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AUTOMORPHISM CRITERIA FOR M^*-GROUPS

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Abstract We prove that the determination of all M^*-groups is essentially equivalent to the determination of finite groups generated by an element of order 3 and an element of order 2 or 3 that admit a particular automorphism. We also show how the second commutator subgroup of an M^*-group G can often be used to construct M^*-groups which are direct products with G as one factor. Several applications of both methods are given.

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1. Introduction

It is well known that the full conformal automorphism group $\mathrm{Aut}(X)$ of a Riemann surface (or complex algebraic curve) X of genus $g \geqslant 2$ satisfies $|\mathrm{Aut}(X)| \leqslant 84(g-1)$. Automorphism groups of Riemann surfaces with this maximal number of automorphisms are called *Hurwitz groups*. The article by Conder [**5**] contains a nice survey of known results about Hurwitz groups.

A corresponding analysis of real algebraic curves has also received a good deal of attention. If X is a real algebraic curve with real points (or bordered Klein surface) of algebraic genus $p \geqslant 2$, then its full automorphism group satisfies $|\mathrm{Aut}(X)| \leqslant 12(p-1)$. Groups isomorphic to the automorphism group of such a real curve with this maximal number of automorphisms are called M^*-groups. Several infinite families of M^*-groups have been discovered. In particular, it is known [**15**] that $\mathrm{PSL}(2,q)$ is an M^*-group if and only if $q \neq 2$, 7, 9, 11 or 3^n, where n is odd; some values of q for which $\mathrm{PGL}(2,q)$ is an M^*-group are also known (see [**11**]). For a summary of known families of M^*-groups, see the survey article [**3**].

In this paper we examine the role played by the first and second commutator subgroups of M^*-groups. We show how the second commutator subgroup of an M^*-group G can often be used to construct M^*-groups which are direct products with G as one factor. Using the first commutator subgroup, we then prove that the determination of all M^*-groups is essentially equivalent to the determination of finite groups generated by an element of order 3 and an element of order 2 or 3 that admit particular automorphisms. This relates M^*-groups to groups such as

$$(m,n,r;s)=\langle a,b \mid a^m=b^n=(ab)^r=[a,b]^s=1\rangle$$

and

$$(l,m \mid n,k)=\langle \alpha,\beta \mid \alpha^l=\beta^m=(\alpha\beta)^n=(\alpha\beta^{-1})^k=1\rangle,$$

which are currently an active area of research. Several applications of these methods are given. For example, we determine the precise values of q for which $\mathrm{PGL}(2,q)$ is an M^*-group.

2. Preliminaries

Real algebraic curves can be viewed as *symmetric Riemann surfaces*, namely, Riemann surfaces which admit an anti-analytic involution, or *symmetry* (see [**1**]). The quotient of a Riemann surface under a symmetry is known as a *Klein surface*. Some topological features of the real curve can be obtained from the associated Klein surface in the following way. The real curve disconnects its complexification if and only if the corresponding Klein surface is orientable, and the set of real points of the curve, which consists of a disjoint union of *ovals*, is homeomorphic to the boundary of the surface. In turn, it is known that the orientability and the number of boundary components of a Klein surface can be obtained from a presentation of its uniformizing *non-Euclidean crystallographic (NEC) group*. The algebraic genus of the Klein surface is defined to be the genus of its Riemann double cover. It is well known [**13**] that if X is a bordered Klein surface of algebraic genus $p \geqslant 2$, and G is its full automorphism group, then $|G| \leqslant 12(p-1)$. If $G = 12(p-1)$, then we say G is an *M^*-group*.

A useful way of constructing Klein surfaces of algebraic genus $p \geqslant 2$ is by considering them as the orbit space of the hyperbolic plane under a group of isometries. An NEC group is a discrete subgroup Γ of the group $\mathrm{PGL}(2,\mathbb{R})$ of orientation-preserving and orientation-reversing isometries of the hyperbolic plane U such that the quotient U/Γ is compact. An NEC group is called a *bordered surface group* if it contains a reflection but does not contain elliptic isometries of finite order. Each compact bordered Klein surface X of algebraic genus $p \geqslant 2$ is the quotient U/Λ for some bordered surface group Λ. Moreover, given a surface X so represented, a finite group G is a group of automorphisms of X if and only if there exists an NEC group Γ and an epimorphism from Γ onto G which has Λ as its kernel. Such an epimorphism, whose kernel is a bordered surface group, is called *smooth*. All groups of automorphisms of bordered Klein surfaces arise in this way.

Assume that Λ is a bordered surface group, $X = U/\Lambda$ has algebraic genus p, and $G = \Gamma^*/\Lambda$ satisfies $|G| = 12(p-1)$, for some NEC group Γ^*. Then G is an M^*-group acting on X and it is well known [**13**] that Γ^* is isomorphic to the abstract group with the presentation

$$\langle c_0, c_1, c_2, c_3 \mid c_0^2 = c_1^2 = c_2^2 = c_3^2 = (c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_0)^3 = 1 \rangle, \tag{2.1}$$

where each c_i is a reflection. For each M^*-group G there is a smooth epimorphism $\theta : \Gamma^* \to G$. Since $\Lambda = \ker(\theta)$ is a bordered surface NEC group, at least one of $\{c_0, c_1, c_2, c_3\}$ must belong to Λ. It is easy to see that neither c_0 nor c_3 can belong to Λ since c_0c_3 has order 3. We may assume, by a trivial change of notation if necessary, that $c_1 \in \Lambda$; so $c_2 \notin \Lambda$. Therefore, writing $\theta(c_0) = \alpha$, $\theta(c_2) = \beta$ and $\theta(c_3) = \gamma$, we see that each M^*-group admits the following (partial) presentation:

$$\langle \alpha, \beta, \gamma \mid \alpha^2 = \beta^2 = \gamma^2 = (\beta\gamma)^2 = (\alpha\gamma)^3 = \cdots = 1 \rangle. \tag{2.2}$$

