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RESEARCH ARTICLE

Equivalency of multi-state survival signatures of multi-state systems of different sizes and its use in the comparison of systems

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Abstract
In this paper, the multi-state survival signature is first redefined for multi-state coherent or mixed systems with independent and identically distributed (*i.i.d.*) multi-state components. With the assumption of independence of component lifetimes at different state levels, transformation formulas of multi-state survival signatures of different sizes are established through the use of equivalent systems and a generalized triangle rule for order statistics from several independent and non-identical distributions. The results obtained facilitate stochastic comparisons of multi-state coherent or mixed systems with different numbers of *i.i.d.* multi-state components. Specific examples are finally presented to illustrate the transformation formulas established here, and also their use in comparing systems of different sizes.

1. Introduction

Signature theory, as an important part in the theory of reliability, contributes fundamentally in describing structures of reliability systems and in facilitating stochastic comparisons of different systems. The concept of system signature, proposed originally by Samaniego [22], is a vector $\boldsymbol{s} = (s_1, \ldots, s_n)$, with element s_i being the probability that the failure of a coherent system is caused by the ith ordered failure from its n independent and identically distributed (*i.i.d.*) components. For coherent systems with the same number of *i.i.d.* components, Kochar *et al.* [13] established that the usual stochastic ordering, hazard rate ordering and likelihood ratio ordering in system signatures lead to corresponding orderings of system lifetimes. More theoretical results and applications of the system signature can be found in the book by Samaniego [23]. Stochastic comparisons of coherent systems have been discussed based on the system signature in different ways recently; for example, with components ordered in hazard rate/reverse hazard rate/likelihood ratio ordering [1], with exchangeable or dependent non-exchangeable components [18], with different types or even different sizes of components [9], with information of system state or number of failed components under single/double monitoring [12], or by taking both performance and cost into consideration [15].

As discussed by Yi and Cui [31], there are many efficient methods for computing the system signature and each has its own advantages and limitations. Several related concepts have also been discussed in the literature; for example, minimal/maximal signature [19], dynamic signature [24], joint signature

[20] and ordered system signature [2] are some important ones among them. Survival signature, as a generalization of the system signature, was originally proposed by Coolen and Coolen-Maturi [5] for the survival of systems with multiple types of components, and are now widely used for studying many practical systems like large complex networks [3]. A similar concept, called joint survival signature, has been presented recently by Coolen-Maturi *et al.* [6] for coherent systems with shared components.

The above concepts have all focused on binary-state systems, while for multi-state systems [16,17] which are more practical in the field of reliability, related discussions on signature theory have also been made in the literature. For example, for multi-state systems with binary-state components, there are some concepts such as multi-dimensional D-spectrum [11], bivariate signature [8], multi-state ordered signature [25] and multi-state joint signature [28]. As for multi-state systems with multi-state components, Eryilmaz and Tuncel [10] introduced multi-state survival signature based on a natural generalization of the survival signature of Coolen and Coolen-Maturi [5]. Related discussions on computational methods for the multi-state survival signature can be found in Yi *et al.* [27,29].

There are many theories and methods that are useful in the study of multi-state systems; for example, Markov and semi-Markov models, universal generating function methods, combined methods and fuzzy methods [16]. However, when it comes to description of their system structures, signature theory has its unique advantages over traditional methods, especially for large complex systems whose structures are too complex to be represented by structural functions. Irrespective of whether one has binary-state systems or multi-state systems, it is known that signatures are vectors or matrices whose dimensions are determined by the number of components (i.e., the system size). This means that signature representations can still be simple even for large complex systems. Moreover, they can be calculated by Monte Carlo simulations no matter how complex the system structures are, and there are also other efficient computational methods available for different types of systems [27,29,31].

Signature and its related concepts play a vital role in stochastic comparisons of systems [4,32]. For systems of same size, stochastic comparisons of them can be carried out directly based on orderings of their signatures [13]. However, for systems of different sizes, some transformation formulas are required to transform the signature of smaller dimension to its counterpart of larger dimension [21]. For binary-state systems [14,21] and multi-state systems with binary-state components [26,30], these transformation formulas have already been established which facilitate stochastic comparisons of those systems of different sizes. But, in the case of multi-state systems with multi-state components, the problem becomes quite complex with different component lifetime distributions at different state levels to be taken care of. For tackling this issue, in this work, we first redefine the concept of multi-state survival signature in Yi *et al.* [27] for multi-state systems with multi-state components, and then establish transformation formulas for multi-state survival signatures of different sizes based on the assumption of independence of component lifetimes at different state levels.

The rest of this paper is organized as follows. In Section 2, the multi-state system survival signature is first redefined for multi-state coherent or mixed systems with multi-state components, and transformation formulas are then established for multi-state survival signatures of different sizes. Some illustrative examples are presented in Section 3 to demonstrate the transformation formulas established here, and then their usefulness in comparing systems of different sizes is demonstrated in Section 4 with numerical examples. Finally, some concluding remarks are made in Section 5.

