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FACTORIZATION OF ANALYTIC FUNCTIONS WITH VALUES IN NON-COMMUTATIVE L_1-SPACES AND APPLICATIONS

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1. Introduction and background. Let X be a Banach space such that X^* is a von Neumann algebra. We prove that X has the analytic Radon-Nikodym property (in short: ARNP). More precisely we show that for any function f in $H^1(X)$ we have

$$\|f(0)\|_X^2 + \tfrac{1}{2}\|f - f(0)\|_{H^1(X)}^2 \leqq \|f\|_{H^1(X)}^2.$$

This implies the ARNP for X as well as for all the Banach spaces which are finitely representable in X. The proof uses a C^*-algebraic formulation of the classical factorization theorems for matrix valued H^1-functions. As a corollary we prove (for instance) that if $A \subset B$ is a C^*-subalgebra of a C^*-algebra B, then every operator from A into H^∞ extends to an operator from B into H^∞ with the same norm. We include some remarks on the ARNP in connection with the complex interpolation method.

Finally we also show that the Banach space c_1 (of all the trace class operators on l_2) fails the "analytic U.M.D. property", while all the (commutative) L_1-spaces have this property.

Let now X be a general complex Banach space, and let $D = \{z \in \mathbf{C} \mid |z| < 1\}$ be the open unit disc. We will denote by $H^\infty(X)$ the space of all bounded analytic functions $f : D \longrightarrow X$ equipped with the norm

$$\|f\|_\infty = \sup_{z \in D} \|f(z)\|.$$

Let m be the normalized Haar measure on the torus $T = \mathbf{R}/2\pi\mathbf{Z}$.

More generally, for $0 < p < \infty$, we will denote by $H^p(X)$ the space of all analytic functions $f : D \longrightarrow X$ such that

$$(1.1) \qquad \|f\|_{H^p(X)} = \sup_{r<1} \left(\int \|f(re^{it})\|^p dm(t) \right)^{1/p} \text{ is finite.}$$

Equipped with this norm, this space becomes a Banach space if $1 \leqq p < \infty$ (a quasi-Banach space if $0 < p < 1$). Note for future reference that since $x \longrightarrow \|x\|^p$ is subharmonic, we have

$$(1.2) \qquad \|f\|_{H^p(X)} = \lim_{r \to 1} \uparrow \left(\int \|f(re^{it})\|^p dm(t) \right)^{1/p}.$$

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A Banach space X is said to have the analytic Radon-Nikodym property (ARNP in short) if every function in $H^\infty(X)$ admits radial limits in almost every (a.e. in short) point of the unit circle.

This property was introduced in [**7**] and has been extensively studied recently (c.f. [**13**], [**14**], [**15**], [**16**], [**18**], [**24**]).

For instance, we recall

THEOREM 1.1. ([**7**], [**13**]) *Let X be a complex Banach space. Then the following properties are equivalent*

(i) *The space X has the* ARNP.

(ii) *For some $1 \leqq p \leqq \infty$, every function in $H^p(X)$ has radial limits in* a.e. *point of the circle.*

(iii) *For all $1 \leqq p \leqq \infty$, every function in $H^p(X)$ has radial limits in* a.e. *point of the circle.*

(iv) *The set of all polynomials with coefficients in X is dense in $H^p(X)$.*

Let $\mathcal{B}$ be the Borel σ-field on the torus $T = \mathbf{R}/2\pi\mathbf{Z}$. Then the above properties are equivalent to

(v) *Every vector measure $\mu : \mathcal{B} \longrightarrow X$ with total variation $|\mu|$ satisfying $|\mu| \ll m$ and such that*

$$\int e^{int} d\mu(t) = 0 \quad \forall n > 0 \tag{1.3}$$

possesses a Radon-Nikodym derivative in $L_1(T, m : X)$.

Note that (v) may be viewed as a Banach space valued analogue of the F. and M. Riesz theorem (c.f. e.g. [**22**] p. 47).

If one omits the condition (1.3), then the above property characterizes the Radon-Nikodym property (RNP in short) for which we refer the reader to [**12**].

It is known that the RNP is closely connected with the martingale convergence theorem. In particular, if $1 \leqq p \leqq \infty$ (resp. $1 < p < \infty$), then X has the RNP if and only if every X- valued martingale which is bounded in $L_p(X)$ converges a.e. (resp. converges in $L_p(X)$) (c.f. [**9**]).

In the analytic case an analogous result holds but the convergence theorem must be restricted to a special class of martingales which we now define.

We consider the infinite dimensional torus $T^{\mathbf{N}}$ equipped with the probability measure $\mathbf{P} = m^{\mathbf{N}}$. Let us denote by $t_0, t_1, t_2, \ldots$ the coordinates of a point t in $T^{\mathbf{N}}$. Let $\mathcal{F}_n$ be the σ-field generated by the coordinates $(t_0, t_1, \ldots, t_n)$.

Let Δ_0 be a constant function with value in X and for each $k \geqq 1$ let $\Delta_k(t_0, t_1, \ldots, t_{k-1})$ be a function in $L_1(T^{\mathbf{N}}, \mathbf{P}; X)$ which depends only on the k first coordinates.

Let

$$M_n = \sum_{1 \leqq j \leqq n} e^{it_j} \Delta_{j-1}(t_0, t_1, \ldots, t_{j-1}).$$

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Then $\{m_n\}$ is clearly an X-valued martingale adapted to $\{\mathcal{F}_n\}$.

All the martingales of this form will be called *analytic* martingales (following [**5**]).

We also use the notion of "Hardy-martingale" which was introduced in [**16**] (but was implicitly considered in [**14**]).

A martingale (M_n) in $L_1(T^{\mathbf{N}}, X)$ is called a Hardy martingale if M_n depends only on the first coordinates $(t_0, t_1, \dots, t_n)$ and if the increments $dM_n = M - n - M_{n-1}$ have a (formal) Fourier series with respect to t_n of the following form:

$$dM_n = \sum_{k>0} e^{ikt_n} \varphi_{k,n}(t_0, t_1, \dots, t_{n-1})$$

where $\varphi_{k,n} \in L_1(T^{\mathbf{N}}; X)$ depends only on the n first coordinates.

Equivalently, this means that, with respect to t_n, M_n coincides for each fixed $t_0, t_1, \dots, t_{n-1}$ with the boundary values of an analytic function in $H^1(X)$.

The following result is the "analytic" version of the above mentioned result of Chatterji.

THEOREM 1.2. *Let X be a complex Banach space. The following are equivalent.*

(i) *The space X has the* ARNP.

(ii) *Every analytic martingale bounded in* $L_1(X)$ *converges* a.s.

(iii) *Every Hardy martingale bounded in* $L_1(X)$ *converges* a.s.

This is due to G. Edgar (c.f. [**14**], [**15**]), see also Garling's paper [**16**] for more information.

Remark. The proof of Theorem 1.3 is of course closely related to the fact that for any f in $H^1(X)$ the martingale $M_t = f(B_t)$ obtained by composing f with the complex valued Brownian motion (starting at 0 and stopped at the boundary of D) is a martingale bounded in $L_1(X)$. By a suitable approximation, (M_t) can be replaced by a Hardy martingale or by a subsequence of an analytic martingale, in order to "test" the radial behaviour of the function f (see [**14**] for more details). For a detailed comparison between the various kinds of martingales mentioned above, we refer to [**19**].

Remark 1.3. We should recall that if a Banach space valued martingale converges in $L_1(X)$, then it converges a.s. Conversely, for an analytic or for a Hardy martingale (M_n), since $x \longrightarrow \|x\|^p$ is subharmonic for every $p > 0$, the random variables $\|M_n\|^{\frac{1}{2}}$ form a submartingale, therefore, by Doob's maximal inequality (c.f. e.g. [**31**]), we have

$$\| \sup_n \|M_n\| \|_{L^1} \leqq 4 \sup_n \|M_n\|_{L_1(X)}.$$

This shows that for a Hardy martingale bounded in $L_1(X)$, the a.s. convergence implies the convergence in $L_1(X)$. A similar result holds for $L_p(X)$ for all $p > 0$. See [**16**] for more information.