For an M^*-group G with presentation (2.2), we define the index of the presentation to be $q = \operatorname{ord}(\alpha\beta)$. Observe that G may have different indexes since it may have different presentations of the form (2.2). The topological type of a Klein surface $X = U/\Lambda$ associated with $G = \Gamma^*/\Lambda$ can be obtained from this presentation. Indeed, the number of boundary components of X equals $|G|/2q$, and X is orientable if and only if all the relators have an even number of letters α, β and γ (see [**8**]). It is easy to observe that if G has presentation (2.2), then G has an alternate presentation defined by $\alpha' := \alpha$, $\beta' := \beta\gamma$, $\gamma' := \gamma$, whose generators satisfy the corresponding relations in (2.2). The index of this new presentation is $q' := \operatorname{ord}(\alpha\beta\gamma)$. We let ${}^{\mathrm{t}}G$ denote G with this new presentation, and note that ${}^{\mathrm{t}}({}^{\mathrm{t}}G)$ is G with its original presentation.

It is known that the quotient of a Hurwitz group is either trivial or a Hurwitz group (see, for example, [**5**]). In the case of M^*-groups, the situation is slightly different. If G is an M^*-group and N is a normal subgroup of G of index greater than 6, then G/N is an M^*-group (see [**11**]).

In this paper, the subgroups of Γ^* of index 4 or less which contain a bordered surface NEC group as a normal subgroup play a crucial role. There are precisely three subgroups of Γ^* of index 2 which contain c_1 [**2**]. They are

(i) the group Γ_1 generated by c_0, c_1, $c_2c_0c_2$ and c_3;

(ii) the group Γ_2 generated by c_2c_3, c_3c_0 and c_1; and

(iii) the group Γ_3 generated by c_3c_0, c_1, c_2 and $c_3c_1c_3$.

There is a unique normal subgroup which contains c_1 and has index 4 in Γ^* (see [**2**, § 3]). It is

(iv) the group Δ generated by c_0c_3, $c_2c_3c_0c_2$ and c_1.

This shows, in particular, that $\Gamma_i \cap \Gamma_j = \Delta$ for all $i \neq j$.

Notation. We state some conventions used throughout the paper. The cyclic group of order n will be denoted by $\mathbb{Z}_n$, and A_n and S_n will denote the alternating and symmetric groups on n letters, respectively. We will let Λ denote a bordered surface NEC group, Γ^* will always denote a group with presentation as given in (2.1), and G will always denote an M^*-group. Generators of Γ^* will be denoted, as above, by c_0, c_1, c_2, c_3. If $\Lambda \leqslant \Gamma^*$, we will always assume that $c_1 \in \Lambda$. In addition, Γ_1, Γ_2, Γ_3 and Δ will have the above meanings. For any group H, the first and second commutator subgroups of H will be denoted by H' and H'', respectively.

Let Λ_{c_1} denote the normal subgroup of Γ^* generated by c_1. If $\Phi \triangleleft \Gamma^*$ and it contains c_1, we let $\bar{\Phi}$ denote Φ/Λ_{c_1}; in particular, $\bar{\Gamma}^* = \Gamma^*/\Lambda_{c_1}$. Note that $\Gamma^*/\Phi \cong \bar{\Gamma}^*/\bar{\Phi}$. With Δ defined as above, we see that $\bar{\Gamma}^*/\bar{\Delta} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. This implies that the commutator subgroup $(\bar{\Gamma}^*)'$ is contained in $\bar{\Delta}$. Note that $\bar{\Gamma}^*/(\bar{\Gamma}^*)'$ is generated by elements of order 2. Since $\bar{\Gamma}^*/(\bar{\Gamma}^*)'$ is abelian, $c_0(\bar{\Gamma}^*)'$ and $c_3(\bar{\Gamma}^*)'$ commute; however, c_0c_3 has order 3. This implies that $c_0c_3 \in (\bar{\Gamma}^*)'$, so $\bar{\Gamma}^*/(\bar{\Gamma}^*)'$ is actually generated by two elements. This yields that $\bar{\Delta} = (\bar{\Gamma}^*)'$. Note that $\bar{\Delta}$ is a free product generated by two elements of order 3. This implies that $\bar{\Delta}/\bar{\Delta}' \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, which yields that $\bar{\Gamma}^*/(\bar{\Gamma}^*)'' \cong S_3 \times S_3$.

For future reference, we will need the following lemma.

Lemma 2.1. *Let H be a group and let N be a normal subgroup of H. Then $(H/N)' = H'N/N$ and $(H/N)'' = H''N/N$.*

3. Commutator subgroups of M^*-groups

Theorem 3.1. *Let G be an M^*-group. Then there exists a normal subgroup N of $S_3 \times S_3$ such that we have the following.*

(i) *$G/G'' \cong (S_3 \times S_3)/N$.*

(ii) *For each $N_1 \triangleleft S_3 \times S_3$ with $N_1 \leqslant N$, let $K = N/N_1$. Then there exists an M^*-group $\hat{G}$ such that*
$$1 \to K \to \hat{G} \to G \to 1$$
is a short exact sequence. Furthermore, $\hat{G}$ contains a subgroup isomorphic to $G'' \times K$.

Proof. Since G is an M^*-group, there exists a smooth epimorphism $\theta : \Gamma^* \to G$, such that $c_1 \in \Lambda := \ker(\theta)$. Since $G \cong \bar{\Gamma}^*/\bar{\Lambda}$, we obtain from Lemma 2.1 that $G' \cong (\bar{\Gamma}^*)'\bar{\Lambda}/\bar{\Lambda}$ and $G'' \cong (\bar{\Gamma}^*)''\bar{\Lambda}/\bar{\Lambda}$. Since $\bar{\Gamma}^*/(\bar{\Gamma}^*)'' \cong S_3 \times S_3$, we define $N := (\bar{\Gamma}^*)''\bar{\Lambda}/(\bar{\Gamma}^*)''$ to obtain that $G/G'' \cong \bar{\Gamma}^*/(\bar{\Gamma}^*)''\bar{\Lambda} \cong (S_3 \times S_3)/N$. This proves (i).

