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WEAK CONVERGENCE OF AN ITERATIVE SCHEME FOR GENERALIZED EQUILIBRIUM PROBLEMS

JIAN-WEN PENG ✉ **and JEN-CHIH YAO**

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Abstract

In this paper, we introduce an iterative scheme using an extragradient method for finding a common element of the set of solutions of a generalized equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of the variational inequality for a monotone, Lipschitz-continuous mapping. We obtain a weak convergence theorem for three sequences generated by this process. Based on this result, we also obtain several interesting results. The results in this paper generalize and extend some well-known weak convergence theorems in the literature.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to R and let $B : C \to H$ be a nonlinear mapping, where R is the set of real numbers. Then, Takahashi and Takahashi [21] considered the following generalized equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) + \langle Bx, y - x \rangle \geq 0, \quad \forall\, y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GEP(F)$. If $B = 0$, the generalized equilibrium problem (1.1) becomes the equilibrium problem for $F : C \times C \to R$, which is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall\, y \in C. \tag{1.2}$$

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The set of solutions of (1.2) is denoted by $EP(F)$.

The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games and others; see, for instance, [5] and [1].

Recall that a mapping $S : C \to H$ is nonexpansive [7] if it holds that

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall\, x, y \in C.$$

We denote the set of fixed points of S by $\mathrm{Fix}(S)$. It is known (see [7]) that $\mathrm{Fix}(S)$ is closed convex, but possibly empty. A mapping A of C into H is called monotone if

$$\langle Ax - Ay, x - y\rangle \ge 0$$

for all $x, y \in C$. A mapping A of C into H is called α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \ge \alpha\|Ax - Ay\|^2$$

for all $x, y \in C$. A mapping $A : C \to H$ is called k-Lipschitz-continuous if there exists a positive real number k such that

$$\|Ax - Ay\| \le k\|x - y\|$$

for all $x, y \in C$. It is easy to see that the class of α-inverse-strongly-monotone mappings does not contain some important classes of mappings even in a finite-dimensional case. For example, if the matrix in the corresponding linear complementarity problem is positively semidefinite, but not positively definite, then the mapping A will be monotone and Lipschitz-continuous, but not α-inverse-strongly monotone.

Let the mapping $A : C \to H$ be monotone and k-Lipschitz-continuous. The variational inequality problem is to find a $x \in C$ such that

$$\langle Ax, y - x\rangle \ge 0$$

for all $y \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

Takahashi and Takahashi [21] introduced the following iterative scheme for finding a common element of the set of solutions of problem (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Starting with an arbitrary $u \in C$ and $x_1 \in C$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, & \forall\, y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S[\alpha_n u + (1 - \alpha_n)u_n], & \forall\, n \in N. \end{cases} \tag{1.3}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.3) converges strongly to $z = P_{\mathrm{Fix}(S)\cap GEP(F)}u$.

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Some methods have been proposed to solve the problem (1.2); see, for instance, [5, 1, 4, 20, 18, 17, 12]. Recently, Combettes and Hirstoaga [4] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. Takahashi and Takahashi [20] introduced the following iterative scheme by the viscosity approximation method for finding a common element of the set of solution (1.2) and the set of fixed points of a nonexpansive mapping in a Hilbert space. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall\, y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n, & \forall\, n \in N. \end{cases} \tag{1.4}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.4) converge strongly to $z \in \mathrm{Fix}(S) \cap EP(F)$, where $z = P_{\mathrm{Fix}(S)\cap EP(F)} f(z)$, where f is a contraction on H. Tada and Takahashi [18] introduced some iterative schemes for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of a nonexpansive mapping and proved a strong convergence theorem and a weak convergence theorem in a Hilbert space. Su *et al.* [17] introduced the following iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for an α-inverse strongly monotone mapping in a Hilbert space. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall\, y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)SP_C(u_n - \lambda_n Au_n), & \forall\, n \in N. \end{cases} \tag{1.5}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{r_n\}$ and $\{\lambda_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.5) converge strongly to $z \in \mathrm{Fix}(S) \cap EP(F) \cap VI(C, A)$, where $z = P_{\mathrm{Fix}(S)\cap EP(F)\cap VI(C,A)} f(z)$.