2. Comparisons of multi-state systems of different sizes

For multi-state coherent systems with *i.i.d.* multi-state components, Yi *et al.* [27] have defined their multi-state survival signature in a matrix form as follows.

Definition 2.1. *Let T_j $(j = 1, \ldots, M)$ be the first time that a multi-state coherent system, having n i.i.d. multi-state components and a state space $\Omega = \{0, \ldots, M\}$ for both the system and the components, enters state $j-1$ or below. Furthermore, for $j = 1, \ldots, M$, let $X_j^{(i)}$ $(i = 1, \ldots, n)$ be i.i.d. random*

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variables with a common absolutely continuous distribution $F_j(x),\ x \geq 0$, *with* $X_j^{(i)}$ *being the first time that component i enters state* $j-1$ *or below. Suppose the system and the components start at perfect functioning state* M, *degrade into imperfect functioning states* $M-1,\ldots,1$ *successively and finally enter the complete failure state* 0. *Then, the multi-state survival signature of the system can be defined as* $\boldsymbol{S} = (\boldsymbol{S}^{(0)},\ldots,\boldsymbol{S}^{(M)})$, *where* $\boldsymbol{S}^{(j)} = (S^{(j)}_{i_1,\ldots,i_M}, 0 \leq i_1,\ldots,i_M \leq n)$ $(j=0,\ldots,M)$ *is the multi-state survival signature at system state level* j, *with*

$$S^{(j)}_{i_1,\ldots,i_M} = P\left\{T_j > t \,\middle|\, m_0(t) = i_0,\ldots,m_{M-1}(t) = i_{M-1}, m_M(t) = i_M := n - \sum_{w=0}^{M-1} i_w\right\}$$

being the conditional probability that the system is in state j or above at time t, given $m_l(t) = i_l$ *components in state l, for all* $l = 0,\ldots,M$.

Usually, as in [14,21,26,30], comparisons of systems of different sizes can be carried out based on the fact that any binary/multi-state system can be regarded as a mixture of several k-out-of-n type systems. For that purpose, it will be better if a consecutive type system has a simple form of signature vector/matrix, which leads to a modified definition of multi-state survival signature as follows.

Definition 2.2. *With notations defined in Definition 2.1, for a multi-state coherent or mixed system with n i.i.d. multi-state components, its multi-state survival signature can be defined as* $\boldsymbol{S} = (\boldsymbol{S}^{(0)},\ldots,\boldsymbol{S}^{(M)})$, *where* $\boldsymbol{S}^{(j)} = (S^{(j)}_{i_1,\ldots,i_M}, 0 \leq i_1,\ldots,i_M \leq n)$ $(j=0,\ldots,M)$ *is the multi-state survival signature at system state level* j, *with*

$$S^{(j)}_{i_1,\ldots,i_M} = P\{T_j > t | m_1(t) = i_1,\ldots,m_M(t) = i_M\}$$

being the conditional probability that the system is in state j or above at time t, given $m_l(t) = i_l$ *components in state l or above, for all* $l = 1,\ldots,M$.

Remark 2.1.

(1) The new definition is different from Definition 2.1 only in the definition of $m_l(t)$ except that it can also be applied for mixed systems. As in the discussions of Yi *et al.* [27], $S^{(j)}_{i_1,\ldots,i_M}$ $(j=0,1,\ldots,M)$ are independent of time t and is defined in a way similar to that in Eryilmaz and Tuncel [10].

(2) $S^{(j)}_{i_1,\ldots,i_M} = S^{(j)}_{\max(i_1,\ldots i_M),\ldots,\max(i_{M-1},i_M),i_M}$ $(j=0,\ldots,M)$, which leads to two ways of representing $\boldsymbol{S}^{(j)}$:

1. Keep all the elements $S^{(j)}_{i_1,\ldots,i_M}, 0 \leq i_1,\ldots,i_M \leq n$, and relabel subscripts $(i_1,\ldots,i_M)$ as $\sum_{j=1}^M i_j(n+1)^{j-1} + 1$. For example, when $n=2$ and $M=2$, we have

$$\boldsymbol{S}^{(j)} = (S^{(j)}_1,\ldots,S^{(j)}_9) = (S^{(j)}_{0,0}, S^{(j)}_{1,0}, S^{(j)}_{2,0}, S^{(j)}_{0,1}, S^{(j)}_{1,1}, S^{(j)}_{2,1}, S^{(j)}_{0,2}, S^{(j)}_{1,2}, S^{(j)}_{2,2})^T,$$

with $S^{(j)}_{0,1} = S^{(j)}_{1,1}, S^{(j)}_{0,2} = S^{(j)}_{1,2} = S^{(j)}_{2,2}$;

2. Delete all $S^{(j)}_{i_1,\ldots,i_M}$ that do not satisfy $0 \leq i_M \leq \cdots \leq i_1 \leq n$, and then relabel subscripts $(i_1,\ldots,i_M)$ according to formula (9) in [7] as

$$1 + \sum\nolimits_{j=1}^{M-1} \sum\nolimits_{l=i_{j+1}}^{i_j-1} \binom{n-l+j-1}{j-1} + \sum\nolimits_{l=0}^{i_M-1} \binom{n-l+M-1}{M-1}.$$