To prove (ii), let $N_1 \triangleleft S_3 \times S_3$ with $N_1 \leqslant N$. Since $N = (\bar{\Gamma}^*)''\bar{\Lambda}/(\bar{\Gamma}^*)''$, there exists an NEC group $\bar{\Delta}_1 \leqslant (\bar{\Gamma}^*)''\bar{\Lambda}$ such that $N_1 \cong \bar{\Delta}_1/(\bar{\Gamma}^*)''$. Since $(\bar{\Gamma}^*)'' \leqslant \bar{\Delta}_1 \leqslant (\bar{\Gamma}^*)''\bar{\Lambda}$, we get $(\bar{\Gamma}^*)''\bar{\Lambda} = \bar{\Delta}_1\bar{\Lambda}$ and $N \cong (\bar{\Gamma}^*)''\bar{\Lambda}/(\bar{\Gamma}^*)'' = \bar{\Delta}_1\bar{\Lambda}/(\bar{\Gamma}^*)''$. Define $\hat{G} = \bar{\Gamma}^*/(\bar{\Lambda} \cap \bar{\Delta}_1)$. Then $\hat{G}$ contains the subgroup
$$\frac{\bar{\Lambda}}{\bar{\Lambda} \cap \bar{\Delta}_1} \cong \frac{\bar{\Lambda}\bar{\Delta}_1}{\bar{\Delta}_1} \cong \frac{(\bar{\Gamma}^*)''\Lambda/(\bar{\Gamma}^*)''}{\bar{\Delta}_1/(\bar{\Gamma}^*)''} \cong \frac{N}{N_1} \cong K.$$

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Furthermore, the subgroups $K \cong \bar{\Lambda}/(\bar{\Lambda} \cap \bar{\Delta}_1)$ and $G'' \cong \bar{\Delta}_1/(\bar{\Lambda} \cap \bar{\Delta}_1)$ are both normal, generate $\bar{\Lambda}\bar{\Delta}_1/(\bar{\Lambda} \cap \bar{\Delta}_1)$ and have trivial intersection. Therefore, $\bar{\Lambda}\bar{\Delta}_1/(\bar{\Lambda}\cap\bar{\Delta}_1) \cong G'' \times K$. This proves (ii). □

Recall that a group is said to be perfect if $G = G'$. If G is perfect, then $G'' = G$; therefore, for each factor group K of $S_3 \times S_3$, there is an M^*-group $\hat{G}$ of order $|G|\,|K|$ which contains a subgroup isomorphic to $G \times K$. Therefore, $\hat{G} \cong G \times K$. This shows the following.

Corollary 3.2. *Let G be a perfect M^*-group. Then $G \times \mathbb{Z}_2$, $G \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $G \times S_3$, $G \times \mathbb{Z}_2 \times S_3$, and $G \times S_3 \times S_3$ are M^*-groups.*

Example 3.3. Recall that $\mathrm{PSL}(2, q)$ is an M^*-group if and only if $q \neq 2$, 7, 9, 11 or 3^n, where n is odd [**15**]. For these values, $\mathrm{PSL}(2, q)$ is simple, therefore $\mathrm{PSL}(2, q) \times \mathbb{Z}_2$, $\mathrm{PSL}(2, q) \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathrm{PSL}(2, q) \times S_3$, $\mathrm{PSL}(2, q) \times \mathbb{Z}_2 \times S_3$ and $\mathrm{PSL}(2, q) \times S_3 \times S_3$ are M^*-groups.

Example 3.4. Using the generators of the alternating group A_n given in [**4**], it was shown [**9**] that A_n is an M^*-group for all but finitely many values of n. For these values we get that $A_n \times \mathbb{Z}_2$, $A_n \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $A_n \times S_3$, $A_n \times \mathbb{Z}_2 \times S_3$ and $A_n \times S_3 \times S_3$ are M^*-groups.

Proposition 3.5. *An M^*-group possesses either zero, one or three subgroups of index 2. An M^*-group possesses at most one normal subgroup of index 4.*

Proof. Let $G = \Gamma^*/\Lambda$ be an M^*-group with presentation (2.2). It can have at most three subgroups of index 2 and one normal subgroup of index 4 since Γ^* has exactly three subgroups of index 2 which contain Λ (namely Γ_1, Γ_2 and Γ_3) and a unique normal subgroup of index 4 which contains Λ (namely Δ). The subgroups of G corresponding to each of Γ_1, Γ_2, Γ_3 and Δ are $G_1 := \langle \alpha, \beta\alpha\beta, \gamma \rangle$, $G_2 := \langle \beta\gamma, \gamma\alpha \rangle$, $G_3 := \langle \beta, \gamma\alpha \rangle$ and $G_4 := \langle \alpha\gamma, \beta\alpha\gamma\beta \rangle$. Note that $G_i = G$ if and only if Λ is not a subgroup of Γ_i. It may occur that an M^*-group has no subgroups of index 2, or exactly one subgroup of index 2, as evidenced by Examples 3.3 and 3.4. However, if G has two different subgroups of index 2, then $\Lambda \leqslant \Gamma_i \cap \Gamma_j = \Delta$ and so Λ is contained in Γ_1, Γ_2 and Γ_3. This yields three subgroups of index 2 in G, and one of index 4. □

Let $\mathrm{Inn}(H)$ denote the group in inner automorphisms of the group H, and let $\mathrm{Inn}_2(H)$ denote its subset consisting of those induced by an element of order 2. If G is an M^*-group, then $[G : G'] = 1$, 2, or 4, so G is a semidirect product of G' with a group of order 1, 2 or 4. We use the following lemmas to precisely determine the structure of such groups. The proofs are left to the reader.

