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A LOWER BOUND FOR A REMAINDER TERM ASSOCIATED WITH THE SUM OF DIGITS FUNCTION

by D. M. E. FOSTER

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For a fixed integer $q \geqq 2$, every positive integer $k = \sum_{r \geq 0} a_r(q,k)q^r$ where each $a_r(q,k) \in \{0,1,2,\ldots,q-1\}$. The sum of digits function $\alpha(q,k) = \sum_{r \geq 0} a_r(q,k)$ behaves rather erratically but on averaging has a uniform behaviour. In particular if $A(q,n) = \sum_{k=1}^{n-1} \alpha(q,k)$, where $n > 1$, then it is well known that $A(q,n) \sim \frac{1}{2}((q-1)/\log q)n \log n$ as $n \to \infty$. For odd values of q, a lower bound is now obtained for the difference $2S(q,n) = A(q,n) - \frac{1}{2}(q-1)[\log n/\log q]n$, where $[\log n/\log q]$ denotes the greatest integer $\leqq \log n/\log q$. This complements an upper bound already found.

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1. Introduction

If $q \geqq 2$ is a fixed integer it is well known that every positive integer k may be expressed uniquely in the form

$$k = \sum_{r=0}^{\infty} a_r(q,k)q^r \text{ where } a_r(q,k) \in \{0,1,\ldots,q-1\} \tag{1.1}$$

and the "sum of digits" function $\alpha(q,k)$ is given by

$$\alpha(q,k) = \sum_{r=0}^{\infty} a_r(q,k), \tag{1.2}$$

both the above sums being finite. It is not difficult to see that, although the behaviour of $\alpha(q,k)$ itself is somewhat erratic, its average behaviour is more regular and has been widely studied.

For an integer $n > 1$, let

$$A(q,n) = \sum_{k=1}^{n-1} \alpha(q,k)$$

and define $A(q,1) = 0$. Behaviour in the special case $n = q^s$ suggests the asymptotic result

$$A(q,n) \sim \frac{\frac{1}{2}(q-1)}{\log q} n \log n \quad \text{as} \quad n \to \infty,$$

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a result obtained by Bush [2] in 1940. Later work by Bellman and Shapiro [1], Mirsky [11] and Drazin and Griffiths [6] gave estimates for the remainder term

$$A(q,n)-\frac{\frac{1}{2}(q-1)}{\log q}n\log n.$$

The case $q=2$ in particular has yielded the most precise results. In this case (and also for $q=3$) the results of Drazin and Griffiths are best possible and they have also been obtained by McIlroy [10] and Shiokawa [12].

In 1975, Delange [5] obtained a very elegant analytical form for the remainder term, involving a periodic, continuous but nowhere differentiable function, thereby generalizing an earlier result concerned with the case $q=2$ of Trollope [14]. In 1977, Stolarsky [13] considered the average of a more general sum of the type

$$A_d(q,n)=\sum_{k=1}^{n-1}\{\alpha(q,k)\}^d$$

when $q=2$ and $d\geqq 0$. (Stolarsky's paper also contains an excellent account of the history of related problems). More recently, Coquet [3] has obtained some very precise estimates for $A_d(q,n)$ using probabilistic techniques.

In the mid 1960s, Trollope [15] also considered the related problem concerned with Cantor representations of integers, and Kirschenhofer and Tichy [8] have since generalised Delange's result, mentioned above, to this situation too. The appropriate remainder term now takes the slightly different form

$$S(q,n)=\left\{A(q,n)-\frac{1}{2}(q-1)\left[\frac{\log n}{\log q}\right]n\right\}\Big/\frac{1}{2}$$

in the special case when the Cantor representation of an integer k becomes a q-adic representation of the form (1.1) for some q. As usual, $[\log n/\log q]$ denotes the greatest integer less than or equal to $\log n/\log q$. (A sum of this type has very recently been considered by Larcher and Tichy [9] in connection with the Gray code number system.) Thus, in the original digits problem, one can consider directly an estimate for $S(q,n)/n$. The best possible upper bound for all $q\geqq 2$ is $q-1$; see [7]. For $q=2$, 3, 4, 5 and 7 the best possible lower bounds are $-2/3$, $-2/7$, $-9/23$, $-7/13$ and $-6/19$ respectively. The proofs for $q=2$ and 3 are contained in [7]. Each of these inequalities is deduced from a more precise result.

Clearly every positive integer $n\not\equiv 0 \pmod q$ is of the form $n=n_m$ where

$$n_m=a_0q^{t_0}+a_1q^{t_0+t_1}+a_2q^{t_0+t_1+t_2}+\cdots+a_mq^{t_0+t_1+t_2+\cdots+t_m}, \tag{1.3}$$

for some $m\in\mathbf{N}\cup\{0\}, t_0=0$, positive integers $t_1,t_2,\ldots,t_m$ and non-zero coefficients

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$a_0, a_1, \ldots, a_m \in \{1, 2, \ldots, q-1\}$. For convenience of notation, given such an integer n introduce

$$n_{-1} = 0,\ n_0 = a_0 \quad \text{and} \quad n_i = a_0 + a_1 q^{t_1} + \cdots + a_i q^{t_1 + \cdots + t_i}$$

for $1 \leqq i \leqq m$. Then it is not difficult to see that $S(q, n_m)$ has the following simple form:

$$S(q, n_m) = \sum_{r=0}^{m} a_r(a_r - 1) q^{t_0 + t_1 + \cdots + t_r} + \sum_{r=0}^{m} \{2a_r - (q-1)t_r\} n_{r-1}. \tag{1.4}$$

It is easily verified that, if $s \in \mathbf{N}$,

$$\frac{S(q, q^s n_m)}{q^s n_m} = \frac{S(q, n_m)}{n_m},$$

so that we may assume that n is of the form (1.3) for some integer $m \geqq 0$. In [7] it was proved that

$$\frac{S(q, n_m)}{n_m} \leqq (q-1)\left\{1 - \frac{m+1}{q^{m+1} - 1}\right\},$$

with equality when $n_m = (q-1)(1 + q + q^2 + \cdots + q^m)$, together with a similar precise result for the lower bound when $q = 2$ and 3 and subsequently (but not published) for $q = 4$, 5 and 7. However the details are rather complicated, slightly more so when q is even, and the object of the present paper is to obtain an asymptotic result for all *odd* values of $q \geqq 9$. Numerical evidence for $5 \leqq q \leqq 13$ obtained by my colleague Mrs M. F. McCall has suggested that the likely critical case for odd q occurs when n_m is of the form

$$n_m^* = n_{m-l} + \tfrac{1}{2}(q-1)(q^{m-l+1} + q^{m-l+2} + \cdots + q^{m-2}) + \tfrac{1}{2}(q - \beta_q) q^{m-1} + q^m,$$

where

$$n_{m-l} = \sum_{r=0}^{m-l} a_r q^{t_0 + \cdots + t_r}$$

and β_q is the unique odd integer satisfying

$$3q - \sqrt{8q^2 - 9q + 1} \leqq \beta_q \leqq 3q - \sqrt{8q^2 - 9q + 1} + 2, \tag{1.5}$$

unless the even integer $8q^2 - 9q + 1$ is a perfect square. If we introduce

$$h_q = \frac{q^2 - 3q - (\beta_q - 1)(\beta_q - 5)}{2(3q - \beta_q + 1)}, \tag{1.6}$$

then, as we shall see later, if $m - l$ remains fixed

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$$\frac{S(q, n_m^*)}{n_m^*} \to -h_q$$

as $m \to \infty$.