For example, when $n=2$ and $M=2$, we have

$$\boldsymbol{S}^{(j)} = (S^{(j)}_1,\ldots,S^{(j)}_6) = (S^{(j}_{0,0}, S^{(j)}_{1,0}, S^{(j)}_{2,0}, S^{(j)}_{1,1}, S^{(j)}_{2,1}, S^{(j)}_{2,2})^T.$$

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In this work, we adopt the latter for the sake of brevity and convenience;

(3) $\boldsymbol{S}^{(j_2)} \leq \boldsymbol{S}^{(j_1)}$ $(0 \leq j_1 < j_2 \leq M)$, namely, $S^{(j_2)}_{i_1,\ldots,i_M} \leq S^{(j_1)}_{i_1,\ldots,i_M}$ for all $0 \leq i_M \leq \cdots \leq i_1 \leq n$. Also, $\boldsymbol{S}^{(0)} = (S^{(0)}_{i_1,\ldots,i_M}, 0 \leq i_M \leq \cdots \leq i_1 \leq n)$, with $S^{(0)}_{i_1,\ldots,i_M} = 1$ for all $0 \leq i_M \leq \cdots \leq i_1 \leq n$, and such a determined matrix can be denoted by $\boldsymbol{S}^{(0)}_n$ for all systems of size n.

Now, for comparing multi-state coherent or mixed systems with multi-state components and of different sizes, we shall assume that the component lifetimes $X^{(1)}_j, \ldots, X^{(n)}_j$ are independent for different j $(j = 1, \ldots, M)$. Then, for establishing a relationship between multi-state survival signatures of two equivalent multi-state systems, an extended triangle rule as in Navarro *et al.* [21] needs to be presented first.

Theorem 2.1. *Suppose the random variables* $X^{(1)}_j, \ldots, X^{(n+1)}_j$ $(j = 1, \ldots, M)$ *are i.i.d. with a common absolutely continuous distribution* $F_j(x)$, $x \geq 0$, *and are independent for different* j. *Then, for* $1 \leq k_{1,j} \leq \cdots \leq k_{r_j,j} \leq n$ $(j = 1, \ldots, M, r_j = 1, \ldots, M)$, *the order statistics vector* $(X^{(k_{i,j}:n)}_j, j = 1, \ldots, M, i = 1, \ldots, r_j)$ *has the same distribution as*

$$(X_j^{(k_{i,j}+I_{\{i>a_j\}}:n+1)}, j = 1, \ldots, M, i = 1, \ldots, r_j)$$

with probability

$$(n+1)^{-M} \prod_{j=1}^{M} \left\{ (k_{1,j})^{I_{\{a_j=0\}}} \left[\prod_{l=1}^{r_j-1} (k_{l+1,j} - k_{l,j})^{I_{\{a_j=l, k_{l+1,j}>k_{l,j}\}}} \right] (n+1-k_{r_j,j})^{I_{\{a_j=r_j\}}} \right\}$$

for all $(a_1, \ldots, a_M) \in \boldsymbol{A} = \{(a_1, \ldots, a_M) : a_j \in \{0, \ldots, r_j\}$, *for all* $j = 1, \ldots, M\}$.

Proof. According to the proof of Theorem 2.1 in Yi *et al.* [26], we find that for any $j = 1, \ldots, M$, the order statistics vector $(X^{(k_{1,j}:n)}_j, \ldots, X^{(k_{r_j,j}:n)}_j)$ has the same distribution as $(X^{(k_{1,j}+1:n+1)}_j, \ldots, X^{(k_{r_j,j}+1:n+1)}_j)$ with probability $k_{1,j}/(n+1)$, as $(X^{(k_{1,j}:n+1)}_j, \ldots, X^{(k_{r_j,j}:n+1)}_j)$ with probability $(n+1-k_{r_j,j})/(n+1)$, and as $(X^{(k_{1,j}:n+1)}_j, \ldots, X^{(k_{l,j}:n+1)}_j, X^{(k_{l+1,j}+1:n+1)}_j, \ldots, X^{(k_{r_j,j}+1:n+1)}_j)$ with probability $(k_{l+1,j} - k_{l,j})/(n+1)$ for all $l = 1, \ldots, r_j - 1$. This result implies that the order statistics vector $(X^{(k_{i,j}:n)}_j,\ i = 1, \ldots, r_j)$ has the same distribution as $(X_j^{(k_{i,j}+I_{\{i>a_j\}}:n+1)},\ i = 1, \ldots, r_j)$ with probability

$$(n+1)^{-1} (k_{1,j})^{I_{\{a_j=0\}}} \left[\prod_{l=1}^{r_j-1} (k_{l+1,j} - k_{l,j})^{I_{\{a_j=l, k_{l+1,j}>k_{l,j}\}}} \right] (n+1-k_{r_j,j})^{I_{\{a_j=r_j\}}}$$

for all $a_j = 0, \ldots, r_j$. With the independence of $X^{(1)}_j, \ldots, X^{(n+1)}_j$ for different j, the required result follows readily. □

