \tilde{y})$. It readily follows that $(\tilde{x}, \tilde{y}) \in V$ and $\phi(\tilde{x}, \tilde{y}) = 0$.

LEMMA 2. *For any positive matrix A, there exist diagonal matrices D_1 and D_2 with positive diagonals such that D_1AD_2 and D_2AD_1 are stochastic.*

Proof. Pick $(\tilde{x}, \tilde{y}) \in V$, which minimizes $\phi(x, y)$. Then $\sum_j \tilde{y}_i\,a_{ij}\,\tilde{x}_j = 1$ for all i while $\sum_j \tilde{x}_i\,a_{ij}\,\tilde{y}_j = k$, a constant, for all i. It readily follows that k and 1 are each maximal eigenvalues for D_1AD_2. Thus $k = 1$.

LEMMA 3. *If A is positive and if D_1AD_2 and D_2AD_1 are both stochastic where D_1 and D_2 are diagonal with positive diagonals, then $D_2 = pD_1$ for some number $p > 0$.*

Proof. Let $D_1 = \mathrm{dg}(x_1, \ldots, x_N)$ and $D_2 = \mathrm{dg}(y_1, \ldots, y_N)$ and set $p_j = y_j/x_j$. If p_j is not constant, then

$$\max_j p_j = p_{i_0} = (\textstyle\sum_j x_{i_0}\,a_{i_0j}\,x_j)^{-1} = (\sum_j x_{i_0}\,a_{i_0j}\,x_j\,p_j)(\sum_j x_{i_0}\,a_{i_0j}\,x_j)^{-1} < \max_j p_j,$$

a contradiction.

Proof of the theorem. From Lemmas 2 and 3, there is a diagonal matrix D_1 with positive diagonal and a positive number p such that pD_1AD_1 is stochastic. The existence part of the theorem follows by taking $D = p^{\frac{1}{2}}D_1$.

Suppose D and C are diagonal matrices with positive diagonals such that DAD and CAC are stochastic. Then $CAC = B = (b_{ij})$ and $DC^{-1}BC^{-1}D$ are both stochastic. If $DC^{-1} = \mathrm{dg}(z_1, \ldots, z_N)$,

$$\max_j z_j = z_{i_0} = (\textstyle\sum_j b_{i_0j}\,z_j)^{-1} \leqslant (\sum_j b_{i_0j})^{-1}(\min_j z_j)^{-1} = (\min_j z_j)^{-1},$$

and similarly $\min_j z_j \geqslant (\max_j z_j)^{-1}$, with equality possible in each case only if z_j is constant for all j. It follows that $z_j = 1$, $j = 1, \ldots, N$, and therefore that $D = C$.

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COROLLARY 1 (Marcus and Newman **1**). *If A is symmetric and has positive entries, there exists a diagonal matrix D with positive main diagonal entries such that DAD is doubly stochastic.*

Proof. This follows at once since DAD is symmetric when A is symmetric.

COROLLARY 2. *Corresponding to each positive $N \times N$ matrix A and each set of positive real numbers $p_1, \ldots, p_N$ there is a unique matrix of the form DAD with row sums $p_1, \ldots, p_N$ where D is a diagonal matrix with a positive main diagonal.*

Proof. Let $D_0 = \mathrm{dg}(p_1, \ldots, p_N)$ and set $B = D_0^{-1}A$. There is a diagonal matrix D with a positive main diagonal such that $S = DBD$ is stochastic. Then $D_0 S = DAD$ has the appropriate row sums. We have used the fact that $D_0 D = DD_0$.

In the proof of Lemma 1 a method is suggested for determining the matrix DAD of the theorem from A by an iterative scheme.

Define diagonal matric sequences $\{X_n\}$ and $\{Y_n\}$ such that

$$X_0 = I, \qquad Y_0 = \mathrm{dg}[(\textstyle\sum_j a_{1j})^{-1}, \ldots, (\sum_j a_{Nj})^{-1}],$$

$$X_{n+1} = D_{1,n} X_n, \; Y_{n+1} = D_{2,n} Y_n,$$

where $D_{1,n}$ and $D_{2,n}$ are diagonal matrices such that

$$D_{1,n}^{-1} u = X_n A Y_n u \quad \text{and} \quad D_{2,n}^{-1} u = Y_n A X_{n+1} u;$$

here u denotes the N-dimensional vector all of whose components equal one. Then $X_n A Y_n \to DAD$.

REFERENCES

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