On the other hand, to solve the variational inequality problem in the finite-dimensional Euclidean R^n, Korpelevich [9] introduced the following so-called extragradient method:

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases} \tag{1.6}$$

for every $n = 0, 1, 2, \ldots,$ where $\lambda \in (0, 1/k)$. He showed that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.6), converge to the same point $z \in VI(C, A)$. The idea of the extragradient iterative process introduced by Korpelevich was successfully generalized and extended not only in Euclidean but also in Hilbert and Banach spaces; see, for example, the recent papers of He *et al.* [8], Gárciga *et al.* [6], Ceng and Yao [3], Solodov and Svaiter [16], and Solodov [15].

Moreover, Zeng and Yao [24] and Nadezhkina and Takahashi [10] introduced some iterative processes by an extragradient method for finding the common element of the set of fixed points of nonexpansive mappings and the set of solutions of variational inequality problem for a monotone, Lipschitz-continuous mapping. Yao and Yao [23] introduced an iterative process based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings and the set of solutions of variational inequality problem for an α-inverse strongly monotone mapping. Plubtieng and Punpaeng [12] introduced an iterative process based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality problem for α-inverse strongly monotone mappings.

In the present paper, by using the extragradient methods, we introduce an iterative process for finding the common element of the set of fixed points of a nonexpansive mapping, the set of solutions of a generalized equilibrium problem and the set of solutions of the variational inequality problem for a monotone, Lipschitz-continuous mapping in a Hilbert space. Then, we obtain a weak convergence theorem for three sequences generated by this process. Based on this result, we also obtain several interesting results. The results in this paper generalize and extend some well-known weak convergence theorems in the literature.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let symbols $\rightarrow$ and $\rightharpoonup$ denote strong and weak convergence, respectively. In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C. It is also known that $P_C x \in C$ and

$$\langle x - P_C(x), P_C(x) - y \rangle \geq 0 \tag{2.1}$$

for all $x \in H$ and $y \in C$.

It is easy to see that (2.1) is equivalent to

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2 \tag{2.2}$$

for all $x \in H$ and $y \in C$. Let A be a monotone mapping of C into H. In the context of the variational inequality problem the characterization of projection (2.1) implies the following:

$$u \in VI(C, A) \Rightarrow u = P_C(u - \lambda Au), \quad \lambda > 0,$$

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and

$$u = P_C(u - \lambda Au) \quad \text{for some } \lambda > 0 \Rightarrow u \in VI(C, A).$$

It is also known that H satisfies the Opial's condition [11], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone, k-Lipschitz-continuous mapping of C into H and let $N_C v$ be normal cone to C at $v \in C$, that is, $N_C v = \{w \in H \mid \langle v - u, w\rangle \geq 0, \forall\, u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$ (see [13]).

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

We recall some lemmas that are needed in the rest of this paper.

LEMMA 2.1 [1]. *Let C be a nonempty closed convex subset of H, let F be a bifunction from $C \times C$ to R satisfying conditions (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall\, y \in C.$$

LEMMA 2.2 [4]. *Let C be a nonempty closed convex subset of H, let F be a bifunction from $C \times C$ to R satisfying conditions (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall\, y \in C \right\}$$

for all $x \in H$. Then, the following statements hold:

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(1) *T_r is single-valued;*
(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(3) *$F(T_r) = EP(F)$;*
(4) *$EP(F)$ is closed and convex.*

LEMMA 2.3 [14]. *Let H be a real Hilbert space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for all $n = 0, 1, 2, \ldots$, and let $\{v_n\}$ and $\{w_n\}$ be sequences in H such that $\limsup_{n\to\infty} \|v_n\| \leq c$, $\limsup_{n\to\infty} \|w_n\| \leq c$, $\limsup_{n\to\infty} \|\alpha_n v_n + (1-\alpha_n)w_n\| = c$ for some $c \geq 0$. Then, $\lim_{n\to\infty} \|v_n - w_n\| = 0$.*

LEMMA 2.4 [22]. *Let H be a real Hilbert space, let D be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in D$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\|,$$

for every $n = 0, 1, 2, \ldots$. Then, the sequence $\{P_D x_n\}$ converges strongly to some $z \in D$.

3. A weak convergence theorem

In this section, we show a weak convergence theorem which solves the problem of finding a common element of the set of solutions of a generalized equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of the variational inequality for a monotone, Lipschitz-continuous mapping in a Hilbert space.

THEOREM 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying conditions (A1)–(A4) and let B be an α-inverse-strongly monotone mapping of C into H. Let A be a monotone and k-Lipschitz-continuous mapping of C into H. Let S be a nonexpansive mapping of C into H such that $\Omega = \mathrm{Fix}(S) \cap VI(C, A) \cap GEP(F) \neq \emptyset$. Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in H, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(u_n - \lambda_n A y_n), \end{cases}$$

for every $n = 0, 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ for some $\gamma, \tau \in (0, 2\alpha)$. Then, $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge weakly to $w \in \Omega$, where $w = \lim_{n\to\infty} P_\Omega x_n$.