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The following lemma was first stated in [**42**]. It is easy to prove using a "gap argument" which was already used in [**14**] (see also [**4**]).

LEMMA 1.4. *Let X be a complex Banach space. Let $\delta > 0$ and $0 < q < \infty$. Assume that for all polynomials f in $H^1(X)$, we have*

$$(1.4) \qquad \|f(0)\|^q + \delta\|f - f(0)\|^q_{H^1(X)} \leqq \|f\|^q_{H^1(X)}.$$

Then, for all X-valued Hardy martingales (M_n), we have for all $k < n$

$$(1.5) \qquad \|M_k\|^q_{L_1(X)} + \delta\|M_n - M_k\|^q_{L_1(X)} \leqq \|M_n\|^q_{L_1(X)}.$$

In particular, X has the ARNP.

Proof. Since this is known, we only sketch the proof for the reader's convenience.

We can clearly assume by approximation that M_n is a trigonometric polynomial in the variable $(t_0, \dots, t_n)$. We introduce a new variable θ in T and given integers $a_{k+1}, a_{k+2}, \dots a_n$, we transform $(t_0, t_1, \dots, t_n)$ by the following formula

$$\varphi_\theta(t_0, t_1, \dots, t_n) = (t_0, t_1, \dots, t_k, t_{k+1} + a_n\theta).$$

Then, let

$$f_t(\theta) = M_n(\varphi_\theta(t_0, t_1, \dots, t_n)).$$

By induction (since all increments are trigonometric polynomials) it is possible to choose $a_{k+1}, \dots, a_n$ such that for each t the negative Fourier coefficients of the function $\theta \longrightarrow f_t(\theta)$ are all zero, so that f_t can be identified with an analytic polynomial with coefficients in X. If we then write (1.4) for f_t and integrate with respect to t, we obtain (1.5). (Note that $T \longrightarrow \varphi_\theta(t)$ preserves the measure on $T^{\mathbf{N}}$.)

The inequality (1.5) clearly implies that if

$$\sup_n \|M_n\|_{L_1(X)} < \infty$$

then (M_n) is a Cauchy (hence a convergent) sequence in $L_1(X)$. Therefore X has the ARNP.

Remark. It is possible to prove the implication (1.4) $\Rightarrow$ ARNP in the case $1 \leqq q < \infty$ without using martingales (c.f. Appendix, Proposition 5.1).

Finally we introduce some notation and basic facts about operator algebras and their preduals. Let H be a (complex) Hilbert space. We denote by $B(H)$ the space of all bounded operators on H. We denote as usual by $H \hat{\otimes} H$ the projective tensor product of H with itself. This space can be identified with the space of all operators $T : H \longrightarrow H$ such that $\mathrm{tr}|T| < \infty$ equipped with the

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norm $\|T\| = \operatorname{tr}|T|$. It is well known that its dual $(H \hat{\otimes} H)^*$ can be identified with the space $B(H)$. If $1 \leqq p < \infty$, we will denote by c_p the space of all operators $T : l_2 \longrightarrow l_2$ such that $\operatorname{tr}|T|^p < \infty$. For $p = 2$ this is just the space of all Hilbert-Schmidt operators. When $p = 1$ we may identify c_1 with $l_2 \hat{\otimes} l_2$. The space c_1 (resp. $H \hat{\otimes} H$) can be identified with the dual of the space of all compact operators on l_2 (resp. H). We note in particular the obvious fact that c_1 has the RNP since it is a separable dual (c.f. [**12**]). Since the RNP is separably determined, this is also true for $H \hat{\otimes} H$.

Following works by many authors ([**29**], [**21**], [**11**], [**27**], etc.), Sarason [**37**] proved the following result.

THEOREM 1.5. *Every function in $H^1(c_1)$ is the product of two functions in $H^2(c_2)$. More precisely, for any F in $H^1(c_1)$ there are g, h in $H^2(c_2)$ such that*

$$\forall z \in D \quad F(z) = g(z)h(z) \tag{1.6}$$

and

$$\|F\|_{H^1(c_1)} = \|g\|_{H^2(c_2)}\|h\|_{H^2(c_2)}. \tag{1.7}$$

In Theorem 2.5 below we will prove an extension of this result with c_1 replaced by the dual A^* of a C^*-algebra A.

Remark 1.6. In the framework of tensor products this can be reformulated as follows:

For any F in $H^1(l_2 \hat{\otimes} l_2)$ there are sequences (g_k) and (h_k) in $H^2(l_2)$ such that

$$\forall z \in D \quad F(z) = \sum_{k=1}^{\infty} g_k(z) \otimes h_k(z) \tag{1.8}$$

and

$$\|F\|_{H^1(l_2 \hat{\otimes} l_2)} = \sum_{k=1}^{\infty} \|g_k\|_{H^2(l_2)}\|h_k\|_{H^2(l_2)}. \tag{1.9}$$

Indeed, let (e_n) denote the canonical basis of l_2. Let us denote by $(g_{ij}(z))$ and $(h_{ij}(z))$ the coefficients of the matrices $g(z)$ and $h(z)$ relative to the basis $(e_i \otimes e_j)$, and similarly for F.

We have by (1.6)

$$f_{ij}(z) = \sum_k g_{ik}(z)h_{kj}(z)$$

hence

$$\begin{aligned} f(z) &= \sum F_{ij}(z) e_i \otimes e_j \\ &= \sum_k g_k(z) \otimes h_k(z) \end{aligned}$$

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where

$$g_k(z) = \sum_i g_{ik} e_i \quad \text{and} \quad h_k(z) = \sum_j h_{kj} e_j.$$

This proves that (1.8) and (1.9) follow from (1.6) and (1.7). (The converse direction is also easy.)

Yet another formulation of theorem 1.5 is that the natural mapping (induced by the tensor product) from $H^2(l_2) \hat{\otimes} H^2(l_2)$ into $H^1(l_2 \hat{\otimes} l_2)$ is onto and maps the closed unit ball onto the closed unit ball.

Remark 1.7. Let us assume that the scalar product $(x|y)$ on H is linear in x and antilinear in y. Usually the natural identification between $(H \hat{\otimes} H)^*$ and $B(H)$ associates to any operator T in $B(H)$ the $\mathbf{R}$-linear functional which maps $x \otimes y$ into $(Tx|y)$. Unfortunately this defines only an $\mathbf{R}$-linear isomorphism between $B(H)$ and $(H \hat{\otimes} H)^*$. This is not convenient when dealing with analytic functions. We need a $\mathbf{C}$-linear identification between $H \hat{\otimes} H$ and the predual of $B(H)$. Of course this is trivial to obtain. We introduce a fixed antilinear isometry $y \longrightarrow \bar{y}$ from H onto H and we may then define a $\mathbf{C}$-linear correspondence between $H \hat{\otimes} H$ and the predual of $B(H)$ as follows: To any

$$S = \sum_{n=1}^{\infty} x_n \otimes y_n$$

in $H \hat{\otimes} H$ (with $\sum \|x_n\| \, \|y_n\| < \infty$) we associate the functional φ_s in $B(H)^*$ defined by

$$\text{(1.10)} \qquad \forall T \in B(H) \quad \varphi_s(T) = \sum_{n=1}^{\infty} (Tx_n|\bar{y}_n).$$

Then the correspondence $s \longrightarrow \varphi_s$ is $\mathbf{C}$-linear.

2. Main results. Let X be a complex Banach space. For $0 < p \leqq \infty$ we let $\tilde{H}^p(X)$ denote the closure in $H^p(X)$ of the set of polynomials with coefficients in X. For any function $f : D \longrightarrow X$ we put

$$f_r(z) = f(rz), \quad z \in D, 0 < r < 1.$$

It is elementary to check that if $f \in H^p(X)$, then $f_r \in \tilde{H}^p(X)$ for every $r \in (0, 1)$.