Remark 2.2. Specifically, for $M = 2$ and $1 \leq k_{1,j} \leq k_{2,j} \leq n$ $(j = 1, 2)$, the order statistics vector $(X^{(k_{1,1}:n)}_1, X^{(k_{2,1}:n)}_1, X^{(k_{1,2}:n)}_2, X^{(k_{2,2}:n)}_2)$ has the same distribution as

$$(X_1^{(k_{1,1}+I_{\{a_1=0\}}:n+1)}, X_1^{(k_{2,1}+I_{\{a_1=0,1\}}:n+1)}, X_2^{(k_{1,2}+I_{\{a_2=0\}}:n+1)}, X_2^{(k_{2,2}+I_{\{a_2=0,1\}}:n+1)})$$

with probability

$$(n+1)^{-2} \prod_{j=1}^{2} [(k_{1,j})^{I_{\{a_j=0\}}} (k_{2,j} - k_{1,j})^{I_{\{a_j=1, k_{2,j}>k_{1,j}\}}} (n+1-k_{2,j})^{I_{\{a_j=2\}}}]$$

for all (a_1, a_2) such that $a_1, a_2 \in \{0, 1, 2\}$.

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With the use of Theorem 2.1, the relationship between multi-state survival signatures for multi-state coherent or mixed systems with different numbers of multi-state components can be discussed. First, we need to introduce matrix $\boldsymbol{k} = (k_{i,j}, i = 1, \ldots, M, j = 1, \ldots, M)$ such that $0 \le k_{i,j} \le k_{\tilde{i},\tilde{j}} \le n$ for any $1 \le i < \tilde{i} \le M, 1 \le \tilde{j} < j \le M$, corresponding to a multi-state $\boldsymbol{k}$-out-of-n:G system, with n *i.i.d.* multi-state components and a state space $\Omega = \{0, \ldots, M\}$ for both the system and the components, being in state i ($i = 1, \ldots, M$) or above if and only if there are at least $k_{i,j}$ ($j = 1, \ldots, M$) components in state j or above. Evidently, the lifetime of such a system can be represented through component lifetimes as

$$T_i = \min(X_1^{(n+1-k_{i,1}:n)}, \ldots, X_M^{(n+1-k_{i,M}:n)}),\ i = 1, \ldots, M,$$

with $X_1^{(n+1:n)} = \cdots = X_M^{(n+1:n)} = +\infty$, and its multi-state survival signature can be given as $\boldsymbol{S}_{\boldsymbol{k}:n} = (\boldsymbol{S}_n^{(0)}, \boldsymbol{S}_{\boldsymbol{k}:n}^{(1)}, \ldots, \boldsymbol{S}_{\boldsymbol{k}:n}^{(M)})$, where $\boldsymbol{S}_{\boldsymbol{k}:n}^{(i)} = \boldsymbol{S}_{\boldsymbol{k}_i:n}^{(i)} = (S_{\boldsymbol{k}_i;i_1,\ldots,i_M}^{(i)}, 0 \le i_M \le \cdots \le i_1 \le n)$ ($i = 1, \ldots, M$) with $\boldsymbol{k}_i = (k_{i,1}, \ldots, k_{i,M})$ and

$$S_{\boldsymbol{k}_i,i_1,\ldots,i_M}^{(i)} = I_{\{i_1 \ge k_{i,1},\ldots,i_M \ge k_{i,M}\}},\ 0 \le i_M \le \cdots \le i_1 \le n.$$

For $j = 1, \ldots, M$, note that $0 \le k_{1,j} \le \cdots \le k_{M,j} \le n$; by denoting r_j for the number of zeros in $k_{1,j}, \ldots, k_{M,j}$, we have $k_{1,j} = \cdots = k_{r_j,j} = 0 < 1 \le k_{r_j+1,j} \le \cdots \le k_{M,j} \le n$, that is,

$$1 \le n + 1 - k_{M,j} \le \cdots \le n + 1 - k_{r_j+1,j} \le n < n + 1 = n + 1 - k_{r_j,j} = \cdots = n + 1 - k_{1,j}.$$