Theorem. *For all odd* $q \geqq 9$,

$$\frac{S(q, n_m)}{n_m} > -h_q.$$

The sequence of odd integers (β_q) increases with q and starts off as follows:

$$\beta_q = \begin{cases} 3 & \text{if} \quad q = 3, 5, 7, \\ 5 & \text{if} \quad q = 9, 11, \ldots, 19, \\ 7 & \text{if} \quad q = 21, 23, \ldots, 31, \\ 9 & \text{if} \quad q = 33, 35, \ldots, 43. \end{cases}$$

When $q = 3$, 5 or 7, as mentioned earlier, there is a more precise result, namely

$$\frac{S(q, n_m)}{n_m} \geqq -h_q(m)$$

where

$$h_3(m) = \frac{6(3^m - 1)}{7.3^{m+1} - 5} \quad \text{and} \quad h_q(m) = \frac{(q^2 - 3q + 4)q^{m-1} - 1}{2\{(3q-2)q^{m-1} - 1\}} \qquad (q = 5, 7).$$

In each of these cases $h_q(m) \uparrow h_q$ as $m \uparrow \infty$.

In the special case when $8q^2 - 9q + 1 = (2x)^2$ for some positive integer x, it may be verified that either choice of $\beta_q = 3q - 2x$ or $3q - 2x + 2$ in (1.5) gives rise to the same value of h_q in (1.6). This case can only arise when $q \equiv 1 \pmod 4$, so that we can express $q = 1 + 4y$ where $y(32y + 7) = x^2$. Let $d = hcf(x, y)$ with $x = x_1 d$ and $y = y_1 d$ for positive coprime integers x_1 and y_1. Then $y_1(32dy_1 + 7) = dx_1^2$ giving $d = y_1 y_2$ for some integer y_2. This leads to $y_2(x_1^2 - 32y_1^2) = 7$ so that $y_2 = 1$ or 7. If $y_2 = 1$, $x_1^2 - 32y_1^2 = 7$ which is insoluble since x_1 must be odd giving $x_1^2 \equiv 1 \pmod 4$. Thus $y_2 = 7$ and we are left with the Pell equation $x_1^2 - 32y_1^2 = 1$ whose general solution for positive integers x_1 and y_1 is given by

$$x_1 + \sqrt{32}\, y_1 = (17 + 3\sqrt{32})^r, \qquad r \in \mathbf{N}.$$

This solution arises from the continued fraction expression of $\sqrt{32}$. (See, for example, Davenport [4]). The two smallest values of q which occur are 253 ($r=1$) and 291313 ($r=2$); and when $q=253$ it is easily verified that $\beta_q=45$ or 47.

It is convenient to prove the theorem in the following equivalent form. Introduce

$$\alpha_q=h_q-1 \quad \text{so that} \quad \alpha_q=\frac{q^2-9q-(\beta_q-1)(\beta_q-7)}{2(3q-\beta_q+1)}. \tag{1.7}$$

Then

$$(\alpha_9,\alpha_{11},\alpha_{13},\alpha_{15},\alpha_{17},\alpha_{19})=\left(\frac{4}{23},\frac{15}{29},\frac{6}{7},\frac{49}{41},\frac{72}{47},\frac{99}{53}\right)$$

and

$$(\alpha_{21},\alpha_{23},\alpha_{25},\alpha_{27},\alpha_{29}\,\alpha_{31})=\left(\frac{42}{19},\frac{23}{9},\frac{200}{69},\frac{81}{25},\frac{290}{81},\frac{341}{87}\right),$$

and so on. For $m\geqq 0$ we define

$$\begin{aligned} T(q,n_m)&=S(q,n_m)+h_q n_m \\ &=S(q,n_m)+(\alpha_q+1)n_m \\ &=\sum_{r=0}^{m} a_r(a_r+\alpha_q)q^{t_0+t_1+\cdots+t_r} \\ &\quad+\sum_{r=0}^{m}\{2a_r-(q-1)t_r\}n_{r-1}. \end{aligned} \tag{1.8}$$

Then the inequality of the theorem is equivalent to

$$T(q,n_m)>0. \tag{1.9}$$

I am very grateful to Mrs McCall for her help in giving me a feel for the general result. I should also like to thank the referee for his very careful corrections of many errors in the original draft.

2.

In this section we prove three lemmas which will be needed in the proof of the theorem.

Lemma 2.1. $3\leqq\beta_q-\alpha_q\leqq 5.$ (2.1)

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Proof. We have, using (1.7) and rearranging,

$$\beta_q-\alpha_q-3=\frac{8q^2-9q+1-(\beta_q-3q)^2}{2(3q-\beta_q+1)}\geqq 0, \qquad \text{using (1.5).}$$

In fact, if $8q^2-9q+1$ is not a perfect square, then as both $8q^2-9q+1$ and β_q-3q are even integers it is clear that

$$\beta_q-\alpha_q-3\geqq\frac{1}{3q-\beta_q+1}>\frac{1}{3q}, \tag{2.2}$$

a result which will be of use later.

Similarly,

$$5-(\beta_q-\alpha_q)=\frac{(3q-\beta_q+2)^2-(8q^2-9q+1)}{2(3q-\beta_q+1)}\geqq 0 \qquad \text{using (1.5).}$$

Once again, if $8q^2-9q+1$ is not a perfect square then

$$5-(\beta_q-\alpha_q)>\frac{1}{3q}. \tag{2.3}$$

Lemma 2.2. *Let e and f be real numbers and let x_0 be a positive integer with $(q-1)x_0-e\geqq 1$. Then, if the inequality*

$$4\{(q-1)x-e\}\leqq fq^{x-1}$$

holds for $x=x_0$, it also holds for all integers $x>x_0$.

Proof. Clearly $fq^{x_0-1}\geqq 4$. If we take $x=x_0+1$ we have

$$\begin{aligned}4\{(q-1)(x_0+1)-e\}&=4\{(q-1)x_0-e\}+4(q-1)\\&\leqq fq^{x_0-1}+4(q-1)\\&\leqq fq^{x_0},\end{aligned}$$

provided that

$$4(q-1)\leqq fq^{x_0-1}(q-1)$$

which is true.

It is convenient at this point to introduce some useful substitutions for $a_{m-1},a_{m-2},\ldots$ chosen to reflect the critical case when $n=n_m^*$. These take the form

$$2a_{m-1}=q-\beta_q+2\delta_1$$

and (2.4)

$$2a_r = q-1-2\delta_{m-r-1}+2\delta_{m-r}(m-2\geqq r\geqq 0)$$

where $\delta_1, \delta_2, \ldots, \delta_m \in \mathbf{Z}$.

The final lemma facilitates some routine calculations which are repeated throughout the proof of the theorem at all the different stages.