Lemma 3.6. *Let H be a group, let A be a group of automorphisms of H and let $G = H \rtimes A$ denote the semidirect product. Then there is an element $(g, \phi) \in G$ which commutes with every element of H if and only if ϕ is the inner automorphism of H induced by g.*

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Lemma 3.7. *Let H be a group, and let $G = H \rtimes \langle \phi_1, \phi_2 \rangle$, where ϕ_1 and ϕ_2 are commuting automorphisms of H which each have order 2. Define $\phi_3 := \phi_1\phi_2$. Then we have the following.*

(i) *The semidirect product G is actually the direct product $H \times \mathbb{Z}_2 \times \mathbb{Z}_2$ if and only if the following both hold.*

  (a) *$\phi_i \in \mathrm{Inn}_2(H)$ for $i = 1, 2, 3$.*

  (b) *There are an h_1 and an h_2, each of order 2, which induce ϕ_1 and ϕ_2, respectively, and $h_1 h_2$ has order 2.*

(ii) *The semidirect product $H \rtimes \langle \phi_1, \phi_2 \rangle$ does not split as in* (i) *but is the direct product $(H \rtimes \langle \phi_i \rangle) \times \mathbb{Z}_2$ if and only if the following both hold.*

  (a) *$\phi_i \notin \mathrm{Inn}_2(H)$.*

  (b) *For some $j \neq i$, $\phi_j \in \mathrm{Inn}_2(H)$. There is an $h_j \in H$ of order 2 which induces ϕ_j by conjugation and $h_j = \phi_i(h_j)$.*

Lemma 3.7 can be restated far more simply if we consider the quotient of H by its centre.

Lemma 3.8. *Let H be a group and let ϕ_1 and ϕ_2 be commuting automorphisms of H which each have order 2. Define $\phi_3 := \phi_1\phi_2$. Let $Z(H)$ denote the centre of H, let $\bar{H} = H/Z(H)$ and let $G = \bar{H} \rtimes \langle \phi_1, \phi_2 \rangle$. Then*

(i) *the semidirect product G is actually the direct product $\bar{H} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ if and only if two of ϕ_1, ϕ_2 and ϕ_3 are inner automorphisms of H; and*

(ii) *the semidirect product G does not split as in* (i) *but is the direct product $(\bar{H} \rtimes \langle \phi_i \rangle) \times \mathbb{Z}_2$ if and only if $\phi_i \notin \mathrm{Inn}(H)$ and, for some $j \neq i$, $\phi_j \in \mathrm{Inn}(H)$.*

Definition 3.9. Let G be as in Lemma 3.7. If (i) holds we say that the pair $\{\phi_i, \phi_j\}$ is a *fully splitting pair for H*, or that $\{\phi_i, \phi_j\}$ *fully splits*. If (ii) holds, we say (ϕ_i, ϕ_j) *is a partially splitting pair for H*, or that (ϕ_i, ϕ_j) *partially splits*. In this case, the automorphisms are listed so that $\phi_i \notin \mathrm{Inn}_2(H)$ and $\phi_j \in \mathrm{Inn}_2(H)$. If neither (i) nor (ii) holds, we say $\{\phi_i, \phi_j\}$ *is not a splitting pair for H*, or that $\{\phi_i, \phi_j\}$ *does not split*.

We will denote the M^*-group G, with a particular choice of generators α, β and γ, as in (2.2) by $G(\alpha, \beta, \gamma)$. Let $\mathcal{M}$ denote the set of all such groups, in other words, $\mathcal{M}$ yields all of the ways in which M^*-groups can be epimorphic images of Γ^*. Let H be a finite group which can be generated by elements a and b, each of order 3, and which possesses the group automorphisms

$$\phi_1 : a \mapsto b,\ b \mapsto a, \quad \text{and} \quad \phi_2 : a \mapsto a^{-1},\ b \mapsto b^{-1}.$$

We let $H(a, b)$ denote H with this particular pair of generators. Let $X\{3, 3\}$ denote the set of all groups $H(a, b)$. We now show that the determination of all M^*-groups is essentially equivalent to the determination of all groups in $X\{3, 3\}$.

Theorem 3.10. *Define $\Psi : \mathcal{M} \to X\{3,3\}$ by $\Psi(G(\alpha,\beta,\gamma)) = G'(\alpha\gamma, \beta\alpha\gamma\beta)$. This map is onto $X\{3,3\}$. For each $H(a,b) \in X\{3,3\}$ with $|H(a,b)| > 6$,*

$$\Psi^{-1}(H(a,b)) = \begin{cases} \{H \rtimes \langle\phi_1,\phi_2\rangle\} & \text{if } \{\phi_1,\phi_2\} \text{ does not split,} \\ \{H \rtimes \langle\phi_i\rangle, (H \rtimes \langle\phi_i\rangle) \times \mathbb{Z}_2\} & \text{if } (\phi_i,\phi_j) \text{ partially splits,} \\ \{H, H \times \mathbb{Z}_2, H \times \mathbb{Z}_2 \times \mathbb{Z}_2\} & \text{if } \{\phi_1,\phi_2\} \text{ fully splits.} \end{cases}$$

Observe that if the generators of each group are not considered, then Ψ maps each M^*-group to its first commutator subgroup.

Proof. Let $G(\alpha,\beta,\gamma)$ be an M^*-group with presentation (2.2). From the introduction, $(\Gamma^*)' = \Delta$ and is generated by c_0c_3, $c_2c_3c_0c_2$ and c_1. Lemma 2.1 yields that G' is generated by the image of Δ, so $G' = \langle\alpha\gamma, \beta\alpha\gamma\beta\rangle$. Note that $G'(\alpha\gamma, \beta\alpha\gamma\beta) \in X\{3,3\}$, since $\alpha\gamma$ and $\beta\alpha\gamma\beta$ each have order 3, conjugation by β interchanges the two generators, and conjugation by γ maps each generator to its inverse. Therefore, the map is well defined.