Then, from Theorem 2.1, an equivalent system of size $n + 1$ for a multi-state $\boldsymbol{k}$-out-of-n:G system has its lifetime as

$$T_i = \min(X_1^{(n+1-k_{i,1}+I_{\{i \le a_1\}}:n+1)}, \ldots, X_M^{(n+1-k_{i,M}+I_{\{i \le a_M\}}:n+1)}),\ i = 1, \ldots, M,$$

with probability

$$\prod_{j=1}^{M} \left\{ (n+1)^{-1} (k_{r_j+1,j})^{I_{\{a_j = r_j\}}} \left[\prod_{l=r_j+1}^{M-1} (k_{l+1,j} - k_{l,j})^{I_{\{a_j = l, k_{l+1,j} > k_{l,j}\}}} \right] (n + 1 - k_{M,j})^{I_{\{a_j = M\}}} \right\}^{I_{\{r_j < M\}}}$$

for all $(a_1, \ldots, a_M) \in \mathscr{A} = \{(a_1, \ldots, a_M) : a_j \in \{r_j, \ldots, M\}$ for all $j = 1, \ldots, M\}$. Note that for $i = 1, \ldots, r_j,\ \ j = 1, \ldots, M$, we have $X_j^{(n+1-k_{i,j}+I_{\{i \le a_j\}}:n+1)} = X_j^{(n+2-k_{i,j}:n+1)} = +\infty$ with $k_{i,j} = 0$. Moreover, the equivalent system of size $n + 1$ has a multi-state survival signature $\boldsymbol{S}_{\boldsymbol{k}:n}^{*}=(\boldsymbol{S}_{n+1}^{(0)}, \boldsymbol{S}_{\boldsymbol{k}:n}^{*(1)}, \ldots, \boldsymbol{S}_{\boldsymbol{k}:n}^{*(M)})$, where $\boldsymbol{S}_{\boldsymbol{k}:n}^{*(i)} = \boldsymbol{S}_{\boldsymbol{k}_i:n}^{*(i)} = (S_{\boldsymbol{k}_i,i_1,\ldots,i_M}^{*(i)}, 0 \le i_M \le \cdots \le i_1 \le n + 1)$ ($i = 1, \ldots, M$) with

$$\begin{aligned} S_{\boldsymbol{k}_i,i_1,\ldots,i_M}^{*(i)} &= \sum_{(a_1,\ldots,a_M) \in \mathscr{A}} \prod_{j=1}^{M} \left\{ (n+1)^{-1} (k_{r_j+1,j})^{I_{\{a_j = r_j\}}} \left[\prod_{l=r_j+1}^{M-1} (k_{l+1,j} - k_{l,j})^{I_{\{a_j = l, k_{l+1,j} > k_{l,j}\}}} \right] \right. \\ &\quad \left. \times (n + 1 - k_{M,j})^{I_{\{a_j = M\}}} \right\}^{I_{\{r_j < M\}}} I_{\{i_1 \ge k_{i,1} + I_{\{i > a_1\}}, \ldots, i_M \ge k_{i,M} + I_{\{i > a_M\}}\}} \\ &= (n+1)^{-M} \prod_{j=1}^{M} [k_{i,j} I_{\{i_1 \ge k_{i,j}+1\}} + (n + 1 - k_{i,j}) I_{\{i_1 \ge k_{i,j}\}}],\ 0 \le i_M \le \cdots \le i_1 \le n + 1. \end{aligned}$$

Now, with the multi-state survival signature of an equivalent system of size $n + 1$ derived above for a multi-state $\boldsymbol{k}$-out-of-n:G system, we are able to obtain the multi-state survival signature of an equivalent system of size $n + 1$ for any multi-state coherent or mixed system with n *i.i.d.* components by regarding it as a mixture of several multi-state $\boldsymbol{k}$-out-of-n:G type systems, as established in the following theorem.

Theorem 2.2. *Let $\boldsymbol{S} = (\boldsymbol{S}^{(0)}, \ldots, \boldsymbol{S}^{(M)})$, where $\boldsymbol{S}^{(i)} = (S^{(i)}_{i_1,\ldots,i_M}, 0 \le i_M \le \cdots \le i_1 \le n)$ $(i = 0, 1, \ldots, M)$, be the multi-state survival signature of a multi-state coherent or mixed system with n i.i.d. multi-state components and a state space $\Omega = \{0, \ldots, M\}$ for both the system and the components. Suppose the component lifetimes $X^{(1)}_j, \ldots, X^{(n)}_j$ $(j = 1, \ldots, M)$ are i.i.d. with a common absolutely continuous distribution $F_j(x)$, $x \ge 0$, and are independent for different j. Then, its equivalent system of size $n+1$ has its multi-state survival signature as*

$$\boldsymbol{S}^* = (\boldsymbol{S}^{(0)}_{n+1}, \boldsymbol{S}^{*(1)}, \ldots, \boldsymbol{S}^{*(M)}) = \sum_{\boldsymbol{k} \in \mathscr{K}} s_{\boldsymbol{k}} \boldsymbol{S}^*_{\boldsymbol{k}:n},$$

where

$$\mathscr{K} = \{(k_{i,j}, i = 1, \ldots, M, j = 1, \ldots, M) : 0 \le k_{i,j} \le k_{\tilde{i},\tilde{j}} \le n \quad \text{for any } 1 \le i < \tilde{i} \le M,\ 1 \le \tilde{j} < j \le M\},$$

and $s_{\boldsymbol{k}}, \boldsymbol{k} \in \mathscr{K}$, can be given as a solution to the set of linear equations