If l is any integer satisfying $2\leqq l\leqq m+1$, for convenience of notation introduce

$$F_m(l-1)=a_m(a_m+\alpha_q)q^{t_1+\cdots+t_m}+\sum_{r=m-l+1}^{m-1} a_r\left\{\sum_{s=r+1}^{m}[2a_s-(q-1)t_s]+a_r+\alpha_q\right\}q^{t_1+\cdots+t_r}.$$

It is useful to rearrange $F_m(l-1)$ as follows. We have

$$F_m(l-1)=\sum_{r=m-l+1}^{m} a_r(a_r+\alpha_q)q^{t_1+\cdots+t_r}+\sum_{r=m-l+1}^{m-1} a_r\left\{\sum_{s=r+1}^{m}[2a_s-(q-1)t_s]\right\}q^{t_1+\cdots+t_r},$$

where the double sum may be expressed as

$$\sum_{s=m-l+2}^{m}[2a_s-(q-1)t_s]\sum_{r=m-l+1}^{s-1} a_r q^{t_1+\cdots+t_r}$$

$$=\sum_{s=m-l+2}^{m}[2a_s-(q-1)t_s]n_{s-1}-n_{m-l}\sum_{s=m-l+2}^{m}[2a_s-(q-1)t_s].$$

Thus

$$F_m(l-1)=\sum_{r=m-l+1}^{m}\{a_r(a_r+\alpha_q)q^{t_1+\cdots+t_r}+[2a_r-(q-1)t_r]n_{r-1}\}$$

$$-n_{m-l}\sum_{r=m-l+1}^{m}[2a_r-(q-1)t_r]. \qquad (2.5)$$

As $n_{-1}=0$, putting $l=m+1$ we have in particular

$$F_m(m)=T(q,n_m). \qquad (2.6)$$

Lemma 2.3. *If* $2\leqq l\leqq m+1$, $t_m=t_{m-1}=\cdots=t_{m-l+2}=1$, $a_m=1$ *and* $a_{m-1}, a_{m-2},\ldots,a_{m-l+1}$ *are given by* (2.4) *then*

$$4F_m(l-1)q^{-(t_1+\cdots+t_{m-l+1})}=4(q-1)\sum_{s=1}^{l-2}q^{l-2-s}k(\delta_s)+4h(\delta_{l-1})+q-7-2\alpha_q+2\beta_q, \quad (2.7)$$

where

$$h(x)=x(x+3+\alpha_q-\beta_q)$$

and

$$k(x)=x(x+4+\alpha_q-\beta_q).$$

In the particular case when $l=2$ the summation

$$\sum_{s=1}^{l-2}$$

is assumed to be vacuous.

Proof. When $l=2$ we have

$$F_m(1)=a_m(a_m+\alpha_q)q^{t_1+\cdots+t_m}+a_{m-1}\{2a_m-(q-1)t_m+a_{m-1}+\alpha_q\}q^{t_1+\cdots+t_{m-1}},$$

together with $t_m=a_m=1$ and $2a_{m-1}=q-\beta_q+2\delta_1$. Thus

$$4F_m(1)q_q^{-(t_1+\cdots+t_{m-1})}=4(1+\alpha_q)q-(q-\beta_q+2\delta_1)(q+\beta_q-2\delta_1-6-2\alpha_q),$$

$$=4\delta_1(\delta_1+3+\alpha_q-\beta_q)+10q-q^2+\beta_q^2-6\beta_q-2\alpha_q+2\alpha_q(3q-\beta_q+1).$$

Using (1.7) to eliminate α_q from the last term we eventually obtain

$$4F_m(1)q^{-(t_1+\cdots+t_{m-1})}=4h(\delta_1)+q-7-2\alpha_q+2\beta_q.$$

When $l=3$, we have

$$F_m(2)=F_m(1)+a_{m-2}\{2(a_m+a_{m-1})-(q-1)(t_m+t_{m-1})+a_{m-2}+\alpha_q\}q^{t_1+\cdots+t_{m-2}},$$

together with $t_m=t_{m-1}=a_m=1$,

$$2a_{m-1}=q-\beta_q+2\delta_1 \quad\text{and}\quad 2a_{m-2}=q-1-2\delta_1+2\delta_2.$$

Using the above simplification of $F_m(1)$ we see that

$$4F_m(2)q^{-(t_1+\cdots+t_{m-2})}=q\{4h(\delta_1)+q-7-2\alpha_q+2\beta_q\}+4a_{m-2}(a_{m-2}-q+4+\alpha_q-\beta_q+2\delta_1).$$

The terms involving a_{m-2} are

$$(q-1-2\delta_1+2\delta_2)(-q-1+2\delta_1+2\delta_2+8+2\alpha_q-2\beta_q)$$

$$=-(q-2\delta_1)^2+(2\delta_2-1)^2+(q-1-2\delta_1+2\delta_2)(8+2\alpha_q-2\beta_q)$$

$$=-(q-1)(q-7-2\alpha_q+2\beta_q)+4(q-1)\delta_1-4h(\delta_1)+4h(\delta_2),$$

following rearrangement. Thus

$$4F_m(2)q^{-(t_1+\cdots+t_{m-2})}$$

$$=4qh(\delta_1)+q(q-7-2\alpha_q+2\beta_q)-(q-1)(q-7-2\alpha_q+2\beta_q)+4(q-1)\delta_1-4h(\delta_1)+4h(\delta_2),$$

and this takes the required form

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$$4F_m(2)q^{-(t_1+\cdots+t_{m-2})}=4(q-1)k(\delta_1)+4h(\delta_2)+q-7-2\alpha_q+2\beta_q.$$

The general case is now quite easy. Assume that (2.7) holds for some integer l satisfying $2\leqq l\leqq m$ and look at $F_m(l)$. We have

$$F_m(l)=F_m(l-1)+a_{m-l}\left\{\sum_{s=m-l+1}^{m}[2a_s-(q-1)t_s]+a_{m-l}+\alpha_q\right\}q^{t_1+\cdots+t_{m-l}}.$$

By the hypothesis of this case, we have

$$t_m=t_{m-1}=\cdots=t_{m-l+1}=1 \quad\text{and}\quad a_m=1$$

together with the usual substitutions for $a_{m-1},\dots,a_{m-l}$. Thus on substituting we obtain

$$F_m(l)=F_m(l-1)+a_{m-l}(a_{m-l}-q+4+\alpha_q-\beta_q+2\delta_{l-1})q^{t_1+\cdots+t_{m-l}}.$$

Using (2.7) we see that

$$\begin{aligned}4q^{-(t_1+\cdots+t_{m-l})}F_m(l)=4(q-1)\sum_{s=1}^{l-2}q^{l-1-s}k(\delta_s)\\+4qh(\delta_{l-1})+q(q-7-2\alpha_q+2\beta_q)\\+4a_{m-l}(a_{m-l}-q+4+\alpha_q-\beta_q+2\delta_{l-1}).\end{aligned}$$

Since $2a_{m-l}=q-1-2\delta_{l-1}+2\delta_l$ we have a similar calculation as in the evaluation of $F_m(2)$, with an obvious change of notation. This leads to

$$4q^{-(t_1+\cdots+t_{m-l})}F_m(l)=4(q-1)\sum_{s=1}^{l-2}q^{l-1-s}k(\delta_s)+4(q-1)k(\delta_{l-1})+4h(\delta_l)+q-7-2\alpha_q+2\beta_q,$$

and this is (2.7) with l replaced by $l+1$.

3. Proof of the theorem

We proceed by induction on the integer $m\geqq 0$. Initially

$$T(q,n_0)=a_0(a_0+\alpha_q)>0,$$

since $a_0\geqq 1$ and $\alpha_q>0\,\forall\, q\geqq 9$. Thus we now choose $m\geqq 1$ and assume that

$$T(q,n_r)>0\ \forall\ 0\leqq r\leqq m-1. \tag{3.1}$$

The proof runs through several stages, ending up with the main inductive step once the critical form for n_m begins to emerge.