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PROOF. Put $t_n = P_C(u_n - \lambda_n Ay_n)$ for every $n = 0, 1, 2, \ldots$. Let $u \in \Omega$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.2. Then

$$u = P_C(u - \lambda_n Au) = T_{r_n}(u - r_n Bu).$$

From $u_n = T_{r_n}(x_n - r_n Bx_n) \in C$ and the α-inverse-strongly monotonicity of B, we have

$$\begin{aligned}
\|u_n - u\|^2 &= \|T_{r_n}(x_n - r_n Bx_n) - T_{r_n}(u - r_n Bu)\|^2 \\
&\le \|x_n - r_n Bx_n - (u - r_n Bu)\|^2 \\
&\le \|x_n - u\|^2 - 2r_n\langle x_n - u, Bx_n - Bu\rangle + r_n^2\|Bx_n - Bu\|^2 \\
&\le \|x_n - u\|^2 - 2r_n\alpha\|Bx_n - Bu\|^2 + r_n^2\|Bx_n - Bu\|^2 \\
&= \|x_n - u\|^2 + r_n(r_n - 2\alpha)\|Bx_n - Bu\|^2 \\
&\le \|x_n - u\|. \qquad (3.1)
\end{aligned}$$

From (2.2), the monotonicity of A, and $u \in VI(C, A)$, we have

$$\begin{aligned}
\|t_n - u\|^2 &\le \|u_n - \lambda_n Ay_n - u\|^2 - \|u_n - \lambda_n Ay_n - t_n\|^2 \\
&= \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n\langle Ay_n, u - t_n\rangle \\
&= \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n(\langle Ay_n - Au, u - y_n\rangle \\
&\quad + \langle Au, u - y_n\rangle + \langle Ay_n, y_n - t_n\rangle) \\
&\le \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\
&\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 \\
&\quad + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\
&= \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\quad + 2\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle.
\end{aligned}$$

Further, since $y_n = P_C(u_n - \lambda_n Au_n)$ and A is k-Lipschitz-continuous, we have

$$\begin{aligned}
\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle &= \langle u_n - \lambda_n Au_n - y_n, t_n - y_n\rangle \\
&\quad + \langle \lambda_n Au_n - \lambda_n Ay_n, t_n - y_n\rangle \\
&\le \langle \lambda_n Au_n - \lambda_n Ay_n, t_n - y_n\rangle \\
&\le \lambda_n k\|u_n - y_n\|\|t_n - y_n\|.
\end{aligned}$$

So,

$$\begin{aligned}
\|t_n - u\|^2 &\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k\|u_n - y_n\|\|t_n - y_n\| \\
&\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\quad + {\lambda_n}^2 k^2\|u_n - y_n\|^2 + \|t_n - y_n\|^2 \\
&= \|u_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2 \\
&\le \|u_n - u\|^2. \qquad (3.2)
\end{aligned}$$

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Therefore from (3.1) and (3.2), $x_{n+1} = \alpha_n x_n + (1-\alpha_n)St_n$ and $u = Su$,

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|\alpha_n x_n + (1-\alpha_n)St_n - u\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|St_n - u\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|t_n - u\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)[\|u_n - u\|^2 + ({\lambda_n}^2k^2 - 1)\|u_n - y_n\|^2] \\ &\le \|x_n - u\|^2 + (1-\alpha_n)({\lambda_n}^2k^2 - 1)\|u_n - y_n\|^2 \\ &\le \|x_n - u\|^2, \end{aligned} \tag{3.3}$$

for every $n = 0, 1, 2, \ldots$. Therefore, there exists $\theta = \lim_{n\to\infty}\|x_n - u\|$ and $\{x_n\}$ is bounded. From (3.1) and (3.2), we also obtain that $\{t_n\}$ and $\{u_n\}$ are bounded.

By (3.3), we have

$$\|u_n - y_n\|^2 \le \frac{1}{(1-\alpha_n)(1-{\lambda_n}^2k^2)}(\|x_n - u\|^2 - \|x_{n+1} - u\|^2). \tag{3.4}$$

Hence, $\|u_n - y_n\| \to 0$.