We now show that Ψ is onto. Given $H(a,b) \in X\{3,3\}$, define $G := H \rtimes \langle\phi_1,\phi_2\rangle$ and define an epimorphism $\theta : \Gamma^* \to G$ by

$$\alpha := \theta(c_0) = (a,e)(e,\phi_2), \qquad \beta := \theta(c_2) = (e,\phi_1), \qquad \gamma := \theta(c_3) = (e,\phi_2). \tag{3.1}$$

It is easily verified that the images of the generators of Γ^* satisfy presentation (2.2), and $\Psi(G(\alpha,\beta,\gamma)) = G'(a,b) = H(a,b)$. This proves that the map is onto, but Lemma 3.7 also shows that G can be rewritten as $H \rtimes \langle\phi_1,\phi_2\rangle$, $(H \rtimes \langle\phi_i\rangle) \times \mathbb{Z}_2$ or $H \times \mathbb{Z}_2 \times \mathbb{Z}_2$ when the pairs are not splitting, partially splitting and fully splitting, respectively. We now show that the three remaining groups listed in the theorem map to $H(a,b)$. Assume that $\{\phi_1,\phi_2\}$ is fully splitting, and let conjugation by h_1 and h_2 induce ϕ_1 and ϕ_2, respectively. Replacing (e,ϕ_1) and (e,ϕ_2) in (3.1) by (h_1,e) and (h_2,e), respectively, yields that $H(\alpha,\beta,\gamma) = H(a,b)$, and it is an M^*-group which maps to $H(a,b)$. Replacing only (e,ϕ_1) in (3.1) by (h_1,e) yields that $H \rtimes \langle\phi_2\rangle = H \times \mathbb{Z}_2$ is an M^*-group which maps to $H(a,b)$. Assume now that (ϕ_i,ϕ_j) is partially splitting, and let h_j be an element of order 2 which induces ϕ_j by conjugation. If $j = 1$ or 2, then replacing (e,ϕ_j) in (3.1) by (h_j,e) yields that $H \rtimes \langle\phi_i\rangle$ maps to $H(a,b)$. If $j = 3$, then replacing (e,ϕ_1) by (h_3,ϕ_2) yields that $H \rtimes \langle\phi_i\rangle$ maps to $H(a,b)$. Therefore, each of the six indicated groups maps to $H(a,b)$.

We now show that the inverse image of $H(a,b)$ contains no groups other than the ones listed. Let $G(\alpha,\beta,\gamma)$ be an M^*-group such that $G' = H(a,b)$. We know that $[G : H] = 1$, 2 or 4. If $G = H$, then conjugation by β, γ and $\beta\gamma$ are inner automorphisms of H, so $\{\phi_1,\phi_2\}$ is a completely splitting pair. If $[G : H] = 2$, assume first that $\beta \notin H$. Then either $\gamma \in H$ or $\beta\gamma \in H$. The first possibility yields that (ϕ_1,ϕ_2) is a partially splitting pair and $G \cong H \rtimes \langle\phi_1\rangle$; the second possibility yields that (ϕ_2,ϕ_3) is a partially splitting pair and $G \cong H \rtimes \langle\phi_2\rangle$. If $\beta \in H$, then $\gamma \notin H$, so (ϕ_2,ϕ_1) is a partially splitting pair and $G \cong H \rtimes \langle\phi_2\rangle$. Finally, if $[G : H] = 4$, then clearly neither β nor γ nor $\beta\gamma$ can be in H. This yields that $\{\beta,\gamma\}$ is not a splitting pair and $G \cong H \rtimes \langle\phi_1,\phi_2\rangle$. □

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All M^*-groups G are generated by three elements; however, if $G = G'$, then $G \in X\{3,3\}$. This yields the following.

Corollary 3.11. *Every perfect M^*-group is generated by two elements of order* 3.

The map Ψ in Theorem 3.10 yields a canonical way of associating generators for an M^*-group with generators for its commutator subgroup. We will always consider the relationship of a group $H(a,b) \in X\{3,3\}$ to an M^*-group G with $G' = H$ in terms of the map Ψ. For example, we will say that there is a unique M^*-group $G(\alpha,\beta,\gamma)$ with commutator subgroup $H(a,b)$ if there is a unique $G(\alpha,\beta,\gamma)$ which maps to $H(a,b)$ under Ψ. It may happen, however, that if we consider H with other generating sets, then there may be several M^*-groups which have H as their commutator subgroup.

Corollary 3.12. *Let $H(a,b) \in X\{3,3\}$. If $\phi_2 \in \mathrm{Inn}(H)$, then $H = H'$. If ϕ_1 or $\phi_3 \in \mathrm{Inn}(H)$, then $[H : H'] = 1$ or 3. In particular, if $[H : H'] \neq 1$ or 3, then $H \rtimes \langle \phi_1, \phi_2 \rangle$ is the only M^*-group which has $H(a,b)$ as its commutator subgroup.*

Proof. If $h_2 \in H$ induces ϕ_2 by conjugation, then $h_2 a h_2^{-1} H' = a^2 H'$ implies $a \in H'$. Similarly, $b \in H'$, therefore $H = H'$. If $j = 1$ or 3, and if $h_j \in H$ induces ϕ_j by conjugation, then $h_j a h_j^{-1} H' = bH'$ or $b^2 H'$. This implies that $H = \langle H', a \rangle$, which implies $[H : H'] = 1$ or 3. The last statement follows from the first two and Theorem 3.10. □

Theorem 3.10 reveals that the determination of all M^*-groups is equivalent to the determination of all groups in $X\{3,3\}$. It also relates M^*-groups to the groups $(3,3 \mid n,k)$ and $(3,3,r;s)$ which belong to families which have been extensively studied (see [**7**] and [**12**] and the references contained therein for an examination of these groups). The group $(l,m \mid n,k)$ has the presentation

$$\langle a,b \mid a^l = b^m = (ab)^n = (ab^{-1})^k = 1 \rangle \tag{3.2}$$

and the $(m,n,r;s)$ group is defined by

$$\langle a,b \mid a^m = b^n = (ab)^r = [a,b]^s = 1 \rangle. \tag{3.3}$$

For each family, it is known precisely when $(l,m \mid n,k)$ and $(m,n,r;s)$ are finite, with the exception of the $(2,3,13;4)$ group, whose finiteness is still an open problem.