Stage 1 ($m \geqq 1$). We can express

$$T(q, n_m) = a_m(a_m + \alpha_q)q^{t_1 + \cdots + t_m} + \{2a_m - (q-1)t_m\}n_{m-1} + T(q, n_{m-1}).$$

By the induction hypothesis (3.1), $T(q, n_{m-1}) > 0$ and it will therefore follow that $T(q, n_m) > 0$ provided that

$$\{(q-1)t_m - 2a_m\}n_{m-1} < a_m(a_m + \alpha_q)q^{t_1 + \cdots + t_m}. \tag{3.2}$$

If $2a_m \geqq (q-1)t_m$, (3.2) follows easily. Thus suppose that $(q-1)t_m > 2a_m$. Since

$$\begin{aligned} n_{m-1} &= a_0 + a_1 q^{t_1} + a_2 q^{t_1+t_2} + \cdots + a_{m-1} q^{t_1+t_2 \cdots + t_{m-1}} \\ &\leqq (q-1)\{1 + q^{t_1} + q^{t_1+t_2} + \cdots + q^{t_1+t_2+\cdots+t_{m-1}}\} \\ &\leqq q^{t_1+t_2+\cdots+t_{m-1}+1} - 1 < q^{t_1+t_2+\cdots+t_{m-1}+1}, \end{aligned}$$

the inequality (3.2) will hold provided that

$$(q-1)t_m - 2a_m < a_m(a_m + \alpha_q)q^{t_m - 1}. \tag{3.3}$$

If $t_m = 2$, (3.3) takes the form

$$2(q-1) < a_m\{(a_m + \alpha_q)q + 2\}$$

which holds easily $\forall\, a_m \geqq 2$. When $a_m = 1$, the inequality becomes $(\alpha_q - 1)q + 4 > 0$. For $q \geqq 15$, $\alpha_q > 1$ and for $9 \leqq q \leqq 13$, $\alpha_q < 1$ and the inequality is equivalent to $q\alpha_q > q - 4$ which holds for $q = 13$. Thus we are left only with the cases $q = 9$ and 11.

If $t_m = 3$, (3.3) becomes

$$3(q-1) \leqq a_m\{(a_m + \alpha_q)q^2 + 2\}$$

which holds for all $a_m \geqq 1$ and $q \geqq 9$. As $(q-1)t_m - 2a_m \geqq 1$ we can now appeal to Lemma 2.2 to deduce that (3.3) holds for all integers $t_m \geqq 3$ when $q \geqq 9$.

Thus we have proved that $T(q, n_m) > 0$ except in the following cases:

$$t_m = 1 \tag{I(i)}$$

or

$$(t_m; a_m) = (2; 1) \text{ when } q = 9 \text{ or } 11. \tag{I(ii)}$$

If $m = 1$, we have

$$T(q, n_1) = a_1(a_1 + \alpha_q)q^{t_1} + a_0\{a_0 + \alpha_q + 2a_1 - (q-1)t_1\}.$$

The only cases which we need consider are I(i) and I(ii). In the first case, $t_1 = 1$ and $a_1 \geqq 1$ so that

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$$T(q,n_1) \geqq (1+\alpha_q)q + a_0(a_0 - q + 3 + \alpha_q).$$

We now make the usual substitution for a_0 namely

$$2a_0 = 2a_{m-1} = q - \beta_q + 2\delta_1,$$

and using the case $l=2$ of Lemma 2.3 we see that

$$4T(q,n_1) \geqq 4h(\delta_1) + q - 7 - 2\alpha_q + 2\beta_q.$$

By Lemma 2.1, $3 \leqq \beta_q - \alpha_q \leqq 5$ so that $q - 7 - 2\alpha_q + 2\beta_q \geqq q - 1$. Also, for integral values of δ_1 the two least values of $h(\delta_1)$ occur when $\delta_1 = 0$ and 1. Clearly $h(0)=0$ and $h(1) = 4 + \alpha_q - \beta_q \geqq -1$. Thus in the worst case

$$4T(q,n_1) \geqq -4 + q - 1 = q - 5 > 0.$$

In the remaining case I(ii) when $q=9$ or 11 we have $(t_m; a_m) = (t_1; a_1) = (2; 1)$. Hence

$$T(q,n_1) = (1+\alpha_q)q^2 + a_0(a_0 - 2q + 4 + \alpha_q).$$

Directly,

$$T(9,n_1) = 81(1+\alpha_9) + a_0(a_0 + \alpha_9 - 14)$$

and

$$T(11,n_1) = 121(1+\alpha_{11}) + a_0(a_0 + \alpha_{11} - 18),$$

with $\alpha_9 = 4/23$ and $\alpha_{11} = 15/29$ and in each case $T(q,n_1) > 0 \; \forall a_0 \geqq 1$.

We can now assume that $m \geqq 2$ and proceed to the next stage of the argument.

Stage 2 ($m \geqq 2$). We subdivide $T(q,n_m)$ further as follows:

$$\begin{aligned} T(q,n_m) = & \sum_{r=m-1}^{m} a_r(a_r+\alpha_q)q^{t_1+\cdots+t_r} \\ & + \{2a_m - (q-1)t_m\}(n_{m-2} + a_{m-1}q^{t_1+\cdots+t_{m-1}}) \\ & + \{2a_{m-1} - (q-1)t_{m-1}\}n_{m-2} + T(q,n_{m-2}). \end{aligned} \tag{3.4}$$

By (3.1), $T(q,n_{m-2}) > 0$ and so $T(q,n_m) > 0$ provided that

$$\begin{aligned} & \{(q-1)(t_m + t_{m-1}) - 2(a_m + a_{m-1})\}n_{m-2} \\ & \leqq q^{t_1+\cdots+t_{m-1}}[a_m(a_m+\alpha_q)q^{t_m} + a_{m-1}\{2a_m - (q-1)t_m + a_{m-1} + \alpha_q\}]. \end{aligned} \tag{3.5}$$

If $(q-1)(t_m+t_{m-1}) - 2(a_m + a_{m-1}) \leqq 0$, the left-hand side of (3.5) is non-positive and the right-hand side is positive since

$$2a_m-(q-1)t_m+a_{m-1}+\alpha_q \geqq (q-1)t_{m-1}-a_{m-1}+\alpha_q \geqq q-1-a_{m-1}+\alpha_q$$

and $\alpha_q>0\ \forall\, q \geqq 9$. Thus we shall assume henceforth that

$$(q-1)(t_m+t_{m-1})-2(a_m+a_{m-1})>0. \tag{3.6}$$

As

$$\begin{aligned} n_{m-2} &\leqq (q-1)(1+q^{t_1}+q^{t_1+t_2}+\cdots+q^{t_1+t_2+\cdots+t_{m-2}}) \\ &\leqq q^{t_1+t_2+\cdots+t_{m-2}+1}-1<q^{t_1+t_2+\cdots t_{m-2}+1}, \end{aligned}$$

it will follow that $T(q,n_m)>0$ provided that

$$\begin{aligned} &q^{-(t_{m-1}-1)}\{(q-1)(t_m+t_{m-1})-2(a_m+a_{m-1})\} \\ &\leqq a_m(a_m+\alpha_q)q^{t_m}+a_{m-1}\{2a_m-(q-1)t_m+a_{m-1}+\alpha_q\}. \end{aligned} \tag{3.7}$$

Case I(i). Putting $t_m=1$, (3.7) takes the form

$$\begin{aligned} &q^{-(t_{m-1}-1)}\{(q-1)t_{m-1}-2(a_m+a_{m-1})+q-1\} \\ &\leqq a_m(a_m+\alpha_q)q+a_{m-1}(a_{m-1}+2a_m-q+1+\alpha_q). \end{aligned} \tag{3.8}$$