By the same process as in (3.2), we also have

$$\begin{aligned}\|t_n - u\|^2 &\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k\|u_n - y_n\|\|t_n - y_n\| \\ &\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \|u_n - y_n\|^2 + {\lambda_n}^2k^2\|t_n - y_n\|^2 \\ &= \|u_n - u\|^2 + ({\lambda_n}^2k^2 - 1)\|y_n - t_n\|^2.\end{aligned}$$

Then, in contrast to (3.3),

$$\begin{aligned}\|x_{n+1} - u\|^2 &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|t_n - u\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)[\|u_n - u\|^2 + ({\lambda_n}^2k^2 - 1)\|y_n - t_n\|^2] \\ &\le \|x_n - u\|^2 + (1-\alpha_n)({\lambda_n}^2k^2 - 1)\|y_n - t_n\|^2 \\ &\le \|x_n - u\|^2\end{aligned}$$

and, rearranging as in (3.4),

$$\|t_n - y_n\|^2 \le \frac{1}{(1-\alpha_n)(1-{\lambda_n}^2k^2)}(\|x_n - u\|^2 - \|x_{n+1} - u\|^2).$$

Hence, $\|t_n - y_n\| \to 0$. From $\|u_n - t_n\| \le \|u_n - y_n\| + \|y_n - t_n\|$ we also have $\|u_n - t_n\| \to 0$. As A is k-Lipschitz-continuous, we have $\|Ay_n - At_n\| \to 0$.

From (3.3) and (3.1), we have

$$\begin{aligned}\|x_{n+1} - u\|^2 &\le \alpha_n^2\|x_n - u\|^2 + (1-\alpha_n)[\|u_n - u\|^2 + (\lambda_n^2k^2 - 1)\|u_n - y_n\|^2] \\ &\le \alpha_n^2\|x_n - u\|^2 + (1-\alpha_n)\|u_n - u\|^2 \\ &\le \alpha_n^2\|x_n - u\|^2 + (1-\alpha_n)[\|x_n - u\|^2 + r_n(r_n - 2\alpha)\|Bx_n - Bu\|^2] \\ &= \|x_n - u\|^2 + (1-\alpha_n)r_n(r_n - 2\alpha)\|Bx_n - Bu\|^2.\end{aligned}$$

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It follows that

$$\begin{aligned}(1-d)\gamma(2\alpha-\tau)\|Bx_n-Bu\|^2 &\le (1-\alpha_n)r_n(2\alpha-r_n)\|Bx_n-Bu\|^2\\ &\le \|x_n-u\|^2-\|x_{n+1}-u\|^2.\end{aligned}$$

Hence, we obtain $\|Bx_n-Bu\|\to 0$.

For $u\in\Omega$, from Lemma 2.2, we have

$$\begin{aligned}\|u_n-u\|^2 &= \|T_{r_n}(x_n-r_nBx_n)-T_{r_n}(u-r_nBu)\|^2\\ &\le \langle T_{r_n}(x_n-r_nBx_n)-T_{r_n}(u-r_nBu),\, x_n-r_nBx_n-(u-r_nBu)\rangle\\ &= \tfrac{1}{2}\{\|u_n-u\|^2+\|x_n-r_nBx_n-(u-r_nBu)\|^2-\|x_n-r_nBx_n\\ &\quad -(u-r_nBu)-(u_n-u)\|^2\}\\ &\le \tfrac{1}{2}\{\|u_n-u\|^2+\|x_n-u\|^2\\ &\quad -\|x_n-r_nBx_n-(u-r_nBu)-(u_n-u)\|^2\}\\ &= \tfrac{1}{2}\{\|u_n-u\|^2+\|x_n-u\|^2-\|x_n-u_n\|^2\\ &\quad +2r_n\langle Bx_n-Bu,\, x_n-u_n\rangle-r_n^2\|Bx_n-Bu\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}\|u_n-u\|^2 \le{}& \|x_n-u\|^2-\|x_n-u_n\|^2\\ &+2r_n\langle Bx_n-Bu,\, x_n-u_n\rangle-r_n^2\|Bx_n-Bu\|^2.\end{aligned}$$

Then, by (3.3) and (3.2),

$$\begin{aligned}\|x_{n+1}-u\|^2 &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|t_n-u\|^2\\ &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|u_n-u\|^2\\ &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)[\|x_n-u\|^2-\|x_n-u_n\|^2\\ &\quad +2r_n\langle Bx_n-Bu,\, x_n-u_n\rangle-r_n^2\|Bx_n-Bu\|^2]\\ &\le \|x_n-u\|^2-(1-\alpha_n)\|x_n-u_n\|^2\\ &\quad +(1-\alpha_n)2r_n\|Bx_n-Bu\|\|x_n-u_n\|.\end{aligned}$$