$$\sum_{\boldsymbol{k}_i = \tilde{\boldsymbol{k}}} s_{\boldsymbol{k}} = s^{(i)}_{\tilde{\boldsymbol{k}}},\ \tilde{\boldsymbol{k}} \in \tilde{\mathscr{K}} = \{(k_1, \ldots, k_M) : 0 \le k_M \le \cdots \le k_1 \le n\},\ i = 1, \ldots, M,$$

with $\boldsymbol{s}^{(i)} = (s^{(i)}_{\tilde{\boldsymbol{k}}}, \tilde{\boldsymbol{k}} \in \tilde{\mathscr{K}}) = \boldsymbol{M}^{-1}\boldsymbol{S}^{(i)}$ and

$$\boldsymbol{M} = (M_{i_1,\ldots,i_M;j_1,\ldots,j_M},\ 0 \le i_M \le \cdots \le i_1 \le n,\ 0 \le j_M \le \cdots \le j_1 \le n)$$

being a matrix with all elements $M_{i_1,\ldots,i_M;j_1,\ldots,j_M} = I_{\{i_1 \ge j_1,\ldots,i_M \ge j_M\}}$, for all $i = 1, \ldots, M$.

Proof. Any multi-state survival signature $\boldsymbol{S} = (\boldsymbol{S}^{(0)}, \ldots, \boldsymbol{S}^{(M)})$ can be regarded as a mixture of multi-state survival signatures $\boldsymbol{S}_{\boldsymbol{k}:n} = (\boldsymbol{S}^{(0)}_n, \boldsymbol{S}^{(1)}_{\boldsymbol{k}_1:n}, \ldots, \boldsymbol{S}^{(M)}_{\boldsymbol{k}_M:n})$ of multi-state $\boldsymbol{k}$-out-of-n:G systems, namely $\boldsymbol{S} = \sum_{\boldsymbol{k} \in \mathscr{K}} s_{\boldsymbol{k}} \boldsymbol{S}_{\boldsymbol{k}:n}$, with

$$\mathscr{K} = \{(k_{i,j}, i = 1, \ldots, M, j = 1, \ldots, M) : 0 \le k_{i,j} \le k_{\tilde{i},\tilde{j}} \le n \quad \text{for any } 1 \le i < \tilde{i} \le M,\ 1 \le \tilde{j} < j \le M\}.$$

Without loss of generality, $s_{\boldsymbol{k}}, \boldsymbol{k} \in \mathscr{K}$, can be given by related marginal distributions

$$s^{(i)}_{\tilde{\boldsymbol{k}}}, \tilde{\boldsymbol{k}} \in \tilde{\mathscr{K}} = \{(k_1, \ldots, k_M) : 0 \le k_M \le \cdots \le k_1 \le n\},\ i = 1, \ldots, M.$$

Consider now $\boldsymbol{S}^{(i)} = \sum_{\tilde{\boldsymbol{k}} \in \tilde{\mathscr{K}}} s^{(i)}_{\tilde{\boldsymbol{k}}} \boldsymbol{S}^{(i)}_{\tilde{\boldsymbol{k}}:n}$ $(i = 1, \ldots, M)$ with

$$\boldsymbol{S}^{(i)}_{\tilde{\boldsymbol{k}}:n} = (S^{(i)}_{\tilde{\boldsymbol{k}};i_1,\ldots,i_M}, 0 \le i_M \le \cdots \le i_1 \le n),\ \tilde{\boldsymbol{k}} = (k_1, \ldots, k_M),$$

and $S^{(i)}_{\tilde{\boldsymbol{k}};i_1,\ldots,i_M} = I_{\{i_1 \ge k_1,\ldots,i_M \ge k_M\}}$ for all $0 \le i_M \le \cdots \le i_1 \le n$. Then, we have

$$S^{(i)}_{i_1,\ldots,i_M} = \sum_{\tilde{\boldsymbol{k}} \in \tilde{\mathscr{K}}} s^{(i)}_{\boldsymbol{k}_i} I_{\{i_1 \ge k_1,\ldots,i_M \ge k_M\}} = \sum_{i_1 \ge k_1,\ldots,i_M \ge k_M, \tilde{\boldsymbol{k}} \in \tilde{\mathscr{K}}} s^{(i)}_{\tilde{\boldsymbol{k}}}.$$

For $i = 1, \ldots, M$, let $\boldsymbol{s}^{(i)} = (s^{(i)}_{\tilde{\boldsymbol{k}}}, \tilde{\boldsymbol{k}} \in \tilde{\mathscr{K}}) = (s^{(i)}_{k_1,\ldots,k_M}, 0 \le k_M \le \cdots \le k_1 \le n)$ be a column vector arranged in the same way as $\boldsymbol{S}^{(i)} = (S^{(i)}_{i_1,\ldots,i_M},\ 0 \le i_M \le \cdots \le i_1 \le n)$, and

$$\boldsymbol{M} = (M_{i_1,\ldots,i_M;j_1,\ldots,j_M},\ 0 \le i_M \le \cdots \le i_1 \le n,\ 0 \le j_M \le \cdots \le j_1 \le n)$$