The groups $(3,3 \mid n,k)$ and $(3,3,r;s)$ admit the automorphism ϕ_1, which interchanges a and b, and the automorphism ϕ_2, which maps each generator to its inverse. If they are finite, then they belong to $X\{3,3\}$ and therefore Theorem 3.10 shows that they provide M^*-groups. In addition, it also shows that the commutator subgroup of each M^*-group can be realized as a finite factor group of the groups $(3,3 \mid n,k)$ or $(3,3,r;s)$ for some n and k and r and s. Using $(3,3 \mid n,k)$ as an example, a normal subgroup of $N \triangleleft (3,3 \mid n,k)$ must be defined in such a way as to ensure that $H \cong (3,3 \mid n,k)/N$ possesses the group automorphisms ϕ_1 and ϕ_2. This can be done by adding in relations of the form $R(a,b) = R(b,a) = R(a^{-1},b^{-1}) = R(b^{-1},a^{-1}) = 1$. Every finite group H obtained in this way will be the commutator subgroup of an M^*-group, and the related M^*-groups can be obtained by Theorem 3.10.

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Example 3.13. It is known [**6**] that $H = (3,3,3;s)$ is a finite group of order $9s^2$. Then $G = H \rtimes \langle \phi_1, \phi_2 \rangle$ is an M^*-group. Clearly, $\mathbb{Z}_3 \times \mathbb{Z}_3$ is an epimorphic image of H. Therefore, $[H : H'] = 9$ and so Corollary 3.12 yields that G is the only M^*-group which has $H(a,b)$ as its commutator subgroup.

Example 3.14. It is known [**6**] that $H = (3,3,4;5)$ is a finite group of order 1080. Using GAP* we determine that ϕ_2 is induced by conjugation by an element $h_2 \in H$ of order 2 and $\phi_1(h_2) = h_2$. Therefore, both $G := (H \rtimes \langle \phi_1 \rangle) \times \mathbb{Z}_2$ and $K := H \rtimes \langle \phi_1 \rangle$ are M^*-groups, and they are the only M^*-groups whose commutator subgroup is $H(a,b)$. It can be shown that $G \cong G^{3,8,10}$, which is a known M^*-group [**8**]. In addition, $K \cong (2,3,8;5)$, so $(2,3,8;5)$ is an M^*-group of order 2160. Using GAP, we determine that K possesses a normal subgroup isomorphic to $\mathbb{Z}_3$ and $K/\mathbb{Z}_3 \cong \mathrm{PGL}(2,9)$. Therefore, $\mathrm{PGL}(2,9)$ is an M^*-group.

We now examine how the index of the presentation of an M^*-group can be obtained from its commutator subgroup.

Proposition 3.15. *Let $H(a,b) \in X\{3,3\}$ and let $q = \mathrm{ord}(ab^{-1})$. Assume that $G(\alpha,\beta,\gamma)$ has $H(a,b)$ as its commutator subgroup. If $[G : H] = 2$ or 4, then the index of the presentation of G is $2q$. If $[G : H] = 1$, then the index is either q or $2q$.*

Proof. Recall that the index of $G(\alpha,\beta,\gamma)$ is defined to be the order of $\alpha\beta$. If $[G : H] = 4$, then $G = H \rtimes \langle \phi_1, \phi_2 \rangle$ with generators as in (3.1). Note that

$$(\alpha\beta)^2 = ab^{-1}, \quad \text{so } (\alpha\beta)^{2k+j} = (ab^{-1})^k (a, \phi_2\phi_1)^j. \tag{3.4}$$

This shows that $\alpha\beta$ has even order, so $\mathrm{ord}(\alpha\beta) = 2\,\mathrm{ord}(ab^{-1})$. Assume now that $[G : H] = 2$. Then $G = H \times \mathbb{Z}_2$ if $\{\phi_1, \phi_2\}$ fully splits, or $G = H \rtimes \langle \phi_i \rangle$ if (ϕ_i, ϕ_j) partially splits. In both cases, generators α, β and γ for G are given in the proof of Theorem 3.10. The same calculation (3.4) yields that $\mathrm{ord}(\alpha\beta) = 2\,\mathrm{ord}(ab^{-1})$. If $G = H$, then the index of G may be odd or even, so the index of G will be either $\mathrm{ord}(ab^{-1})$ or $2\,\mathrm{ord}(ab^{-1})$. □

In § 1 we noted that if $G(\alpha,\beta,\gamma)$ is an M^*-group, then ${}^{\mathrm{t}}G = G(\alpha, \beta\gamma, \gamma)$ is also an M^*-group with index $\mathrm{ord}(\alpha\beta\gamma)$. It is easy to observe that $G'(\alpha,\beta,\gamma) = H(a,b)$ if and only if $G'(\alpha,\beta\gamma,\gamma) = H(a,b^{-1})$.

4. Subgroups of index 2

We continue with the notation of the previous section with one small modification. Let $G(\alpha,\beta,\gamma) \in \mathcal{M}$ be an M^*-group. Let $H(a,b)$ now denote a finite group generated by a and b, of orders 2 and 3, respectively, which possesses the automorphism

$$\phi : a \mapsto a, \ b \mapsto b^{-1}.$$

Let $X\{2,3\}$ denote the set of all such groups $H(a,b)$. We now show that the determination of all M^*-groups is essentially equivalent to the determination of all groups in $X\{2,3\}$.

* The GAP Group, *GAP—groups, algorithms, and programming*, v. 4.1 (2000) (http://www.gap-system.org).