When $a_m=2$, we shall prove that (3.8) holds for all integers $1 \leqq a_{m-1} \leqq q-1$ and $t_{m-1} \geqq 1$. Since the coefficient of a_m on the right-hand side of (3.8) is positive and that on the left-hand side is negative, the inequality will then clearly follow for all integers $a_m \geqq 2$. But first we make the usual substitution $2a_{m-1}=q-\beta_q+2\delta_1$ and then, after some rearrangement, (3.8) becomes

$$\begin{aligned} &4q^{-(t_{m-1}-1)}\{(q-1)t_{m-1}-2a_m+\beta_q-2\delta_1-1\} \\ &\leqq 4\delta_1(\delta_1+2a_m+1+\alpha_q-\beta_q)+2q-q^2+\beta_q^2-2\beta_q+4a_m\{(a_m+1)q-\beta_q\} \\ &\quad+2\alpha_q\{2(a_m-1)q-1\}+2\alpha_q(3q-\beta_q+1). \end{aligned}$$

Using (1.7) to eliminate α_q from the last term on the right-hand side we obtain the following inequality:

$$\begin{aligned} &4q^{-(t_{m-1}-1)}\{(q-1)t_{m-1}-2a_m+\beta_q-2\delta_1-1\} \\ &\leqq 4\delta_1(\delta_1+2a_m+1+\alpha_q-\beta_q)+4a_m\{(a_m+1)q-\beta_q\} \\ &\quad+4(a_m-1)\alpha_q q-7q-7-2\alpha_q+6\beta_q. \end{aligned} \tag{3.9}$$

Putting $a_m=2$ as indicated earlier, (3.9) becomes

$$4q^{-(t_{m-1}-1)}\{(q-1)t_{m-1}-5+\beta_q-2\delta_1\}$$

$$\leqq 4\delta_1(\delta_1+5+\alpha_q-\beta_q)+17q-7+2\alpha_q(2q-1)-2\beta_q. \tag{3.10}$$

By (3.6) the integer $(q-1)t_{m-1}-5+\beta_q-2\delta_1>0$, and consequently the validity of (3.10) for $t_{m-1}=1$ will follow for all integers $t_{m-1}>1$ by Lemma 2.2. Putting $t_{m-1}=1$ in (3.10) leads to the inequality

$$4\delta_1(\delta_1+7+\alpha_q-\beta_q)+13q+17+2(2q-1)\alpha_q-6\beta_q\geqq 0. \tag{3.11}$$

As $3\leqq\beta_q-\alpha_q\leqq 5$, the least value of $\delta_1(\delta_1+7+\alpha_q-\beta_q)$ occurs when $\delta_1=-1$ if $4<\beta_q-\alpha_q\leqq 5$ and when $\delta_1=-2$ if $3\leqq\beta_q-\alpha_q\leqq 4$. In the first case (3.11) becomes

$$q-7+2(6q-\beta_q)+2(2q-3)\alpha_q\geqq 0$$

which holds easily as $\beta_q<6q$, and in the second case (3.11) becomes

$$13q-23+2(2q-5)\alpha_q+2\beta_q\geq 0$$

which again causes no problems.

The rest of Case I(i) is concerned with verifying (3.9) when $a_m=1$ for all integers $t_{m-1}\geqq 2$. Once again we take $t_{m-1}=2$ and then appeal to Lemma 2.2 to obtain its validity for the integral values of $t_{m-1}>2$. First putting $a_m=1$ in (3.9) we have

$$4q^{-(t_{m-1}-1)}\{(q-1)t_{m-1}-3+\beta_q-2\delta_1\}$$

$$\leqq 4\delta_1(\delta_1+3+\alpha_q-\beta_q)+q-7-2\alpha_q+2\beta_q. \tag{3.12}$$

When $t_{m-1}=2$, (3.12) rearranges to the form

$$4q\,\delta_1(\delta_1+3+\alpha_q-\beta_q+2q^{-1})+q^2-15q+2(\beta_q-\alpha_q)q-4\beta_q+20\geqq 0. \tag{3.13}$$

Since $3\leqq\beta_q-\alpha_q\leqq 5$ and $q\geqq 9$ we have

$$-\frac{2}{9}\leqq\beta_q-\alpha_q-3-2q^{-1}<2,$$

and it follows that the least value of $\delta_1(\delta_1+3+\alpha_q-\beta_q+2q^{-1})$ for integral δ_1 occurs when $\delta_1=0$ or 1. Putting $\delta_1=0$ we need to check the inequality

$$q^2-15q+2(\beta_q-\alpha_q)q-4\beta_q+20\geqq 0.$$

As $\beta_q-\alpha_q\geqq 3$ it will suffice to prove that

$$4\beta_q\leqq q(q-9)+20. \tag{3.14}$$

If $\beta_q=5$ we have $9\leqq q\leqq 19$ and there is no problem. Thus suppose that $\beta_q\geqq 7$ giving $q\geqq 21$. By (1.5), $\beta_q\leqq 3q-\surd(8q^2-9q+1)+2$ and it will suffice to verify that

$$q(q-21)+4\surd(8q^2-9q+1)+12\geqq 0$$

which is easily true $\forall q \geqq 21$. Now putting $\delta_1 = 1$ we need to check the inequality

$$q^2 + \{2(\alpha_q - \beta_q) + 1\}q - 4\beta_q + 28 \geqq 0.$$

As $\alpha_q - \beta_q \geqq -5$, it will suffice to show that

$$4\beta_q \leqq q(q-9) + 28,$$

and this has already been proved with 28 replaced by 20.

Thus in Case I(i) we are left only with $a_m = 1$ and $t_{m-1} = 1$.

Case I(ii). Putting $t_m = 2$ and $a_m = 1$ in (3.7) leads to the inequality

$$(q-1)t_{m-1} - 2a_{m-1} + 2q - 4 \leqq (1+\alpha_q)q^{t_{m-1}+1} + a_{m-1}(a_{m-1} - 2q + 4 + \alpha_q)q^{t_{m-1}-1}. \quad (3.15)$$

For $q = 9$ and 11, (3.15) is easily verified for all $1 \leqq a_{m-1} \leqq q-1$ when $t_{m-1} = 1$. As the right-hand side increases exponentially with t_{m-1}, there is no problem when $t_{m-1} \geqq 2$.

Summing up, if $m \geqq 2$ we have proved the result except in the following case:

$$(t_m, t_{m-1}; a_m) = (1, 1; 1). \quad \text{II}$$

If $m = 2$, by (2.6) we have $T(q, n_2) = F_2(2)$. Then, using condition II together with the usual substitutions for a_1 and a_0, we see from Lemma 2.3 that $T(q, n_2) > 0$ is equivalent to

$$4(q-1)k(\delta_1) + 4h(\delta_2) + q - 7 - 2\alpha_q + 2\beta_q > 0 \quad (3.16)$$

where as usual

$$k(\delta_1) = \delta_1(\delta_1 + 4 + \alpha_q - \beta_q) \quad \text{and} \quad h(\delta_2) = \delta_2(\delta_2 + 3 + \alpha_q - \beta_q).$$

As $3 \leqq \beta_q - \alpha_q \leqq 5$, $k(\delta_1) \geqq k(0) = 0$ for all integers δ_1 and (3.16) is therefore a consequence of

$$4h(\delta_2) + q - 7 - 2\alpha_q + 2\beta_q > 0,$$

an inequality which has already been seen to be true at the end of stage 1 in the case when $m = 1$.