Hence,

$$\begin{aligned}(1-d)\|x_n-u_n\|^2 &\le (1-\alpha_n)\|x_n-u_n\|^2\\ &\le \|x_n-u\|^2-\|x_{n+1}-u\|^2\\ &\quad +(1-\alpha_n)2r_n\|Bx_n-Bu\|\|x_n-u_n\|.\end{aligned}$$

Since $\|Bx_n-Bu\|\to 0$, $\{x_n\}$ and $\{u_n\}$ are bounded, we obtain $\|x_n-u_n\|\to 0$. From $\|t_n-x_n\|\le\|t_n-u_n\|+\|x_n-u_n\|$ we also have $\|t_n-x_n\|\to 0$.

For $u\in\Omega$, since $\|St_n-u\|\le\|t_n-u\|\le\|x_n-u\|$, we have

$$\limsup_{n\to\infty}\|St_n-u\|\le\theta.$$

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Furthermore,

$$\lim_{n\to\infty} \|\alpha_n(x_n - u) + (1-\alpha_n)(St_n - u)\| = \lim_{n\to\infty} \|x_{n+1} - u\| = \theta.$$

By Lemma 2.3, we obtain $\lim_{n\to\infty} \|St_n - x_n\| = 0$.

Since

$$\|Sx_n - x_n\| \le \|Sx_n - St_n\| + \|St_n - x_n\| \le \|x_n - t_n\| + \|St_n - x_n\|,$$

we have

$$\|Sx_n - x_n\| \to 0. \tag{3.5}$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{ni} \rightharpoonup w$. From $\|x_n - u_n\| \to 0$, we obtain that $u_{ni} \rightharpoonup w$. Since $\{u_{ni}\} \subset C$ and C is closed and convex, we obtain $w \in C$.

First, we show $w \in GEP(F)$. By $u_n = T_{r_n}(x_n - r_n Bx_n)$, we know that

$$F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall\, y \in C.$$

It follows from condition (A2) that

$$\langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n), \quad \forall\, y \in C.$$

Hence,

$$\langle Bx_{n_i}, y - u_{n_i}\rangle + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \ge F(y, u_{n_i}), \quad \forall\, y \in C. \tag{3.6}$$

For t with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we obtain $y_t \in C$. So, from (3.6) we have

$$\begin{aligned}\langle y_t - u_{n_i}, By_t\rangle &\ge \langle y_t - u_{n_i}, By_t\rangle - \langle y_t - u_{n_i}, Bx_{n_i}\rangle \\ &\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(y_t, u_{n_i}) \\ &= \langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle + \langle y_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle \\ &\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(y_t, u_{n_i}).\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Bu_{n_i} - Bx_{n_i}\| \to 0$. Further, from the inverse-strongly monotonicity of B, we have $\langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle \ge 0$. So, from condition (A4) we have

$$\langle y_t - w, By_t\rangle \ge F(y_t, w), \tag{3.7}$$

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as $i \to \infty$. From conditions (A1) and (A4) and (3.7), we also have

$$\begin{aligned} 0 = F(y_t, y_t) &\le tF(y_t, y) + (1-t)F(y_t, w) \\ &\le tF(y_t, y) + (1-t)\langle y_t - w, By_t\rangle \\ &= tF(y_t, y) + (1-t)t\langle y - w, By_t\rangle \end{aligned}$$

and, hence,

$$0 \le F(y_t, y) + (1-t)\langle y - w, By_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F(w, y) + \langle y - w, Bw\rangle \ge 0. \tag{3.8}$$

This implies that $w \in GEP(F)$.

We next show that $w \in \mathrm{Fix}(S)$. Assume $w \notin \mathrm{Fix}(S)$. Since $x_{n_i} \rightharpoonup w$ and $w \ne Sw$, from the Opial theorem and (3.5) we have

$$\begin{aligned} \liminf_{i\to\infty} \|x_{n_i} - w\| &< \liminf_{i\to\infty} \|x_{n_i} - Sw\| \\ &\le \liminf_{i\to\infty} \{\|x_{n_i} - Sx_{n_i}\| + \|Sx_{n_i} - Sw\|\} \\ &\le \liminf_{i\to\infty} \|x_{n_i} - w\|. \end{aligned}$$

This is a contradiction. So, we obtain $w \in \mathrm{Fix}(S)$.