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Theorem 4.1. *Define $\Psi : \mathcal{M} \to X\{2,3\}$ by $\Psi : G(\alpha,\beta,\gamma) \mapsto \langle \beta\gamma, \alpha\gamma\rangle$. This map is onto $X\{2,3\}$. For each $H(a,b) \in X\{2,3\}$ with $|H(a,b)| > 6$,*

$$\Psi^{-1}(H) = \begin{cases} \{H \rtimes \langle\phi\rangle\} & \text{if } \phi \notin \mathrm{Inn}_2(H), \\ \{H, H \times \mathbb{Z}_2,\} & \text{if } \phi \in \mathrm{Inn}_2(H). \end{cases}$$

Proof. Recall from § 1 that Γ_2 is a subgroup of index 2 in Γ^*. Let G be an M^*-group with presentation (2.2). The subgroup of G corresponding to Γ_2 is the subgroup generated by $\beta\gamma$ and $\alpha\gamma$ and these elements have orders 2 and 3, respectively. In addition, conjugation by γ maps $\beta\gamma \mapsto \beta\gamma$ and $\alpha\gamma \mapsto (\alpha\gamma)^{-1}$. Therefore, the map Ψ is well defined.

Given any $H(a,b) \in X\{2,3\}$, it is easy to see that $H \rtimes \langle\phi\rangle$ with generators $\alpha := (b,\phi)$, $\beta := (a,\phi)$, $\gamma := (e,\phi)$ is an M^*-group. This shows that ϕ is onto and it shows that $H \rtimes \langle\phi\rangle$ is in the inverse image of $H(a,b)$. In addition, if $G(\alpha,\beta,\gamma)$ is any M^*-group with $\Psi(G) = H(a,b)$ and $[G : H] = 2$, then $G \cong \langle\beta\gamma,\gamma\alpha\rangle \rtimes \langle\gamma\rangle$, so $G \cong H \rtimes \langle\phi\rangle$. If ϕ is an inner automorphism by an element of order 2, then Lemma 3.6 yields that G is actually a direct product. The only other possibility is if $G = \langle\beta\gamma,\gamma\alpha\rangle = H$. In this case, γ is generated by these elements, therefore $\phi \in \mathrm{Inn}_2(H)$. This proves the theorem. □

Theorem 4.1 reveals that the determination of all M^*-groups is equivalent to the determination of all groups in $X\{2,3\}$. These are the finite quotient groups of the modular group $\mathrm{PSL}(2,\mathbb{Z}) = \langle a, b \mid a^2 = b^3 = 1\rangle$ that admit the automorphism ϕ. This can be done by adding in relations of the form $R(a,b) = R(a,b^{-1}) = 1$ to $\mathrm{PSL}(2,\mathbb{Z})$.

Notice that we have examined the subgroups of index 2 of M^*-groups corresponding to Γ_2; it is reasonable to ask if an analysis of Γ_3 might have yielded different results. However, the map Ψ' corresponding to Γ_3 is $G(\alpha,\beta,\gamma) \mapsto \langle\beta,\alpha\gamma\rangle$. It is easy to observe that this is the same as

$$G(\alpha,\beta,\gamma) \mapsto {}^{\mathrm{t}}G(\alpha,\beta,\gamma) = G(\alpha,\beta\gamma,\gamma) \mapsto \Psi(G(\alpha,\beta\gamma,\gamma)) = \langle\beta,\alpha\gamma\rangle.$$

Therefore, $G \in \Psi^{-1}(H)$ if and only if ${}^{\mathrm{t}}G \in \Psi'^{-1}(H)$.

The map corresponding to Γ_1 yields subgroups of index 2 that are generated by three elements a, b and c of order 2 such that bc and ac both have order 3 and which admit $\phi : a \mapsto b \mapsto a,\ c \mapsto c$ as an automorphism (see [**2**]). Using this criterion to determine M^*-groups appears to be more difficult than determining M^*-groups by using the presentation (2.2). In addition, less things are known about this family of groups. For these reasons, we do not examine this map further.

Corollary 4.2. *An M^*-group which possesses at most one subgroup of index 2 is generated by two elements of orders 2 and 3, respectively.*

Proof. The indices of $\langle\beta\gamma,\alpha\gamma\rangle$ and $\langle\beta,\alpha\gamma\rangle$ in $G(\alpha,\beta,\gamma)$ are 1 or 2. The hypothesis implies that $G = \langle\beta\gamma,\alpha\gamma\rangle$ or $G = \langle\beta,\alpha\gamma\rangle$, and we are done. □

5. Applications

Theorems 3.10 and 4.1 can be used to find M^*-groups associated with particular groups. Theorem 4.1, in particular, provides more flexibility than using presentation (2.2) in determining M^*-groups; only two generators need to be determined, the third generator is replaced by an automorphism which does not need to be an inner automorphism. In addition, by finding sets of generators for which the corresponding automorphisms can be chosen to be either inner or not, several M^*-groups can be determined simultaneously.

Example 5.1. Let H be a finite group with $|H| > 6$. The following is a procedure, based on Theorem 4.1, that determines M^*-groups which are related to H. A similar procedure can be defined using Theorem 3.10.

(i) Is H generated by elements a and b of orders 2 and 3, respectively?

(ii) Does H possess the automorphism $\phi : a \mapsto a$, $b \mapsto b^{-1}$?

(iii) Can a and b be chosen so that ϕ is an inner automorphism?

(iv) Is ϕ an automorphism induced by conjugation by an element of order 2?

Affirmative answers to (i) and (ii) yield that $H \rtimes \langle\phi\rangle$ is an M^*-group. An affirmative answer to (iii) yields, in addition, that $H/Z(H)$ is an M^*-group. An affirmative answer to (iv) yields, in addition, that H is an M^*-group.

Let $q = p^n$, where p is a prime. Singerman showed that $\mathrm{PSL}(2, q)$ is an M^*-group if and only if $q \neq 2, 7, 9, 11$ or 3^n, where n is odd [**15**]. Singerman's analysis, in conjunction with Theorems 3.10 and 4.1, allows us to determine all values of q for which $\mathrm{PGL}(2, q)$ is an M^*-group. We may assume that $q \neq 2^n$, since $\mathrm{PGL}(2, 2^n) \cong \mathrm{PSL}(2, 2^n)$.