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be an invertible matrix with all elements $M_{i_1,\ldots,i_M;j_1,\ldots,j_M} = I_{\{i_1 \geq j_1,\ldots,i_M \geq j_M\}}$, so that we have $\boldsymbol{S}^{(i)} = \boldsymbol{M}\boldsymbol{s}^{(i)}$, which yields $\boldsymbol{s}^{(i)} = \boldsymbol{M}^{-1}\boldsymbol{S}^{(i)}$. Then, the values of $s_{\boldsymbol{k}}, \boldsymbol{k} \in \mathscr{K}$, can be obtained by solving the set of equations $\sum_{\boldsymbol{k}_i=\tilde{\boldsymbol{k}}} s_{\boldsymbol{k}} = s^{(i)}_{\tilde{\boldsymbol{k}}}, \tilde{\boldsymbol{k}} \in \tilde{\mathscr{K}}, \ i = 1,\ldots,M$. Note that even when the values of $s_{\boldsymbol{k}}, \boldsymbol{k} \in \mathscr{K}$, are not unique, all of them lead to the same $\boldsymbol{S}^*$ since $\boldsymbol{S}^*$depends only on $s^{(i)}_{\tilde{\boldsymbol{k}}}, \tilde{\boldsymbol{k}} \in \tilde{\mathscr{K}}, \ i = 1,\ldots,M$. This completes the proof of the theorem. □

Remark 2.3. Specifically, for $M = 2$, if the multi-state survival signature of the original system of size 4 is written as

$$\boldsymbol{S} = (\boldsymbol{S}^{(0)}, \boldsymbol{S}^{(1)}, \boldsymbol{S}^{(2)}) = \begin{pmatrix} S^{(0)}_{0,0} & S^{(0)}_{1,0} & S^{(0)}_{2,0} & S^{(0)}_{3,0} & S^{(0)}_{4,0} & S^{(0)}_{1,1} & S^{(0)}_{2,1} & S^{(0)}_{3,1} & S^{(0)}_{4,1} & S^{(0)}_{2,2} & S^{(0)}_{3,2} & S^{(0)}_{4,2} & S^{(0)}_{3,3} & S^{(0)}_{4,3} & S^{(0)}_{4,4} \\ S^{(1)}_{0,0} & S^{(1)}_{1,0} & S^{(1)}_{2,0} & S^{(1)}_{3,0} & S^{(1)}_{4,0} & S^{(1)}_{1,1} & S^{(1)}_{2,1} & S^{(1)}_{3,1} & S^{(1)}_{4,1} & S^{(1)}_{2,2} & S^{(1)}_{3,2} & S^{(1)}_{4,2} & S^{(1)}_{3,3} & S^{(1)}_{4,3} & S^{(1)}_{4,4} \\ S^{(2)}_{0,0} & S^{(2)}_{1,0} & S^{(2)}_{2,0} & S^{(2)}_{3,0} & S^{(2)}_{4,0} & S^{(2)}_{1,1} & S^{(2)}_{2,1} & S^{(2)}_{3,1} & S^{(2)}_{4,1} & S^{(2)}_{2,2} & S^{(2)}_{3,2} & S^{(2)}_{4,2} & S^{(2)}_{3,3} & S^{(2)}_{4,3} & S^{(2)}_{4,4} \end{pmatrix}^T.$$

Then, the multi-state survival signature $\boldsymbol{S}^* = (\boldsymbol{S}^{(0)}_5, \boldsymbol{S}^{*(1)}, \boldsymbol{S}^{*(2)})$ of its equivalent system of size 5 is given by $\boldsymbol{S}^* = \sum_{\boldsymbol{k}\in\mathscr{K}} s_{\boldsymbol{k}} \boldsymbol{S}^*_{\boldsymbol{k}:n}$, where $s_{\boldsymbol{k}}, \boldsymbol{k} \in \mathscr{K}$, with

$$\mathscr{K} = \{(k_{i,j}, i = 1,2, j = 1,2) : 0 \leq k_{1,2} \leq k_{1,1}, k_{2,2} \leq k_{2,1} \leq 4\},$$

are given by the marginal distributions $\boldsymbol{s}^{(i)} = \boldsymbol{M}^{-1}\boldsymbol{S}^{(i)}$ $(i = 1,2)$ by solving

$$\begin{cases} \sum\limits_{(k_{11},k_{12})=(k_1,k_2)} s_{k_{11},k_{12};k_{21},k_{22}} = s^{(1)}_{k_1,k_2}, \ (k_1,k_2) \in \tilde{\mathscr{K}}, \\ \sum\limits_{(k_{21},k_{22})=(k_1,k_2)} s_{k_{11},k_{12};k_{21},k_{22}} = s^{(2)}_{k_1,k_2}, \ (k_1,k_2) \in \tilde{\mathscr{K}}, \end{cases}$$

with $\tilde{\mathscr{K}} = \{(k_1,k_2) : 0 \leq k_2 \leq k_1 \leq 4\}$ and

$$\boldsymbol{M} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 1&1&1&0&0&1&1&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&1&1&1&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&0&0&0&0&0&0 \\ 1&1&1&0&0&1&1&0&0&1&0&0&0&0&0 \\ 1&1&1&1&0&1&1&1&0&1&1&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1&1&1&0&0&0 \\ 1&1&1&1&0&1&1&1&0&1&1&0&1&0&0 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&0 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix}.$$