THEOREM 2.1. *Let H be any Hilbert space and let $X = H \hat{\otimes} H$. Then*

$$\text{(2.1)} \qquad \forall f \in H^1(X) \quad \|f(0)\|_X^2 + \frac{1}{2}\|f - f(0)\|_{H^1(X)}^2 \leqq \|f\|_{H^1(X)}^2.$$

Proof. Note that by (1.2) it is enough to prove (2.1) with f exchanged by $f_r, 0 < r < 1$, so by the remarks preceding Theorem 2.1 it suffices to prove

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(2.1) for polynomials with coefficients in X. Therefore we may as well assume that $H = l_2$ or H is finite dimensional. Then, by Theorem 1.5 (c.f. [**29**] for the finite dimensional case and [**37**] for the general case), every f in $H^1(c_1)$ can be written as a product $f(z) = g(z)h(z)(z \in D)$ where g, h are in $H^2(c_2)$ and satisfy

$$(2.2) \qquad \|f\|_{H^1(c_1)} + \|g\|_{H^2(c_2)}\|h\|_{H^2(c_2)}.$$

Let us denote simply $\|f\|_1$ instead of $\|f\|_{H^1(c_1)}$ and $\|g\|_2$ instead of $\|g\|_{H^2(c_2)}$. Also, in the following inequalities we will identify $f(0)$ with the constant function taking the value $f(0)$.

Then we can write

$$f - f(0) = g(h - h(0)) + (g - g(0))h(0),$$

hence, by Cauchy-Schwarz and the triangle inequality,

$$\begin{aligned}\|f - f(0)\|_1^2 &\leqq 2(\|g(h - h(0)\|_1^2 + \|(g - g(0)h(0)\|_1^2)\\ &\leqq 2\|g\|_2^2\|h - h(0)\|_2^2 + 2\|g - g(0)\|_2^2\|h(0)\|_2^2.\end{aligned}$$

On the other hand,

$$\|f(0)\|_1 = \|g(0)h(0)\|_1 \leqq \|g(0)\|_2\|h(0)\|_2.$$

Therefore we find

$$\begin{aligned}\|f(0)\|_1^2 + \frac{1}{2}\|f - f(0)\|_1^2 &\leqq (\|g(0)\|_2^2 + \|g - g(0)\|_2^2)\|h(0)\|_2^2\\ &\quad + \|g\|_2^2\|h - h(0)\|_2^2,\end{aligned}$$

hence, by Parseval's identity,

$$\begin{aligned}&\leqq \|g\|_2^2\|h(0)\|_2^2 + \|g\|_2^2\|h - h(0)\|_2^2\\ &\leqq \|g\|_2^2\|h\|_2^2.\end{aligned}$$

By (2.2) this concludes the proof.

Remark. It is easy to check that the inequality (2.1) also holds with $H^p(X)$ instead of $H^1(X)$ for $1 \leqq p \leqq 2$. [Hint: write $f = gh$ with

$$\|g\|_{H^r(c_2)}\|h\|_{H^2(c_2)} = \|f\|_{H^p(c_1)} \quad \text{and} \quad \frac{1}{p} = \frac{1}{2} + \frac{1}{r},$$

then proceed as above but use

$$\|(g - g(0))h(0)\|_{H^p(c_1)} \leqq \|g - g(0)\|_{H^2(c_2)}\|h(0)\|_2$$

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and observe that since $r \leqq 2$ we have $\|g\|_{H^2(c_2)} \leqq \|g\|_{H^r(c_2)}$.]

Remark. In the particular case $f(z) = x + zy \quad (x, y \in X)$, the inequality (2.1) is known, with X any non-commutative L_1-space and the constant $\frac{1}{2}$ is best possible. This is due to the first author (c.f. [**10**]). We refer to [**10**] for more information on the notion of "uniform PL-convexity" which corresponds to inequalities analogous to (2.1) but restricted to polynomials of degree 1. We should mention that it is not known whether uniform PL-convexity implies the ARNP. In particular, the following question is open: Assume that a Banach space X satisfies for some $\delta > 0$ and $q < \infty \, \forall x, y \in X$

$$(\|x\|^q + \delta \|y\|^q)^{1/q} \leqq \int \|x + e^{it} Y\| dm(t),$$

then does X have the ARNP? Of course, for a positive answer it suffices to show that X satisfies (1.4).

Remark. Let Y, X be Banach spaces. We say that Y is finitely representable (in short f.r.) in X if for any $\epsilon > 0$ and any finite dimensional subspace $E \subset Y$ there is a subspace $F \subset X$ which is $(1 + \epsilon)$-isomorphic to E. In that case it is easy to see that all the inequalities of a finite dimensional nature which are true for Y must be true also for X. In particular, if X satisfies (2.1), then Y also does (recall that it suffices to consider polynomials in (2.1)).

For example (by the local reflexivity principle, c.f. [**28**] p. 34) the bidual X^{**} is f.r. in X. Therefore, for any Hilbert space H, the space $H = B(H)^* = (H \hat{\otimes} H)^{**}$ satisfies (2.1) and therefore, by Lemma 1.4, $B(H)^*$ has the ARNP.

This is the non-commutative version of the well known fact that L_1 has the ARNP.

We wish to replace in this statement $B(H)$ by any von Neumann (or C^*) algebra and X by any non-commutative L_1-space. But it is apparently an open problem whether every non-commutative L_1-space is f.r. in c_1, so that we cannot replace directly $B(H)^*$ by any non-commutative L_1-space in the preceding reasoning. Actually, by Gelfand's theorem, any von Neumann algebra A may be viewed as a weakly closed C^*-subalgebra of $B(H)$. Therefore, any non-commutative L_1-space X can be viewed as the predual of such an algebra, or equivalently as a quotient space $H \hat{\otimes} H/N$. Indeed, we may identify $A = X^*$ with a weakly closed *-subalgebra of $B(H)$. Letting N be the preannihilator of A in $H \hat{\otimes} H$, it is known that X must be isometric to $H \hat{\otimes} H/N$ (c.f. [**34**] p. 55 or [**23**] §7.1). Let

$$q : H \hat{\otimes} H \longrightarrow H \hat{\otimes} H/N$$

be the quotient mapping. To prove Theorem 2.1 in the general case, we need to be able to lift the elements of $H^1(H \hat{\otimes} H/N)$ up into elements of $H^1(H \hat{\otimes} H)$. This is what we do in the next result.

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Let us say that an operator $u : Y \longrightarrow Z$ (between Banach spaces) is a *metric surjection* if it is onto and if it maps the open unit ball of Y onto the open unit ball of Z. Equivalently, $u^* : Z^* \longrightarrow Y^*$ is an isometric embedding.

THEOREM 2.2. *Let X be a non-commutative L_1-space identified with a quotient $H \hat{\otimes} H/N$ (as explained above) via a quotient mapping $q : H \hat{\otimes} H \longrightarrow H \hat{\otimes} H/N$. Consider f in $\tilde{H}^1(X)$ and $\epsilon > 0$. Then* (i) *there are functions g_n, h_n in $H^2(H)$ such that*

$$\sum_1^\infty \|g_n\|_{H^2(H)}\|h_n\|_{H^2(H)} \leqq (1+\epsilon)\|f\|_{H^1(X)}$$

and

$$(2.3) \qquad \forall z \in D \quad f(z) = q\left(\sum_{n=1}^\infty g_n(z) \otimes h_n(z)\right).$$

(ii) *There is a function F in $\tilde{H}^1(H \hat{\otimes} H)$ such that*

$$\|F\|_{H^1(H\hat{\otimes}H)} \leqq (1+\epsilon)\|f\|_{H^1(X)}.$$

and

$$\forall z \in D \quad q(F(z)) = f(z).$$

Equivalently, if we denote by $Q : \tilde{H}^1(H \hat{\otimes} H) \longrightarrow \tilde{H}^1(X)$, the mapping canonically associated to q, then Q is a metric surjection.