We now take $m \geqq 3$ and let l be any integer satisfying $3 \leqq l \leqq m$.

Stage l ($3 \leqq l \leqq m$). As a consequence of stages 1 and 2 we now make the inductive assumption that

$$t_m = t_{m-1} = \cdots = t_{m-l+2} = 1 \quad \text{and} \quad a_m = 1,$$

together with the substitutions in (2.4) for $a_{m-1}, a_{m-2}, \ldots, a_{m-l+1}$. Following the customary procedure we express

$$T(q,n_m)=\sum_{r=m-l+1}^{m} a_r(a_r+\alpha_q)q^{t_1+\cdots+t_r}+\sum_{r=m-l+1}^{m}\{2a_r-(q-1)t_r\}n_{r-1}+T(q,n_{m-l}). \tag{3.17}$$

By (3.1), $T(q,n_{m-l})>0$ and we therefore have to prove that

$$\left[\sum_{r=m-l+1}^{m}\{(q-1)t_r-2a_r\}\right]n_{m-l}$$

$$\leqq a_m(a_m+\alpha_q)q^{t_1+\cdots+t_m}+\sum_{r=m-l+1}^{m-1} a_r\left\{\sum_{s=r+1}^{m}[2a_s-(q-1)t_s]+a_r+\alpha_q\right\}q^{t_1+\cdots+t_r}. \tag{3.18}$$

The expression on the right hand side of (3.18) is simply the $F_m(l-1)$ of Lemma 2.3, and accordingly we obtain the following simplification of (3.18), namely

$$4q^{-(t_1+\cdots+t_{m-l+1})}\{(q-1)t_{m-l+1}-3+\beta_q-2\delta_{l-1}\}n_{m-l}$$

$$\leqq 4(q-1)\sum_{s=1}^{l-2}q^{l-2-s}k(\delta_s)+4h(\delta_{l-1})+q-7-2\alpha_q+2\beta_q. \tag{3.19}$$

For each integral value of δ_s, $k(\delta_s)\geqq k(0)=0$ and $4h(\delta_{l-1})+q-7-2\alpha_q+2\beta_q>0$ as we have already seen. Thus for each $1\leqq s\leqq l-2$ the right-hand side of (3.19) is positive. Hence it will suffice to consider the case when

$$(q-1)t_{m-l+1}-3+\beta_q-2\delta_{l-1}>0.$$

As $n_{m-l}<q^{t_1+\cdots+t_{m-l+1}}$, (3.19) is a consequence of

$$4q^{-(t_{m-l+1}-1)}\{(q-1)t_{m-l+1}-3+\beta_q-2\delta_{l-1}\}$$

$$\leqq 4(q-1)\sum_{s=1}^{l-2}q^{l-2-s}k(\delta_s)+4h(\delta_{l-1})+q-7-2\alpha_q+2\beta_q. \tag{3.20}$$

We now prove that (3.20) holds for $t_{m-l+1}=2$, and then it will follow, from Lemma 2.2, that it holds for all integers $t_{m-l+1}>2$. Hence putting $t_{m-l+1}=2$ in (3.20) and rearranging we obtain the inequality

$$4(q-1)\sum_{s=1}^{l-2}q^{l-1-s}k(\delta_s)+4q\delta_{l-1}(\delta_{l-1}+3+\alpha_q-\beta_q+2q^{-1})$$

$$+q^2-15q+2(\beta_q-\alpha_q)q-4\beta_q+20\geqq 0.$$

Once again using the fact that $k(\delta_s)\geqq 0$ for all integers δ_s, we are left with inequality (3.13), with δ_1 replaced by δ_{l-1}, whose validity has already been established. Thus it remains to consider the case when $t_{m-l+1}=1$, when (3.20) takes the form

$$4(q-1)\sum_{s=1}^{l-2} q^{l-2-s}k(\delta_s)+4\delta_{l-1}(\delta_{l-1}+5+\alpha_q-\beta_q)-3q+9-2\alpha_q-2\beta_q\geqq 0. \tag{3.21}$$

We now have to run through the possible values of l, namely $3,4,\dots,m$. Initially $l=3$, so that $m\geqq 3$, giving stage 3, and (3.21) becomes

$$4(q-1)k(\delta_1)+4\delta_2(\delta_2+5+\alpha_q-\beta_q)-3q+9-2\alpha_q-2\beta_q\geqq 0. \tag{3.22}$$

This splits into two cases.

(i) $3\leqq\beta_q-\alpha_q\leqq 4$. For integral values of δ_1, the least two values of $k(\delta_1)$ occur when $\delta_1=0$ and -1, and $\min[k(\delta_1):\delta_1\in\mathbf{Z}-\{0,-1\}]=k(1)$. Also $\delta_2(\delta_2+5+\alpha_q-\beta_q)$ has a minimum value for integral δ_2 when $\delta_2=-1$. We now show that (3.22) holds for all integral values of δ_2 when $\delta_1\neq 0$ or -1. This will follow from the validity of the inequality for $(\delta_1,\delta_2)=(1,-1)$. In this case, after some rearrangement, (3.22) becomes

$$17q-27-(4q-10)(\beta_q-\alpha_q)-4\beta_q\geqq 0.$$

As $\beta_q-\alpha_q\leqq 4$ it will be sufficient to verify that

$$17q-27-4(4q-10)-4\beta_q\geqq 0,$$

that is

$$4\beta_q\leqq q+13. \tag{3.23}$$

As $\beta_q\leqq 3q-\sqrt{(8q^2-9q+1)}+2$, we have to verify that $11q-5\leqq 4\sqrt{(8q^2-9q+1)}$ or equivalently $q(7q-34)\geqq 9$, which is obviously true for all $q\geqq 9$. Thus we are left with the cases $\delta_1=0$ or -1.

(ii) $4<\beta_q-\alpha_q\leqq 5$. For integral values of δ_1, the two least values of $k(\delta_1)$ occur when $\delta_1=0$ and 1 and $\min[k(\delta_1):\delta_1\in\mathbf{Z}-\{0,1\}]=k(-1)$. Also $\delta_2(\delta_2+5+\alpha_q-\beta_q)$ has a minimum value when $\delta_2=0$. We now test (3.22) with $(\delta_1,\delta_2)=(-1,0)$ and obtain the inequality

$$(4q-2)(\beta_q-\alpha_q)-15q+21-4\beta_q\geqq 0.$$

As $\beta_q-\alpha_q>4$ it will suffice to verify that

$$4(4q-2)-15q+21-4\beta_q\geqq 0$$

or equivalently

$$4\beta_q\leqq q+13,$$

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which is (3.23) again. The cases $\delta_1 = 0$ or 1 remain.

If $l = m = 3$ we have to consider the case when $t_3 = t_2 = t_1 = 1$ and $a_3 = 1$ together with $\delta_1 = 0$ or -1 when $3 \leqq \beta_q - \alpha_q \leqq 4$ or $\delta_1 = 0$ or 1 when $4 < \beta_q - \alpha_q \leqq 5$. By (2.6), $T(q, n_3) = F_3(3)$ and so, with the usual substitutions for a_2, a_1 and a_0, it follows from Lemma 2.3 that the condition $T(q, n_3) > 0$ is equivalent to

$$4(q-1)\{qk(\delta_1) + k(\delta_2)\} + 4h(\delta_3) + q - 7 - 2\alpha_q + 2\beta_q > 0,$$

once again easily seen to be true for all integers δ_1, δ_2 and δ_3.