Finally, we show $w \in VI(C, A)$. Let

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

where $N_C v$ is the normal cone to C at $v \in C$. We have already mentioned that in this case the mapping T is maximal monotone, and $0 \in Tv$ if and only if $v \in VI(C, A)$. Let $(v, g) \in G(T)$. Then $Tv = Av + N_C v$ and, hence, $g - Av \in N_C v$. So, we have $\langle v - t, g - Av\rangle \ge 0$ for all $t \in C$. On the other hand, from $t_n = P_C(u_n - \lambda_n Ay_n)$ and $v \in C$ we have

$$\langle u_n - \lambda_n Ay_n - t_n, t_n - v\rangle \ge 0$$

and, hence,

$$\left\langle v - t_n, \frac{t_n - u_n}{\lambda_n} + Ay_n \right\rangle \ge 0.$$

Therefore,

$$\begin{aligned} \langle v - t_{n_i}, g\rangle &\ge \langle v - t_{n_i}, Av\rangle \\ &\ge \langle v - t_{n_i}, Av\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i} \right\rangle \\ &= \left\langle v - t_{n_i}, Av - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\ &= \left\langle v - t_{n_i}, Av - At_{n_i} + At_{n_i} - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \end{aligned}$$

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$$= \langle v - t_{n_i}, Av - At_{n_i} \rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i} \rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle$$
$$\geq \langle v - t_{n_i}, At_{n_i} - Ay_{n_i} \rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle.$$

Hence, we obtain $\langle v - w, g \rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $w \in T^{-1}0$ and, hence, $w \in VI(C, A)$. This implies $w \in \Omega$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z$. Then $z \in \Omega$. Let us show $w = z$. Assume that $w \neq z$. From the Opial condition,

$$\begin{aligned}\lim_{n\to\infty} \|x_n - w\| &= \liminf_{i\to\infty} \|x_{n_i} - w\| < \liminf_{i\to\infty} \|x_{n_i} - z\| \\ &= \lim_{n\to\infty} \|x_n - z\| = \liminf_{j\to\infty} \|x_{n_j} - z\| \\ &< \liminf_{j\to\infty} \|x_{n_j} - w\| = \lim_{n\to\infty} \|x_n - w\|.\end{aligned}$$

This is a contradiction. Thus, we have $w = z$. This implies that $x_n \rightharpoonup w \in \Omega$. Since $\|x_n - u_n\| \to 0$, we have $u_n \rightharpoonup w \in \Omega$. Since $\|y_n - u_n\| \to 0$, we also have $y_n \rightharpoonup w \in \Omega$.

Now put $w_n = P_\Omega x_n$. We show that $w = \lim_{n\to\infty} w_n$.

From $w_n = P_\Omega x_n$ and $w \in \Omega$, we have

$$\langle w - w_n, w_n - x_n \rangle \geq 0.$$

From (3.3) and Lemma 2.4, we know that $\{w_n\}$ converges strongly to some $w_0 \in \Omega$. Then

$$\langle w - w_0, w_0 - w \rangle \geq 0$$

and, hence, $w = w_0$. The proof is now complete. □

4. Applications

By Theorem 3.1, we can obtain the following interesting weak convergence theorems in a real Hilbert space.

THEOREM 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying conditions (A1)–(A4) and B be an α-inverse-strongly monotone mapping of C into H. Let S be a nonexpansive mapping of C into H such that* $\mathrm{Fix}(S) \cap GEP(F) \neq \emptyset$. *Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in H, \\ F(u_n, y) + \langle Bx_n, y - u_n \rangle + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall\, y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S u_n, \end{cases}$$

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for every $n = 0, 1, 2, \ldots$. *If* $\{\alpha_n\} \subset [c, d]$ *for some* $c, d \in (0, 1)$ *and* $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\alpha)$. *Then,* $\{x_n\}$ *and* $\{u_n\}$ *converge weakly to* $w \in \mathrm{Fix}(S) \cap GEP(F)$, *where* $w = \lim_{n\to\infty} P_{\mathrm{Fix}(S)\cap GEP(F)} x_n$.