Theorem 5.2. $\mathrm{PGL}(2, q)$ *is an M^*-group if and only if $q \neq 2$ or* 5.

We seek the values of q for which $\mathrm{PSL}(2, q) \in X\{2, 3\}$ and ϕ is an outer automorphism. For these q, Theorem 4.1 yields that $\mathrm{PSL}(2, q) \rtimes \langle\phi\rangle \cong \mathrm{PGL}(2, q)$ is an M^*-group. Matrices $A, B \in \mathrm{PSL}(2, q)$ of orders 2 and 3, respectively, can be defined that have the property that they generate $\mathrm{PSL}(2, q)$ if $\gamma := \mathrm{tr}(AB)$ is 'admissible' (see [**15**] for the precise definition). There are at most $p^{n-1} + \varepsilon p^{n/2} + 11$ inadmissible values for γ, with $\varepsilon = 1$ if n is even and $\varepsilon = 0$ otherwise [**15**]. For an admissible γ, there exists $Z \in \mathrm{PSL}(2, q)$ such that $Z^2 = (AZ)^2 = (BZ)^2 = 1$ if and only if $3 - \gamma^2$ is a square in $\mathrm{GF}(p^n)$; Singerman was concerned with this case in [**15**]. However, if $3 - \gamma^2$ is not a square, then Z satisfies the same relations but $Z \in \mathrm{PGL}(2, q) \setminus \mathrm{PSL}(2, q)$. This implies that if $3 - \gamma^2$ is a non-square, then conjugation by Z is an outer automorphism ϕ of $\mathrm{PSL}(2, q)$, and therefore $\mathrm{PGL}(2, q) \cong \mathrm{PSL}(2, q) \rtimes \langle\phi\rangle$. We focus our attention on this case.

Assume first that $p > 3$. Then there are at least $(p^n - 3)/2$ values of γ for which $3 - \gamma^2$ is not a square in $\mathrm{GF}(p^n)$ [**15**]. If $q = p > 23$ or $q \neq 5^2, 7^2$, then $(p^n - 3)/2 > p^{n-1} + \varepsilon p^{n/2} + 11$. So for these values of q there exists an admissible γ such that $3 - \gamma^2$ is not a square; therefore, $\mathrm{PGL}(2, q)$ is an M^*-group.

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Table 1. *Admissible values for $\gamma = z^k$ with $3 - \gamma^2$ a non-square*

q	13	17	25	49
k	4	1	1	1

Also $\mathrm{PGL}(2,q)$ is an M^*-group for $q = 3$, 7, 11, 19 or 23, since in these cases q is prime and $q \equiv 3 \pmod 4$ (see [**11**]). Example 3.14 shows that $\mathrm{PGL}(2,9)$ is an M^*-group. Using GAP to determine all possible generators of $\mathrm{PGL}(2,5)$ yields that $\mathrm{PGL}(2,5)$ is not an M^*-group. For $q = 13$, 17, 5^2 and 7^2, Table 1 below indicates a value for which γ is admissible and $3 - \gamma^2$ is a non-square. If z is a generator for $\mathrm{GF}(q)^*$, then the table lists a power k such that $z^k = \gamma$.

The remaining case to consider is $q = 3^n$, with $n > 2$. If n is odd, then $3 - \gamma^2 = -\gamma^2$ is a non-square since -1 is a non-square [**15**]. So $\mathrm{PGL}(2,3^n)$ is an M^*-group if $n > 2$ is odd.

We now consider $q = 3^n$ with $n > 2$ and even. We show that $\mathrm{PGL}(2,q)$ is an M^*-group in this case; however, this cannot be done using Theorem 4.1. For this case, $\mathrm{PSL}(2,q)$ is a subgroup of index 2 of $\mathrm{PGL}(2,q)$ which corresponds to Γ_1, and therefore $\mathrm{PSL}(2,q)$ is not generated by two elements A and B of orders 2 and 3 for which ϕ is an outer automorphism. We employ Theorem 3.10 instead.

We define matrices B_1, B, S and T, respectively, by

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 2w+1 \\ 2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \mathrm{i} & \mathrm{i} \\ 0 & -\mathrm{i} \end{pmatrix}. \tag{5.1}$$

Lemma 2.10 of [**10**] yields that $\gamma := \mathrm{tr}(B_1 B) = 1 + 2w$ is admissible if $\gamma^2 \neq 0, 1, 2, 3, 4$, $\gamma^2 \pm \gamma - 1 \neq 0$ and γ does not belong to a proper subfield of $\mathrm{GF}(p^n)$. In (5.1), we choose 'i' to be a square root of -1, and w to be a non-square of $\mathrm{GF}(3^n)$ such that $1 + 2w$ is admissible. Clearly, this can be done if $n \geqslant 4$. Note that B, B_1 and T are in $\mathrm{PSL}(2,3^n)$, have orders 3, 3 and 2, respectively, and the choice of w guarantees that B_1 and B generate $\mathrm{PSL}(2,3^n)$. Note that $TB_1T^{-1} = B_1^{-1}$ and $TBT^{-1} = B^{-1}$. Since $\det S = -4w$ is a non-square in $\mathrm{GF}(3^n)$, $S \notin \mathrm{PSL}(2,3^n)$. In addition, S has order 2, $SB_1S^{-1} = B$ and $SBS^{-1} = B_1$. Therefore, $\mathrm{PSL}(2,3^n)$ possesses the automorphisms required in Theorem 3.10, so $\mathrm{PSL}(2,3^n) \rtimes \langle S \rangle \cong \mathrm{PGL}(2,3^n)$ is an M^*-group.

Example 5.3. Using Theorem 3.10 and the known description of automorphisms of $\mathrm{PSL}(2,q)$ (see [**14**]), it can be proved that if $\mathrm{PSL}(2,q)$ is the commutator subgroup of an M^*-group G, then $G = \mathrm{PSL}(2,q)$, $G = \mathrm{PGL}(2,q)$, or G is a direct product which contains either $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$ as a factor.

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