Summing up, if $l = 3$ the only cases which remain to be considered occur when

$$t_m = t_{m-1} = t_{m-2} = 1 \quad \text{and} \quad a_m = 1$$

together with

$$\delta_1 = \begin{cases} 0 \quad \text{or} \quad -1 & \text{if } 3 \leqq \beta_q - \alpha_q \leqq 4 \\ 0 \quad \text{or} \quad 1 & \text{if } 4 < \beta_q - \alpha_q \leqq 5. \end{cases} \tag{3.24}$$

We next take $l = 4$, so that $m \geqq 4$, giving stage 4, and then (3.21) becomes

$$4(q-1)\{qk(\delta_1) + k(\delta_2)\} + 4\delta_3(\delta_3 + 5 + \alpha_q - \beta_q) - 3q + 9 - 2\alpha_q - 2\beta_q \geqq 0. \tag{3.25}$$

When $\delta_1 = 0$ repetition of the argument at stage 3 leaves

$$\delta_2 = \begin{cases} 0 \quad \text{or} \quad -1 & \text{if } 3 \leqq \beta_q - \alpha_q \leqq 4 \\ 0 \quad \text{or} \quad 1 & \text{if } 4 < \beta_q - \alpha_q \leqq 5. \end{cases}$$

When $\delta_1 = -1$, so that $3 \leqq \beta_q - \alpha_q \leqq 4$, (3.25) becomes

$$4q(q-1)(\beta_q - \alpha_q - 3) - 3q + 9 - 2\alpha_q - 2\beta_q + 4(q-1)k(\delta_2) + 4\delta_3(\delta_3 + 5 + \alpha_q - \beta_q) \geqq 0.$$

In this case, $\min[k(\delta_2) : \delta_2 \in \mathbf{Z} - \{0, -1\}] = k(1)$ and $\delta_3(\delta_3 + 5 + \alpha_q - \beta_q)$ has a minimum value when $\delta_3 = -1$. Putting $(\delta_2, \delta_3) = (1, -1)$ in (3.25) leads to the inequality

$$4(q-1)^2(\beta_q - \alpha_q - 3) + 5q - 15 + 6(\beta_q - \alpha_q) - 4\beta_q \geqq 0,$$

which is true. When $\delta_1 = 1$, so that $4 < \beta_q - \alpha_q \leqq 5$, (3.25) becomes

$$4q(q-1)(5 + \alpha_q - \beta_q) - 3q + 9 - 2\alpha_q - 2\beta_q + 4(q-1)k(\delta_2) + 4\delta_3(\delta_3 + 5 + \alpha_q - \beta_q) \geqq 0. \tag{3.26}$$

This time, for integral values of δ_2, the two least values of $k(\delta_2)$ occur when $\delta_2 = 0$ or 1 and the $\min[k(\delta_2) : \delta_2 \in \mathbf{Z} - \{0, 1\}] = k(-1)$. Also $\delta_3(\delta_3 + 5 + \alpha_q - \beta_q)$ has a minimum value when $\delta_3 = 0$ so that putting $(\delta_2, \delta_3) = (-1, 0)$ in (3.26) gives

$$4(q-1)^2(5 + \alpha_q - \beta_q) + 5q + 1 + 2(\beta_q - \alpha_q) - 4\beta_q \geqq 0,$$

which is true.

Thus if $l=4$ the only cases which remain to be considered occur when

$$t_m=t_{m-1}=t_{m-2}=t_{m-3}=1 \quad \text{and} \quad a_m=1$$

together with

$$(\delta_1,\delta_2)=\begin{cases}(0,0),(0,-1),(-1,0) \quad \text{or} \quad (-1,-1) & \text{if } 3\leqq\beta_q-\alpha_q\leqq 4\\ (0,0),(0,1),(1,0) \quad \text{or} \quad (1,1) & \text{if } 4<\beta_q-\alpha_q\leqq 5.\end{cases} \tag{3.27}$$

If $l=m=4$, we can assume that $t_4=t_3=t_2=t_1=1$ and $a_4=1$ together with condition (3.27). In this case, using Lemma 2.3 again, the condition $T(q,n_4)=F_4(4)>0$ is equivalent to

$$4(q-1)\{q^2k(\delta_1)+qk(\delta_2)+k(\delta_3)\}+4h(\delta_4)+q-7-2\alpha_q+2\beta_q>0$$

which is true.

At stage 5 we have $l=5$, so that $m\geqq 5$. Then the inequality (3.21) takes the form

$$4(q-1)\{q^2k(\delta_1)+qk(\delta_2)+k(\delta_3)\}+4\delta_4(\delta_4+5+\alpha_q-\beta_q)-3q+9-2\alpha_q-2\beta_q\geqq 0. \tag{3.28}$$

If $\delta_1=0$, (3.28) is essentially the same as (3.25) with an obvious change of notation. Thus repetition of the argument at stage 4 leaves the following values of (δ_2,δ_3) namely

$$(\delta_2,\delta_3)=\begin{cases}(0,0),(0,-1),(-1,0) \quad \text{or} \quad (-1,-1) & \text{if } 3\leqq\beta_q-\alpha_q\leqq 4\\ (0,0),(0,1),(1,0) \quad \text{or} \quad (1,1) & \text{if } 4<\beta_q-\alpha_q\leqq 5.\end{cases} \tag{3.29}$$

If $\delta_1=-1$, so that $3\leqq\beta_q-\alpha_q\leqq 4$, (3.28) becomes

$$4q^2(q-1)(\beta_q-\alpha_q-3)-3q+9-2\alpha_q-2\beta_q$$

$$+4(q-1)\{qk(\delta_2)+k(\delta_3)\}+4\delta_4(\delta_4+5+\alpha_q-\beta_q)\geqq 0. \tag{3.30}$$

The left-hand side of (3.30) takes a minimum value when $(\delta_2,\delta_3,\delta_4)=(0,0,-1)$ and in this case the inequality becomes

$$4q^2(q-1)(\beta_q-\alpha_q-3)-3q-7-6\alpha_q+2\beta_q\geqq 0. \tag{3.31}$$

If $8q^2-9q+1$ is not a perfect square, $\beta_q-\alpha_q-3>1/(3q)$ by (2.2), and (3.31) will be a consequence of

$$4q(q-1)-9q-21-18\alpha_q+6\beta_q>0,$$

or equivalently

$$4q^2-13q-21+18(\beta_q-\alpha_q)-12\beta_q>0.$$

As $\beta_q-\alpha_q\geqq 3$ it will suffice to show that

$$4q^2-25q+33+12(q-\beta_q)>0$$

which is easily true. If $8q^2-9q+1$ is a perfect square and $\beta_q-\alpha_q=3$, (3.30) reduces to

$$4(q-1)\{qk(\delta_2)+k(\delta_3)\}+4\delta_4(\delta_4+5+\alpha_q-\beta_q)-3q+9-2\alpha_q-2\beta_q\geqq 0,$$

which is (3.25) again with an obvious change of notation. Hence the values of (δ_2,δ_3) which remain are

$$(\delta_2,\delta_3)=(0,0),(0,-1),(-1,0) \quad \text{or} \quad (-1,-1). \tag{3.32}$$

If $\delta_1=1$, so that $4<\beta_q-\alpha_q\leqq 5$, (3.28) is

$$4q^2(q-1)(5+\alpha_q-\beta_q)-3q+9-2\alpha_q-2\beta_q$$

$$+4(q-1)\{qk(\delta_2)+k(\delta_3)\}+4\delta_4(\delta_4+5+\alpha_q-\beta_q)\geqq 0. \tag{3.33}$$