PROOF. Putting $A = 0$, by Theorem 3.1 we obtain the desired result. □

THEOREM 4.2 [18, Theorem 4.1]. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C \times C$ *to* R *satisfying conditions (A1)–(A4). Let* S *be a nonexpansive mapping of* C *into* H *such that* $\mathrm{Fix}(S) \cap EP(F) \neq \emptyset$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by*

$$\begin{cases} x_0 = x \in H, \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S u_n, \end{cases}$$

for every $n = 0, 1, 2, \ldots$. *If* $\{\alpha_n\} \subset [c, d]$ *for some* $c, d \in (0, 1)$ *and* $\{r_n\} \subset [\gamma, +\infty)$ *for some* $\gamma > 0$. *Then,* $\{x_n\}$ *and* $\{u_n\}$ *converge weakly to* $w \in \mathrm{Fix}(S) \cap EP(F)$, *where* $w = \lim_{n\to\infty} P_{\mathrm{Fix}(S)\cap EP(F)} x_n$.

PROOF. Putting $A = 0$ and $B = 0$, by Theorem 3.1 we obtain the desired result. □

THEOREM 4.3 [10, Theorem 4.1]. *Let* H *be a real Hilbert space. Let* A *be a monotone and* k*-Lipschitz-continuous mapping of* C *into* H. *Let* S *be a nonexpansive mapping of* H *into itself such that* $VI(C, A) \cap \mathrm{Fix}(S) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 = x \in H, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S(x_n - \lambda_n A(x_n - \lambda_n A x_n)), \end{cases}$$

for every $n = 0, 1, 2, \ldots$. *If* $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 1/k)$, $\{\alpha_n\} \subset [c, d]$ *for some* $c, d \in (0, 1)$. *Then,* $\{x_n\}$ *converges weakly to* $w \in VI(C, A) \cap \mathrm{Fix}(S)$, *where* $w = \lim_{n\to\infty} P_{VI(C,A)\cap \mathrm{Fix}(S)} x_n$.

PROOF. Putting $C = H$, $F = 0$ and $B = 0$, then $P_H = I$, by Theorem 3.1 we obtain the desired result.

Let $P : H \to 2^H$ be a maximal monotone mapping. Then, for any $x \in H$ and $r > 0$, consider $J_r^P x = \{z \in H \mid z + rBz \ni x\}$. Such J_r^P is called the resolvent of B and is denoted by $J_r^P = (I + rP)^{-1}$. □

THEOREM 4.4 [10, Theorem 4.2]. *Let* H *be a real Hilbert space. Let* A *be a monotone and* k*-Lipschitz-continuous mapping of* C *into* H. *Let* $P : H \to 2^H$ *be a maximal monotone mapping such that* $A^{-1}(0) \cap P^{-1}(0) \neq \emptyset$. *Let* J_r^P *be the resolvent of* P *for each* $r > 0$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 = x \in H, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_r^P(x_n - \lambda_n A(x_n - \lambda_n A x_n)), \end{cases}$$

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for every $n = 0, 1, 2, \ldots$. *If* $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 1/k)$, $\{\alpha_n\} \subset [c, d]$ *for some* $c, d \in (0, 1)$. *Then,* $\{x_n\}$ *converges weakly to* $w \in A^{-1}(0) \cap P^{-1}(0)$, *where* $w = \lim_{n\to\infty} P_{A^{-1}(0)\cap P^{-1}(0)} x_n$.

PROOF. Putting $C = H$, $F = 0$ and $B = 0$, then we have $P_H = I$, $A^{-1}(0) = VI(C, H)$ and $F(J_r^P) = P^{-1}(0)$. We also know that J_r^P is nonexpansive (see [19]); by Theorem 3.1 we obtain the desired result. □

A mapping T of a closed convex subset C into itself is pseudocontractive if there holds that

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2$$

for all $x, y \in C$; see [2]. Obviously, the class of pseudocontractive mappings is more general than the class of nonexpansive mappings. Now we prove a weak convergence theorem of a new iterative process for finding a common element of the set of solutions of a generalized equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of fixed points of a Lipschitz pseudocontractive mapping.

THEOREM 4.5. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C \times C$ *to* R *satisfying conditions (A1)–(A4) and let* B *be an* α*-inverse-strongly monotone mapping of* C *into* H. *Let* T *be a pseudocontrative and* m*-Lipschitz-continuous mapping of* C *into itself. Let* S *be a nonexpansive mapping of* C *into itself such that* $\mathrm{Fix}(S) \cap \mathrm{Fix}(T) \cap GEP(F) \neq \emptyset$. *Let* $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{cases} x_1 = x \in H, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall\, y \in C, \\ y_n = u_n - \lambda_n(u_n - Tu_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(u_n - \lambda_n(y_n - Ty_n)), \end{cases}$$

for every $n = 1, 2, \ldots$. *If* $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 1/(m + 1))$, $\{\alpha_n\} \subset [c, d]$ *for some* $c, d \in (0, 1)$ *and* $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\alpha)$. *Then,* $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *converge weakly to* $w \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \cap GEP(F)$, *where* $w = \lim_{n\to\infty} P_{\mathrm{Fix}(S)\cap\mathrm{Fix}(T)\cap GEP(F)} x_n$.