Remark. Taking into account Remark 1.7 above, (2.3) means that for all T in $X^* \subset B(H)$ we have

$$\langle f(z), T\rangle = \sum_{n=1}^\infty (Tg_n(z), \overline{h_n(z)}).$$

Proof of Theorem 2.2. Let us denote by P the linear subspace of $H^1(X)$ formed by all the polynomials with coefficients in $q(H \otimes H)$.

Let us denote by $\| \ \|_1$ the norm in $H^1(X)$, and by $\| \ \|_2$ the norm in $H^2(H)$.

Clearly, for every f in P there are polynomials with coefficients in Hg_i, h_i such that

$$\forall z \in D \quad f(z) = q\left(\sum_{i=1}^n g_i(z) \otimes h_i(z)\right).$$

We introduce a norm on P by setting

$$\|f\| = \inf\left\{\sum_1^n \|g_i\|_2\|h_i\|_2\right\}$$

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where the infimum runs over all possible representations.

Note that we have obviously $\|f\|_1 \leqq \|f\|$ and $\| \ \|$ is indeed a norm on P.

The main point of the proof of Theorem 2.2 is to check that actually this "new" norm $\|f\|$ coincides with $\|f\|_1$. Using duality, we will show that this follows rather directly from known results in the theory of vectorial Hankel operators due to Parrott [**33**]. (Cf. also [**36**]. These results are closely related also to Arveson's distance formula for which we refer to [**1**].)

To explain this more precisely, we need to identify the dual spaces to $P(X)$ equipped with the norms $\| \ \|_1$ and $\| \ \|$.

Let us denote by Λ the space of all sequences $a = (a_n)_{n \geqq 0}$ with $a_n \in X^* \subset B(H)$ such that the Hankel matrix $\mathcal{H}_a$ with coefficcients $(\mathcal{H}_a)_{ij} = a_{i+j} (i \geqq 0, j \geqq 0)$ defines a bounded operator on $l_2(H)$. By definition, we set $\|a\| = \|\mathcal{H}_a\|$. Let us denote by $\mathcal{X}$ (resp. $\mathcal{X}_1$) the normed space obtained by equipping P with the norm $\| \ \|$ (resp. $\| \ \|_1$).

We may introduce a duality between P and Λ as follows. Let (f_n) denote the Taylor coefficients of an element f in P. Then for all a in Λ, we define

$$\langle a, f\rangle = \sum_{n=0}^{\infty} \langle a_n, f_n\rangle.$$

(Note that this sum is finite.)

With this duality, we have

$$\begin{aligned} \|a\|_{\mathcal{X}^*} &= \sup \langle a, q(g \otimes h)\rangle \\ &= \sup \sum_{ij} (a_{ij} g_j, \bar{h}_i) \\ &= \|\mathcal{H}_a\| \end{aligned}$$

where each of the above supremum runs over all g, h in P such that $\|g\|_2 \leqq 1$ and $\|h\|_2 \leqq 1$. (Of course we use $\|g\|_2 = (\sum \|g_j\|^{\frac{1}{2}})$.)

This shows that Λ can be naturally identified isometrically with the dual of $\mathcal{X}$.

Similarly, let us denote by $\tilde{\Lambda}$ the space of all the double sequences $\alpha = (\alpha_n)_{n \in \mathbf{Z}}$ with $\alpha_n \in X^* \subset B(H)$ such that the matrix T_α defined by

$$(2.4) \qquad (T_\alpha)_{ij} = a_{i+j} \quad \forall i, j \in \mathbf{Z}$$

defines a bounded operator on $l_2(\mathbf{Z}, H)$. By definition, we set $\|\alpha\|_{\tilde{\Lambda}} = \|T_\alpha\|$.

Here again it is simple to check that $L_1(T, X)^* = \tilde{\Lambda}$ isometrically. Equivalently, this means that the natural mapping from $L_2(H) \hat{\otimes} L_2(H)$ into $L_1(X)$ is a metric surjection. This can be viewed as a consequence of the identity $L_1(X) = L_1 \hat{\otimes} X$ and the fact that every scalar function with L_1-norm 1 is the product of two functions with L_2-norm 1. Let us now return to our original problem to show that $\mathcal{X}$ coincides with $\mathcal{X}_1$, or simply that $\|f\| \leqq \|f\|_1$ for all f in P. To prove

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that it suffices to show that every a in the unit ball of X^* defines an element in the unit ball of X_1^*. Equivalently, it is enough to show that for any $a = (a_n)_{n \geqq 0}$ in the unit ball of $\Lambda = X^*$, there is an $\alpha = (\alpha_n)_{n \in \mathbf{Z}}$ in the unit ball of $L_1(X)^* = \tilde{\Lambda}$ which is such that $\langle \alpha, f \rangle = \langle a, f \rangle$ for all f in P. Clearly this means that $\alpha_n = a_n$ for all $n \geqq 0$.

We have thus reduced our problem to the fact that every Hankel matrix with coefficients in a von Neumann algebra X^* can be completed to a matrix with coefficients in X^* of the form (2.4) and of the same norm. This is precisely what Parrott shows in [**33**] (see the last lines of §3 in [**33**]). There, he gives an explicit inductive construction of the coefficients α_{-1}, α_{-2}, etc. which can be added to the sequence $a = (a_n)_{n \geqq 0}$ in order to form an extended sequence with the desired property $\|T_\alpha\| = \|\mathcal{H}_a\|$.

This allows us to conclude that X and X_1 are identical. Since their completions must be also identical, we obtain (i) and (ii) immediately follows from (i) by setting

$$F(z) = \sum_{n=1}^{\infty} g_n(z) \otimes h_n(z).$$

COROLLARY 2.3. *Let X be an arbitrary non-commutative L_1-space.*

(i) *The inequality* (2.1) *holds for any f in $H^1(X)$.*

(ii) *The space X has the* ARNP.

(iii) *The preceding Theorem* 2.2 *is valid for any f in $H^1(X)$.*

Proof. (i) Consider f in $\tilde{H}^1(X)$. By the second part of Theorem 2.2 and by Theorem 2.1, it is easy to check that f satisfies (2.1). By (1.2) and the fact that $f_r \in \tilde{H}^1(X)$, $0 < r < 1$, (2.1) also holds for any f in $H^1(X)$.

(ii) This follows from Lemma 1.4 (see also Proposition 5.1 in the appendix).

(iii) Since X has the ARNP, we have $H^1(X) = \tilde{H}^1(X)$ by Theorem 1.1, so that Theorem 2.2 also holds for any f in $H^1(X)$.

We note in passing that Sarason's result (Theorem 1.5) follows from Theorem 2.2 up to a factor $1+\epsilon$ in the norm estimates (cf. Remark 1.6). Since this is enough to prove Theorem 2.1, we might claim that our paper is self-contained, except for the main results in [**33**] (or the alternate proof in [**36**]). Note however that these alternate routes to Sarason's result have been known for a long time, in particular since [**32**], at least for the $(1+\epsilon)$-version of Sarason's result. The dual approach to factorization problems was first exploited in [**32**] to deduce the vectorial Nehari Theorem from a result of Sz. Nagy-Foias [**39**]. Later, Parrott [**33**] observed that his result yields a Nehari theorem for essentially bounded weak-* measureable functions f with values in a von Neumann algebra $M \subset B(H)$. Namely, the distance (in the L_∞-norm) of f to $H^\infty(M)$ is equal to the norm of the vectorial Hankel operator determined by f. This can be viewed as a dual formulation to the first part of Theorem 2.2.

We now use an ultraproduct technique to get rid of ϵ in Theorem 2.2.

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PROPOSITION 2.4. *The conclusion of Theorem* 2.2 *holds with* $\epsilon = 0$.