The left-hand side of (3.33) takes a minimum value when $(\delta_2,\delta_3,\delta_4)=(0,0,0)$ and in this case the inequality becomes

$$4q^2(q-1)(5+\alpha_q-\beta_q)-3q+9-2\alpha_q-2\beta_q\geqq 0. \tag{3.34}$$

If $8q^2-9q+1$ is not a perfect square, $5+\alpha_q-\beta_q>1/(3q)$ by (2.3), and consequently (3.34) follows provided that

$$4q^2-13q+27-6(\alpha_q-\beta_q)-12\beta_q>0.$$

As $\beta_q-\alpha_q>4$, it will suffice to show that

$$q(4q-25)+51+12(q-\beta_q)>0,$$

which is true. On the other hand, if $8q^2-9q+1$ is a perfect square and $\beta_q-\alpha_q=5$, (3.33) takes the form

$$4(q-1)\{qk(\delta_2)+k(\delta_3)\}+4\delta_4(\delta_4+5+\alpha_q-\beta_q)-3q+9-2\alpha_q-2\beta_q\geqq 0.$$

Once again this is essentially the same as (3.25) and so the values of (δ_2,δ_3) which remain are

$$(\delta_2,\delta_3)=(0,0),(0,1),(1,0) \quad \text{or} \quad (1,1). \tag{3.35}$$

In the special case when $l=m=5$ the condition $T(q,n_5)=F_5(5)>0$ is equivalent to

$$4(q-1)\{q^3k(\delta_1)+q^2k(\delta_2)+qk(\delta_3)+k(\delta_4)\}+4h(\delta_5)+q-7-2\alpha_q+2\beta_q>0,$$

which is true.

Summing up stage 5, we have proved that $T(q,n_m)>0$ except in the following cases:

$$t_m=t_{m-1}=\cdots=t_{m-4}=1 \quad \text{and} \quad a_m=1$$

together with $\delta_1=0$ and (δ_2,δ_3) satisfying (3.29). In addition when $8q^2-9q+1$ is a perfect square

$$\delta_1=-1 \quad \text{and} \quad (\delta_2,\delta_3)\in\{(0,0),(0,-1),(-1,0),(-1,-1)\} \text{ when } 3\leqq\beta_q-\alpha_q\leqq 4$$

or

$$\delta_1=1 \quad \text{and} \quad (\delta_2,\delta_3)\in\{(0,0),(0,1),(1,0),(1,1)\} \quad \text{when } 4<\beta_q-\alpha_q\leqq 5.$$

By now the critical form for n_m is beginning to emerge. When $8q^2-9q+1$ is not a perfect square, at the end of stage l for $5\leqq l\leqq m$ we are left with the following situation:

$$t_m=t_{m-1}=\cdots=t_{m-l+1}=1 \quad \text{and} \quad a_m=1$$

together with

$$\delta_1=\delta_2=\cdots=\delta_{l-4}=0 \quad \text{and}$$

$$(\delta_{l-3},\delta_{l-2})=\begin{cases}(0,0),(0,-1),(-1,0) \text{ or } (-1,-1) & \text{if } 3\leqq\beta_q-\alpha_q\leqq 4\\ (0,0),(0,1),(1,0) \text{ or } (1,1) & \text{if } 4<\beta_q-\alpha_q\leqq 5.\end{cases}$$

In this case, by (2.5) and (3.17), we have

$$T(q,n_m)=F_m(l-1)+n_{m-l}\sum_{r=m-l+1}^{m}[2a_r-(q-1)t_r]+T(q,n_{m-l}).$$

Using the usual substitutions for a_r $(r=m-l+1,\dots,m)$ together with $t_m=\cdots=t_{m-l+1}=1$ we obtain

$$\sum_{r=m-l+1}^{m}[2a_r-(q-1)t_r]=4-q-\beta_q+2\delta_{l-1}.$$

Then, application of Lemma 2.3 gives

$$T(q,n_m)=\tfrac{1}{4}q^{t_1+\cdots+t_{m-l+1}}\{4(q-1)[qk(\delta_{l-3})+k(\delta_{l-2})]+4h(\delta_{l-1})+q-7-2\alpha_q+2\beta_q\}$$
$$+[4-q-\beta_q+2\delta_{l-1}]n_{m-l}+T(q,n_{m-l}). \tag{3.36}$$

Since $\delta_1,\dots,\delta_n$ are all linearly bounded in terms of q and m, it follows from (1.8) and (3.36) that

$$\frac{S(q,n_m)}{n_m}+h_q=\frac{T(q,n_m)}{n_m}\to 0 \text{ as } m\to\infty \text{ if } m-l \text{ remains fixed,}$$

and this is the critical form mentioned earlier. If $l<m$ we proceed to stage $l+1$, and so on until $l=m$ when the process comes to a halt yielding no further critical cases. In particular, when $l=m$, at the end of stage m we have to consider $T(q,n_m)$ subject to the conditions $t_m=t_{m-1}=\cdots=t_1=1$ and $a_m=1$ together with $\delta_1=\delta_2=\cdots=\delta_{m-4}=0$ and $\delta_{m-3}, \delta_{m-2}\in\{0,1,-1\}$. Using Lemma 2.3 gives

$$\begin{aligned}4T(q,n_m)&=4F_m(m),\\ &=4(q-1)\{q^2k(\delta_{m-3})+qk(\delta_{m-2})+k(\delta_{m-1})\}\\ &\quad+4h(\delta_m)+q-7-2\alpha_q+2\beta_q.\end{aligned}$$

As we have seen $k(x)\geqq 0\ \forall x\in\mathbf{Z}$ and so

$$\begin{aligned}4T(q,n_m)&\geqq 4h(\delta_m)+q-7-2\alpha_q+2\beta_q\\ &>0\end{aligned}$$

by the same argument as that towards the end of stage 1.

When $8q^2-9q+1$ is a perfect square, it is clear that additional critical cases arise. In particular, when $\beta_q-\alpha_q=3$ we have $k(\delta)=\delta(\delta+1)$. In this case at the end of stage l $(5\leqq l\leqq m)$ we are left with

$$\delta_1,\ldots,\delta_{l-4},\delta_{l-3},\delta_{l-2}\in\{0,-1\}.$$

As $k(\delta_i)=0$ for $1\leqq i\leqq l-2$ and $h(\delta_{l-1})=\delta_{l-1}(\delta_{l-1}+3+\alpha_q-\beta_q)=\delta_{l-1}^2$ this leads to

$$\begin{aligned}T(q,n_m)&=\tfrac{1}{4}q^{t_1+\cdots+t_{m-l+1}}[4\delta_{l-1}^2+q-1]\\ &\quad+[4-q-\beta_q+2\delta_{l-1}]n_{m-l}+T(q,n_{m-l}).\end{aligned}$$

and once again

$$\frac{S(q,n_m)}{n_m}+h_q=\frac{T(q,n_m)}{n_m}\to 0 \text{ as } m\to\infty$$

and $m-l$ remains constant. If $l<m$ we can proceed to the next stage as before. When $\beta_q-\alpha_q=5$, $k(\delta)=\delta(\delta-1)$ and the critical cases occur when $\delta_1,\ldots,\delta_{l-2}\in\{0,1\}$; otherwise the situation is similar.

November 1989. Since this paper was submitted for publication the analogous theorem for even values of q has been obtained. A statement of the result and brief outline of the proof will follow.

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Mathematical Institute
University of St Andrews
North Haugh
St Andrews, KY16 9SS