PROOF. Let $A = I - T$. Let us show the mapping A is monotone and $(m + 1)$-Lipschitz-continuous. From the definition of the mapping A, we have

$$\begin{aligned} \langle Ax - Ay, x - y\rangle &= \langle x - y - Tx + Ty, x - y\rangle \\ &= \|x - y\|^2 - \langle Tx - Ty, x - y\rangle \\ &\ge \|x - y\|^2 - \|x - y\|^2 = 0. \end{aligned}$$

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So, A is monotone. We also have

$$\begin{aligned}\|Ax - Ay\|^2 &= \|(I-T)x - (I-T)y\| \\ &= \|x-y\|^2 + \|Tx - Ty\|^2 - 2\langle x-y, Tx - Ty\rangle \\ &\le \|x-y\|^2 + m\|x-y\|^2 + 2\|x-y\|\|Tx-Ty\| \\ &\le \|x-y\|^2 + m\|x-y\|^2 + 2m\|x-y\|^2 = (m+1)^2\|x-y\|^2.\end{aligned}$$

So, we have $\|Ax - Ay\| \le (m+1)\|x-y\|$ and A is $(m+1)$-Lipschitz-continuous. It is easy to check that $\mathrm{Fix}(T) = VI(C, A)$. By Theorem 3.1 we obtain the desired result. □

THEOREM 4.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a pseudocontractive and m-Lipschitz-continuous mapping of C into itself. Let $P : H \to 2^H$ be a maximal monotone mapping such that* $\mathrm{Fix}(T) \cap P^{-1}(0) \neq \emptyset$. *Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in H, \\ y_n = x_n - \lambda_n(x_n - Tx_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)J_r^P(x_n - \lambda_n(y_n - Ty_n)), \end{cases}$$

for every $n = 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/(m+1))$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ for some $\gamma, \tau \in (0, 2\alpha)$. Then, $\{x_n\}$ and $\{y_n\}$ converge weakly to $w \in \mathrm{Fix}(T) \cap P^{-1}(0)$, where $w = \lim_{n\to\infty} P_{\mathrm{Fix}(T)\cap P^{-1}(0)}x_n$.

PROOF. Let $C = H$, $F(x, y) = 0$ for $x, y \in H$, we have $u_n = x_n$ for every $n = 0, 1, 2, \ldots$. We also know that J_r^P is nonexpansive, $F(J_r^B) = B^{-1}(0)$ and $P_H = I$, by Theorem 4.5 we obtain the desired result. □

THEOREM 4.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let B be an α-inverse-strongly monotone mapping of C into H. Let A be a monotone and k-Lipschitz-continuous mapping of C into H. Let S be a nonexpansive mapping of C into H such that* $\mathrm{Fix}(S) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. *Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in H, \\ u_n = P_C(x_n - r_n Bx_n), \\ y_n = P_C(u_n - \lambda_n Au_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)SP_C(u_n - \lambda_n Ay_n), \end{cases}$$

for every $n = 0, 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ for some $\gamma, \tau \in (0, 2\alpha)$. Then, $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge weakly to $w \in \mathrm{Fix}(S) \cap VI(C, A) \cap VI(C, B)$, where $w = \lim_{n\to\infty} P_{\mathrm{Fix}(S)\cap VI(C,A)\cap VI(C,B)}x_n$.

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PROOF. In Theorem 3.1, put $F = 0$. Then, we obtain that

$$\langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in C,\ \forall\, n \in N.$$

This implies that

$$\langle y - u_n, u_n - (x_n - r_n Bx_n)\rangle \geq 0, \quad \forall\, y \in C, \forall\, n \in N.$$

So, we obtain that $u_n = P_C(x_n - r_n Bx_n)$ for all $n \in N$. Then we obtain the desired result from Theorem 3.1. □

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JIAN-WEN PENG, College of Mathematics and Computer Science,
Chongqing Normal University, Chongqing 400047, People's Republic of China
e-mail: jwpeng6@yahoo.com.cn

JEN-CHIH YAO, Department of Applied Mathematics,
National Sun Yat-sen University Kaohsiung, Taiwan 804, R.O.C.