Proof. Note first that $\tilde{H}^1(X) = H^1(X)$ and $\tilde{H}^1(H \hat{\otimes} H) = H^1(H \otimes H)$ by Corollary 2.3. Let $N \subseteq B(H)$ and $X = N_*$ be as in Theorem 2.2 and let $f \in H_1(X)$. For each $m \in \mathbf{N}$ we can choose sequences $(g_k^{(m)})_{m=1}^{\infty}$ and $(h_k^{(m)})_{m=1}^{\infty}$ of functions in $H_2(H)$ such that

$$\sum_{k=1}^{\infty} \|g_k^{(m)}\|_{H^2(H)}^2 \leqq \left(1 + \frac{1}{m}\right) \|f\|_{H^1(X)}$$

$$\sum_{k=1}^{\infty} \|h_k^{(m)}\|_{H^2(H)}^2 \leqq \left(1 + \frac{1}{m}\right) \|f\|_{H^1(X)}$$

and

$$f(z) = q\left(\sum_{k=1}^{\infty} g_k^{(m)}(z) \otimes h_k^{(m)}(z)\right), \quad z \in D.$$

Put

$$\mathcal{H} = \bigoplus_{k=1}^{\infty} H$$

and define $g_m, h_m \in H_2(\mathcal{H})$ by

$$g_m(z) = (g_k^{(m)}(z))_{m=1}^{\infty}, \quad z \in D$$

$$h_m(z) = (\overline{h_k^{(m)}(\bar{z})})_{m=1}^{\infty}, \quad z \in D$$

For $a \in N$, let $\rho(a)$ denote the operator on $\mathcal{H}$, obtained by letting a act on each component in the direct sum $\bigoplus_{k=1}^{\infty} H$. Then for all $a \in N$,

$$\langle f(z), a\rangle \sum_{k=1}^{\infty} (a g_k^{(m)}(z), \overline{h_k^{(m)}(z)}) = (\rho(a) g_m(z), h_m(\bar{z})).$$

Let $\mathcal{U}$ be a free ultrafilter on $\mathbf{N}$ and let $\mathcal{H}_{\mathcal{U}}$ denote the ultrapower of $\mathcal{H}$ corresponding to $\mathcal{U}$. We can define a $*$-representation $\pi : N \longrightarrow B(\mathcal{H}_{\mathcal{U}})$ by

$$\pi(a)x = (\rho(a)x_m)_{m=1}^{\infty}$$

when $(x_m)_{m=1}^{\infty}$ is a representing sequence of $x \in \mathcal{H}_{\mathcal{U}}$. Let $g, h \in H^2(\mathcal{H}_{\mathcal{U}})$ be the functions with representing sequences $(g_m(z))_{m=1}^{\infty}$ and $h_m(z))_{m=1}^{\infty}$, then $\|g\|_2^2$ and $\|h\|_2^2$ are both dominated by

$$\lim_{\mathcal{U}} \left(1 + \frac{1}{m}\right) \|f\|_1 = \|f\|_1$$

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and

$$\langle f(z), a\rangle = (\pi(a)g(z), h(\bar{z})), \quad z \in D.$$

The representation π is in general not normal. However, following [**40**, pp. 127–128], the representation splits uniquely into a direct sum

$$\pi = \pi_n \oplus \pi_s$$

(the normal and singular parts of π).

Let

$$\mathcal{H}_{\mathcal{U}} = \mathcal{H}_{\mathcal{U}}^n \oplus \mathcal{H}_{\mathcal{U}}^s$$

be the corresponding direct sum decomposition of $\mathcal{H}_{\mathcal{U}}$ into two $\pi(N)$-invariant subspaces, and let

$$g(z) = g_n(z) + g_s(z)$$
$$h(z) = h_n(z) + g_s(z)$$

be the corresponding decomposition of $g(z)$ and $h(z)$. Then

$$\langle f(z), a\rangle = (\pi_n(a)g_n(z), h_n(\bar{z})) + (\pi_s(a)g_s(z), g_s(\bar{z})) \tag{2.5}$$

because the $\pi(N)$-invariance of $\mathcal{H}_{\mathcal{U}}^n$ and $\mathcal{H}_{\mathcal{U}}^s$ implies that the two cross terms vanish. (2.5) defines a splitting of $f(z)$ into a normal and a singular part. However, since $f(z) \in N_*$, the singular part vanishes, i.e.,

$$\langle f(z), a\rangle = \langle \pi_n(a)g_n(z), h_n(\bar{z}))\rangle.$$

Note that $g_n, h_n \in H_2(\mathcal{H}_{\mathcal{U}}^n)$ and

$$\|g_n\|_2^2 \leqq \|f\|_1, \|h_n\|_2^2 = \|f\|_1.$$

Since $\|\pi_n\| \leqq 1$, we have $\|g_n\|_2 \|h_n\|_2 \leqq \|f\|_1$, so in fact

$$\|g_n\|_2^2 = \|h_n\|_2^2 = \|f\|_1.$$

By [**40**, Theorem IV 5.5 (p. 222)], the normal representation π_n is spatially isomorphic to a subrepresentation of the representation $a \longrightarrow a \otimes 1_K$ for some Hilbert space K. Thus by (2.6) we can choose $\breve{g}, \breve{h} \in H^2(H \otimes K)$, such that for $a \in N$ and $z \in D$,

$$\langle f(z), a\rangle = ((a \otimes 1_K)\breve{g}(z), \breve{h}(\bar{z}))$$

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and

$$\|\breve{g}\|_2^2 = \|\breve{h}\|_2^2 = \|f\|.$$

Let $(e_i)_{i\in I}$ be an orthonormal basis for K. Then we can identify $H \times K$ with $\bigoplus_{i\in I} H$, such that the action of $a \otimes 1_K$ on $H \otimes K$ is given by multiplication by a in each component of $\bigoplus_{i\in I} H$. With this identification

$$\breve{g}(z) = (g_i(z))_{i\in I}$$
$$\breve{h}(z) = (h_i(z))_{i\in I}$$

where $g_i, h_i \in H^2(H)$,

$$\sum_{i\in 1} \|g_i\|_2^2 = \sum_{i\in 1} \|h_i\|_2^2 = \|f\|_1 \tag{2.7}$$

and

$$\langle f(z), a\rangle = \sum_{i\in 1} (ag_i(z), h_i(\bar{z})), \quad a \in N.$$

Equivalently,

$$f(z) = \sum_{i\in 1} g_i(z) \otimes \overline{h_i(\bar{z})}.$$

By (2.7), g_i and h_i vanish except for countably many $i \in I$. This proves Theorem 2.2 with $\epsilon = 0$.

From the above proof we can extract:

COROLLARY 2.5. (1) *Let A be a C^*-algebra and let $f \in H^1(A^*)$. Then there exists a $*$-representation π of A on a Hilbert space $\mathcal{H}$ and $g, h \in H^2(\mathcal{H})$ such that for all $a \in A$*

$$\langle f(z), a\rangle = (\pi(a)g(z), h(\bar{z})), \quad z \in D \tag{2.8}$$

and

$$\|g\|_2^2 = \|h\|_2^2 = \|f\|_1. \tag{2.9}$$

(2) *Let N be a von Neumann algebra and let $f \in H^1(N_*)$, then there is a normal $*$-representation π of N on a Hilbert space $\mathcal{H}$ and $g, h \in H^2(\mathcal{H})$, such that* (2.8) *and* (2.9) *holds (for $a \in N$).*

Proof. Let N be a von Neumann algebra. Since N_* has ARNP, $H^1(N_*) = \tilde{H}^1(N_*)$, so (2) is contained in the proof of Proposition 2.4. In fact, it follows from the proof of Proposition 2.4 that if N is already realized as a von Neumann

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algebra on a Hilbert space H, then π can be chosen to be a countable multiple of the identity representation, i.e., $\pi(a) = a \otimes 1_K$, where 1_K denotes the identity operator on a separable Hilbert space K. (1) follows immediately from (2) by considering the von Neumann algebra $N = A^{**}$.

Remark 2.6. Recently, Blasco and Pelcyński [**3**] studied the class of Banach spaces such that every bounded multiplier from H^1 into l^1 is bounded from $H^1(X)$ into $l_1(X)$. This class of Banach spaces were denoted spaces of $(H^1 - l^1)$ Fourier type. It follows immediately from Theorem 2.2 that every non-commutative L_1-space X is a space of $(H^1 - l^1)$ Fourier type in the sense of [**3**]. In particular X satisfies Hardy's inequality and Paley's inequality. This answers questions left open in [**3**] where this is proved for $X = c_1$.

It is natural to ask whether the lifting property expressed in Theorem 2.2 is valid with H^p instead of H^1. This follows from a general fact. To state this in full generality we introduce some terminology. Let X be a Banach space and let $0 < p \leqq \infty$. Recall that we denote by $\tilde{H}^p(X)$ the closure of the set of all polynomials (with coefficients in X) in $H^p(X)$. Note that $\tilde{H}^\infty(X)$ coincides with the space of all analytic functions $f : D \longrightarrow X$ which extend continuously to $\bar{D}$. Let Z be another Banach space. Let $u : Z \longrightarrow X$ be an operator. We will say that u is an $\tilde{H}^p$-surjection (resp. a metric $\tilde{H}^p$-surjection) if the natural map

$$\tilde{u} : \tilde{H}^p(Z) \longrightarrow \tilde{H}^p(X)$$

associated to u is a surjection (resp. a metric surjection).

We define similarly the notion of H^p-surjection and metric H^p-surjection.

Then we can state the following (which was observed independently by N. Kalton).

THEOREM 2.7. *Let $u : Z \longrightarrow X$ be as above with $\|u\| = 1$. If u is a metric $\tilde{H}^p$-surjection for some $1 \leqq p \leqq \infty$, then the same is true for all $0 < p \leqq \infty$. A similar statement also holds for H^p-surjections.*

Proof. Let $0 < p < q \leqq \infty$. We claim that if u is a metric $\tilde{H}^q$-surjection then it is a metric $\tilde{H}^p$-surjection. This is very easy to check using outer functions. Indeed, let r be such that $1/p = 1/q + 1/r$, let $\epsilon > 0$ and consider f in $\tilde{H}^p(X)$ with norm 1. By classical results we can find a function φ in H^r such that

$$(2.8) \qquad |\varphi(\cdot)| = (\|f(\cdot)\|_X + \epsilon)^{p/r} \quad \text{on the circle, and } \varphi^{-1} \in H^\infty.$$

We have then $\|\varphi\|_r \leqq 1 + \epsilon$ and we can write f as a product $f = g \cdot \varphi$ with g in $\tilde{H}^q(X)$. Actually $g = \varphi^{-1} f$ and by (2.8) we have

$$\|g\|_{H^q(X)} \leqq 1.$$

By our hypothesis there is G in $\tilde{H}^q(Z)$ such that $u(G) = g$ and $\|G\|_q \leqq (1+\epsilon)$. Now let $F = \varphi G$. We have $u(F) = f$ and by Hölder

$$\|F\|_{H^p(Z)} \leqq \|G\|_q \|\varphi\|_r \leqq (1+\epsilon)^2.$$

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This proves the above claim.

To prove the converse we use duality. We first note that by a simple approximation argument, we may assume (in the metric case) that Z and X are finite dimensional normed spaces. Now consider $1 \leqq p \leqq q \leqq \infty$. Let p', q' be the conjugate exponents so that $1 \leqq q' < p' \leqq \infty$. Then, saying that u is a metric $\tilde{H}^p$-surjection is equivalent to saying that $u^* : X^* \longrightarrow Z^*$ induces naturally an isometric embedding

$$\tilde{u}^* : \tilde{H}^p(X)^* \longrightarrow \tilde{H}^p(Z)^*.$$

We may identify $\tilde{H}^p(X)^*$ with $L_{p'}(X^*)/H_0^{p'}(X^*)$. We can then repeat an argument similar to the first part of the proof to show that this property for p' implies the same for all $q' < p'$. We leave the easy details to the reader. This completes the proof for metric surjections. The case of surjections is identical.

Remarks. (i) It is easy to check that a (metric) H^p-surjection is a fortiori a (metric) $\tilde{H}^p$-surjection.

(ii) By the proof of the above claim, if $0 < p < q \leqq \infty$ and if Z has the ARNP, then necessarily X also has the ARNP. This follows clearly from known results (c.f. Proposition 1.1) since $H^p(Z) = \tilde{H}^p(Z)$ implies $H^p(X) = \tilde{H}^p(X)$, if u is an H^p-surjection.

(iv) If X and Z have the ARNP, then u is a (metric) H^p-surjection if and only if it is a (metric) $\tilde{H}^P$-surjection. Therefore, Theorem 2.8 is valid also with H^p instead of $\tilde{H}^p$ in that case.

(v) Assume that Z and X are dual spaces ($Z = (Z_*)^*, X = (X_*)^*$) and that u is weak-* continuous (i.e., u is the adjoint of an operator $u_* : X_* \longrightarrow Z_*$). Let $0 < p \leqq \infty$. Then if u is a (metric) $\tilde{H}^p$-surjection, it is a (metric) H^p-surjection. Indeed, if $f \in H^p(X)$, let $f_r(z) = f(rz)$ for all z in Δ and $0 < r < 1$. Clearly $f_r \in \tilde{H}^p(X)$ for every $r < 1$. Therefore if u is a metric $\tilde{H}^p$-surjection, for every $0 < r < 1$ there is g_r in $H^p(Z)$ such that

$$u(g_r) = f_r \quad \text{and} \quad \|g_r\| \leqq \frac{1}{r}\|f\|.$$

Let U be a non trivial ultrafilter refining the net corresponding to $r \longrightarrow 1$.

Let us write the Taylor expansion

$$g_r(z) = \sum_{n \geqq 0} g_r(n) z^n.$$

Let

$$F(n) = \lim_U g_r(n)$$

(the limit being in the weak-* topology $\sigma(Z, Z_*)$). Then it is easy to check that the function $F + \sum_{n \geqq 0} z^n nF(n)$ in $H^p(Z)$ with $\|F\| \leqq \|f\|$ and satisfies $u(F) = f$.

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This proves the above claim that u is a metric H^p-surjection. (Actually, the associated map $\tilde{u} : H^p(Z) \longrightarrow H^p(X)$ maps the closed unit ball onto the closed unit ball.)

In particular, the preceding yields

COROLLARY 2.8. *Let X be a non-commutative L_1-space, and let $q : H \hat{\otimes} H \longrightarrow X$ be the quotient mapping described above (c.f. Remark before Theorem 2.2). Then q is a metric H^p-surjection for $0 < p < \infty$ and a metric $\tilde{H}^\infty$-surjection. Moreover, $q^{**} : B(H) \longrightarrow X^{**}$ is a metric H^∞-surjection.*

We also have the following extension theorem

COROLLARY 2.9. *Let $A \subset B$ be a C^*-subalgebra of a C^*-algebra B. Then every operator $u : A \longrightarrow H^\infty$ admits an extension $\tilde{u} : B \longrightarrow H^\infty$ with $\|\tilde{u}\| = \|u\|$.*

Proof. It clearly suffices (by Gelfand's theorem) to prove this for $B = B(H)$. Since H^∞ is a dual space, every $u : A \longrightarrow H^\infty$ extends to the bidual A^{**} which is a von Neumann algebra. Thus we may assume that A is a von Neumann subalgebra of $B(H)$.

Let Z be any Banach space. Clearly the space $B(Z, H^\infty)$ of all bounded operators from Z into H^∞ can be identified with the space $H^\infty(Z^*)$. Therefore the extension theorem reduces to the fact that the natural map

$$H^\infty(B(H)^*) \longrightarrow H^\infty(A^*)$$

is a metric surjection. This follows from the previous corollary. (Note that by a simple weak $*$-convergence argument, we can indeed obtain $\tilde{u}$ with $\|\tilde{u}\| = 1$.)

Remark. Let us denote by T_p the subspace of c_p formed by all the upper triangular matrices in c_p. There is a non-commutative analogue of the identity $H^1 = H^2 \times H^2$, namely the identity $T_1 = T_2 \times T_2$ with a similar control of the norms (c.f. [**38**]).

Now consider (2.1) in the simplest case $X = \mathbf{C}$. It is natural to try to write down a non-commutative analogue of this inequality. Let $D : c_1 \longrightarrow c_1$ be the operator which maps a matrix to the diagonal matrix with the same coefficients as x on the diagonal. Then the following seems to be a natural non-commutative analogue of (2.1): For all x in T_1 we have

$$\|D(x)\|_1^2 + \frac{1}{2}\|x - D(x)\|_1^2 \leqq \|x\|_1^2.$$

This can be proved by the same argument as for Theorem 2.1 above.

3. Remarks on complex interpolation. Although it is not surprising, we would like to emphasize here that the ARNP appears natural within the context of the complex interpolation method. (We refer to [**2**] for background, notation and definition of the complex interpolation method.) By the Riemann mapping

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theorem, in the definition of the ARNP we may replace the open unit disc by other open subsets of $\mathbf{C}$, in particular if we wish by the strip

$$S = \{z \in \mathbf{C} \quad | \quad 0 < \operatorname{Re} z < 1\}.$$

Thus, if X has the ARNP, every bounded analytic function $f : S \longrightarrow X$ admits non-tangential limits a.e. on the boundary of S.

Let A_0, A_1 be an interpolation couple of complex Banach spaces. We refer to [**2**] for the definition of the spaces $(A_0, A_1)_\theta$ and $(A_0, A_1)^\theta$.

Then we wish to formulate the following

PROPOSITION 3.1. *If $A_0 \subset A_1$ and if A_1 has the* ARNP, *then $(A_0, A_1)_\theta = (A_0, A_1)^\theta$ with equal norms.*

The proof follows immediately from the preceding remarks and §4.3 in [**2**].

The preceding statement allows to weaken the classical reflexivity assumption in the following standard situation of an interpolation couple with a Hilbertian "midpoint space".

Let X be a complex Banach space.

Let $i : X^* \longrightarrow X$ be linear (i.e., $\mathbf{C}$-linear throughout the sequel), injective, with dense range and norm 1. We assume that there is an involution $\xi \longrightarrow \xi^*$ on X^* such that $\forall \xi \in X^*$ $i(\xi)(\xi^*) \geqq 0$, and that i is symmetric, i.e.,

$$i(\xi)(\eta) = i(\eta)(\xi) \quad \forall \xi, \eta \in X^*.$$

Then it is well known that we may view X^* as continuously embedded into Hilbert space H by an injective mapping $j : X^* \longrightarrow H$ of norm 1. To define H we simply complete the prehilbertian space X^* equipped with the scalar product

$$\forall \xi, \eta \in X^* \quad (\xi, \eta) = i(\xi)(\eta^*).$$

Then $j : X^* \longrightarrow H$ is the inclusion map.

Since $\|i\| = 1$ we have $\|j\| = 1$ and j^* actually has range into X, so we may consider j^* as an operator from H^* into X :

We denote by $\varphi : H \longrightarrow H^*$ the linear isometry defined by the identity

$$\forall \xi \in X^* \quad \forall \eta \in X^* \quad \langle \varphi(j(\xi)), j(\eta) \rangle = i(\xi)(\eta).$$

Then we have $i = j^* \varphi j$.

Again, we wish to formulate

PROPOSITION 3.2. *In the preceding situation we can view (X^*, X) as an interpolation couple by identifying X^* with $i(X^*) \subset X$. Then, if X^{**} has the* ARNP, *we have*

$$(X^*, X)_{\frac{1}{2}} = H \text{ with equal norms,}$$

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where we identify H with $j^*\varphi(H)$.

Proof. The argument is well known, it goes back to the early days of interpolation (c.f. e.g. [**17**]) but with different assumptions on X such as reflexivity. We briefly recall this argument.

Let $I = (X^*, X)_{\frac{1}{2}} = (X, X^*)_{\frac{1}{2}}$.

Consider the sesquilinear form $u : X^* \times X \longrightarrow \mathbf{C}$ defined by $u(\xi, x) = \xi^*(x)$. Then u is linear in x, antilinear in ξ and of norm 1 both from $X^* \times X$ into $\mathbf{C}$ and from $X \times X^*$ into $\mathbf{C}$. By the basic interpolation theorem, u has norm 1 also from $I \times I$ into $\mathbf{C}$. Hence we have $\forall \xi_1, \xi_2 \in I$

$$|u(\xi_1, \xi_2)| \leqq \|\xi_1\|_I \|\xi_2\|_I,$$

in particular $\forall \xi \in I$,

$$\|\xi\|_H^2 = |u(\xi, \xi)| \leqq \|\xi\|_I^2.$$

Therefore, (3.1) $\|\xi\|_H \leqq \|\xi\|_I$ for all ξ in I.

By duality we have

$$(3.2) \qquad \|\xi\|_{I^*} \leqq \|\xi\|_{H^*} \quad \text{for all } \xi \in H^*.$$

But it is well known that

$$I^* = (X^*, X^{**})^{\frac{1}{2}},$$

hence, by Proposition 3.1, if X^{**} has the ARNP,

$$I^* = (X^*, X^{**})_{\frac{1}{2}}$$

and since $X^* \subset X$,

$$I^* = (X^*, X)_{\frac{1}{2}}.$$

Therefore we conclude from (3.1) and (3.2) that $I = H$ isometrically, with the natural identifications.

Remarks. The typical illustration of the preceding statement is the case of the inclusion $L_\infty \longrightarrow L_1$ over a probability space. The non-commutative situation has also been considered (c.f. [**25**], [**41**], and see also [**35**]). Since all the abstract L_1-spaces (commutative or not) have the ARNP(as well as their biduals), we have thus an "abstract" proof that $(L_\infty, L_1)_{\frac{1}{2}} = L_2$ which makes sense equally well in the commutative or non-commutative setting. The present remark provides a missing reference for the complex case of an assertion made in [**35**] (p. 124 line 6 from bottom).

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4. Unconditionality of analytic martingale differences. Recently the unconditionality of martingale differences has been considered in the Banach space valued case in close connection with singular integrals (c.f. [**4**], [**8**]). A Banach space X is called UMD if for some (or all) $1 < p < \infty$, all the X-valued martingale difference sequences in $L_p(X)$ are unconditional in $L_p(X)$. We will denote below simply by $\| \ \|_p$ the norm in $L_p(X)$.

It is natural in our context to consider the same property but restricted to analytic martingales. This notion was already considered by Garling in [**16**] (and also implicitly in [**5**]). We will say that X has the analytic UMD property (in short: AUMD) if there is a $0 < p < \infty$ and a constant C such that for all X-valued analytic martingales (M_n) and all choices of signs $\epsilon = \pm 1$, we have for all n

$$\|\Sigma_1^n \epsilon_k dM_k\|_p \leqq C\|\Sigma_1^n dM_k\|_p$$

(recall $dM_k = M_k - M_{k-1}$).

In particular this implies a fortiori for all $n \geqq 1$

$$\text{(4.1)} \qquad \|\sum_{1\leqq j\leqq n} M_{2j} - M_{2j-1}\|_p \leqq C\|M_{2n}\|_p$$

(only "even" increments are kept on the left).

Garling observed that if this holds for some $1 < p < \infty$ then it holds for all $0 < p < \infty$ (c.f. [**16**]). Xu observed (see [**16**]) that if the above holds then it also holds for all Hardy martingales (instead of analytic ones).

By known results (c.f. [**5**]), all L_1-spaces are AUMD (but not UMD). In fact, the proof of [**5**] even shows that the AUMD property is inherited by all the quotient spaces of the form L_1/R with R a reflexive subspace of L_1. On the other hand, it is easy to see that L_1/H^1 is not AUMD. In this section we wish to point that the AUMD property of L_1-spaces does not extend to the non-commutative case.

THEOREM 4.1. *The space* $c_1 = l_2 \hat{\otimes} l_2$ *fails the* AUMD *property.*

Proof. We use the main trianngle projection $P : c_2 \longrightarrow c_2$ which maps a matrix x to the upper triangular matrix with the same coefficients as x above the diagonal and zero elsewhere. It is well known (c.f. e.g. [**26**], [**20**]) that P is unbounded on c_1.

Let us denote by (e_{ij}) the canonical basis in c_1. Now let

$$x = \sum_{i,j\leqq n} x_{ij}e_{ij}$$

be an $n \times n$ matrix in c_1. Let (z_i) be a sequence of elements in T (identified with the boundary of D).

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We have clearly (in the c_1-norm)

$$\|x\| = \|\sum_{i,j\leqq n} z_{2i+1}x_{ij}z_{2j}e_{ij}\|. \tag{4.2}$$

Indeed, the element on the right is obtained by multiplying x by the diagonal operators with coefficients $(z_{2i+1})_i$ and $(z_{2j})_j$ on the left and right respectively.

Let

$$M(z_1, z_2, \ldots) = \sum_{i,j\leqq n} z_{2i+1}x_{ij}z_{2j}e_{ij}.$$

By an elementary computation one can check that

$$M = \sum_{j=1}^{n} \Delta_j$$

where

$$\Delta_j = z_{2j}\left(\sum_{i<j} x_{ij}z_{2i+1}e_{ij}\right) + z_{2j+1}\left(\sum_{i\leqq j} x_{ji}z_{2i}e_{ji}\right).$$

Let M_k be the conditional expectation of M with respect to the σ-field generated by $(z_1, \ldots, z_k)$. Clearly (M_k) is an analytic martingale with values in c_1. Assume that c_1 has the AUMD property. Then, applying (4.1) to the above martingale and using (4.2) to get rid of the L_p-norms with respect ot (z_i), we obtain simply

$$\|\sum_{i<j} x_{ij}e_{ij}\|_{c_1} \leqq C\|x\|_{c_1},$$

where x is arbitrary in c_1.

Clearly this implies that the main triangle projection P is bounded in c_1. This contradiction completes the proof.

5. Appendix. Here we present a direct proof of (1.4)$\Rightarrow$ARNPin the case $1 \leqq q < \infty$. Recall the notation $f_r(z) = f(rz)$ for $0 < r < 1, z \in D$.

PROPOSITION 5.1. *Let $1 \leqq q < \infty$ and let $\delta > 0$. If X is a Banach space with the property that:*

(i) *For every polynomial f with coefficients in X:*

$$\|f(0)\|^q + \delta\|f - f(0)\|^q_{H^1(X)} \leqq \|f\|^q_{H^1(X)}.$$

Then

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(ii) *For every polynomial* f *with coefficients in* X *and every* $r \in (0, 1)$:

$$\|f_r\|^q_{H^1(X)} + \delta\|f - f_r\|^q_{H^1(X)} \leqq \|f\|^q_{H^1(X)}.$$

Moreover,

(iii) X *has* ARNP.

Proof. Assume (i) and let f be a polynomial with coefficients in X, and let $r \in (0, 1)$. By the Poisson integration formula,

$$f(re^{it}) = \int_T f(e^{is})P(r, s - t)dm(s),$$

where

$$P(r, t) = (1 - r^2)(1 - 2r\cos t + r^2)^{-1}$$

is the Poisson kernel. For $a \in D$, we let τ_a be the Möbius transformation of D given by

$$\tau_a(z) = (z + a)(1 + \bar{a}z)^{-1}.$$

Note that τ_a extends continuously to a transformation of $\bar{D}$ given by the same formula.

It is clear that $f \circ \tau_a \in \tilde{H}^1(X)$, the closure in $H^1(X)$ of the set of polynomials with coefficients in X. Hence we can apply (i) to $f \circ \tau_a$, i.e.,

$$\|f(a)\|^q + \delta\|f \circ \tau_a - f(a)\|^q_{H^1(X)} \leqq \|f \circ \tau_a\|^q_{H^1(X)}. \tag{5.1}$$

Using that $(\tau_a)^{-1} = \tau_{\_a}$, the Radon-Nikodym derivative of the transformation

$$e^{it} = \tau_a^{-1}(e^{iu})$$

can easily be computed, namely

$$\frac{dt}{du} = \frac{1 - |a|^2}{|1 - \bar{a}e^{iu}|^2} = P(r, \theta - u)$$

when $a = re^{i\theta}$. Hence (5.1) is equivalent to:

$$\|f(re^{i\theta})\|^q + \delta\left(\int_T \|f(e^{i\theta})\|P(r, \theta - u)dm(u)\right)^q \leqq \left(\int_T \|f(e^{iu})\|P(r, \theta - u)dm(u)\right)^q. \tag{5.2}$$

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Since the function $(x, y) \longrightarrow (x^q + \delta y^p)^{1/q}$ from $\mathbf{R}^2$ to $\mathbf{R}$ is convex, we get by averaging the q'th root of (5.2) with respect to θ that

$$(5.3) \qquad \|f_r\|^q_{H^1(X)} + \delta \left(\int_T \int_T \|f(e^{iu}) - f(re^{i\theta}\| P(r, \theta - u) dm(u) dm(\theta) \right)^q \leqq \|f\|^q_{H^1(X)}.$$

Here we have used that

$$\int_T P(r, \theta - u) dm(\theta) = 1.$$

The Poisson integral formula applied to f_r yields

$$f(r^2 e^{iu}) = \int_T f(re^{i\theta}) P(r, \theta - u) dm(u),$$

so by the convexity of the norm in $L^1(T, X)$:

$$\int_T \|f(e^{iu}) - f(re^{i\theta})\| P(r, \theta - u) dm(\theta) \geqq \|f(e^{iu}) - f(r^2 e^{iu})\|$$

for every $u \in \mathbf{R}$. Inserting this in (5.3) we have

$$(5.4) \qquad \|f_r\|^q_{H^1(X)} + \delta \|f - f_{r^2}\|^q_{H^1(X)} \leqq \|f\|^q_{H^1(X)}.$$

Since $r \longrightarrow \|f_r\|^q_{H^1(X)}$ is an increasing function on (0, 1), also

$$\|f_{r^2}\|^q_{H^1(X)} + \delta \|f - f_{r^2}\|^q_{H^1(X)} \leqq \|f\|^q_{H^1(X)},$$

so, by substituting r^2 with r, (ii) follows.

(ii) $\Rightarrow$ (iii). Assume (ii). By (1.2) and the remarks preceding Theorem 2.1 it follows easily that the inequality

$$\|f_r\|^q_{H^1(X)} + \delta \|f - f_r\|^q_{H^1(X)} \leqq \|f\|^q_{H^1(X)}$$

remains valid for any $f \in H^1(X)$. Thus for every $f \in H^1(X)$ and every $r \in (0, 1)$.

$$\|f - f_r\|^q_{H^1(X)} \leqq \frac{1}{\delta} (\|f\|^q_{H^1(X)} - \|f_r\|^q_{H^1(X)}).$$

Hence

$$\lim_{r \longrightarrow 1} \|f - f_r\|_{H^1(X)} = 0.$$

Since $f_r \in \tilde{H}^1(X)$ for $r \in (0, 1)$ it follows that $f \in \tilde{H}^1(X)$. Hence, by Theorem 1.1, X has the ARNP.

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Remark. Actually, the proof of (2.1) given above can be very easily adapted to give a direct proof of (ii) in Proposition 5.1 with $\delta = \frac{1}{2}$ and $q = 2$ in the case $x = H \hat{\otimes} H$.

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Addendum. After we had essentially completed this paper, we received a preprint by Paul Muhly (cf. [**30**]]) where he obtains independently results similar to Corollary 2.5 and Remark 2.6. More precisely, he proves a von Neumann algebra version of Sarason's Theorem [**30**, Lemma 1.2 and Theorem 2.3] which contains our Corollary 2.5 (2). Note that our key inequality (2.1) for non-commutative L_1-spaces can easily be derived from Corollary 2.5 (2) by making slight changes in the proof of Theorem 2.1.

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