Also, note that $\boldsymbol{S}^*_{\boldsymbol{k}:4} = (\boldsymbol{S}^{(0)}_5, \boldsymbol{S}^{*(1)}_{\boldsymbol{k}_1:4}, \boldsymbol{S}^{*(2)}_{\boldsymbol{k}_2:4})$, where $\boldsymbol{S}^{(0)}_5 = \underbrace{(1,\ldots,1)^T}_{21}$ and $\boldsymbol{S}^{*(i)}_{\boldsymbol{k}_i:4} = (S^{*(i)}_{\boldsymbol{k}_i;i_1,i_2}, 0 \leq i_2 \leq i_1 \leq 5)$ $(i = 1,2)$, with

$$S^{*(i)}_{\boldsymbol{k}_i;i_1,i_2} = \frac{(5-k_{i,1})(5-k_{i,2})}{25} I_{\{i_1 \geq k_{i,1}, i_2 \geq k_{i,2}\}} + \frac{(5-k_{i,1})k_{i,2}}{25} I_{\{i_1 \geq k_{i,1}, i_2 \geq k_{i,2}+1\}} + \frac{k_{i,1}(5-k_{i,2})}{25} I_{\{i_1 \geq k_{i,1}+1, i_2 \geq k_{i,2}\}} + \frac{k_{i,1}k_{i,2}}{25} I_{\{i_1 \geq k_{i,1}+1, i_2 \geq k_{i,2}+1\}}, \ 0 \leq i_2 \leq i_1 \leq 5.$$

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In Theorems 2.1 and 2.2, we have considered the equivalence of multi-state survival signatures of multi-state systems of sizes n and $n+1$. Instead of using the theorem repeatedly, a more general result is presented now for the equivalence of multi-state survival signatures of multi-state systems of sizes n and $n+l$ $(l=1,2,\ldots)$.

Theorem 2.3. *Suppose the random variables* $X_j^{(1)},\ldots,X_j^{(n+l)}$ $(j=1,\ldots,M)$ *are i.i.d. with a common absolutely continuous distribution* $F_j(x), x\geq 0$, *and are independent for different* j. *Then, for* $1\leq k_{1,j}\leq\cdots\leq k_{r_j,j}\leq n$ $(j=1,\ldots,M, r_j=1,\ldots,M)$, *the order statistics vector* $(X_j^{(k_{i,j}:n)}, j=1,\ldots,M, i=1,\ldots,r_j)$ *has the same distribution as*

$$(X_j^{(h_{i,j}:n+1)}, j=1,\ldots,M, i=1,\ldots,r_j)$$

with probability

$$\binom{n+l}{n}^{-M}\prod_{j=1}^{M}\left\{\binom{h_{1,j}-1}{k_{1,j}-1}\left[\prod_{l=1}^{r_j-1}\binom{h_{l+1,j}-h_{l,j}-1}{k_{l+1,j}-k_{l,j}-1}^{I_{\{k_{l+1,j}>k_{l,j}\}}}\right]\binom{n+l-h_{r_j,j}}{n-k_{r_j,j}}\right\}$$

for all $\boldsymbol{h}\in\mathscr{H}_{\boldsymbol{k}}$, *with*

$$\begin{aligned}\mathscr{H}_{\boldsymbol{k}}=\{&(h_{i,j}, j=1,\ldots,M, i=1,\ldots,r_j): 1\leq h_{1,j}\leq\cdots\leq h_{r_j,j}\leq n+l, k_{1,j}\leq h_{1,j},\\ &k_{2,j}-k_{1,j}\leq h_{2,j}-h_{1,j},\ldots,k_{r_j,j}-k_{r_j-1,j}\leq h_{r_j,j}-h_{r_j-1,j}, h_{r_j,j}\leq k_{r_j,j}+l,\\ &I_{\{k_{2,j}>k_{1,j}\}}=I_{\{h_{2,j}>h_{1,j}\}},\ldots,I_{\{k_{r_j,j}>k_{r_j-1,j}\}}=I_{\{h_{r_j,j}>h_{r_j-1,j}\}}\ \textit{for all } j\}.\end{aligned}$$

Proof. According to the proof of Theorem 2.4 in Yi *et al.* [30], we find that for any $j=1,\ldots,M$, the order statistics vector $(X_j^{(k_{1,j}:n)},\ldots,X_j^{(k_{r_j,j}:n)})$ has the same distribution as $(X_j^{(h_{1,j}:n)},\ldots,X_j^{(h_{r_j,j}:n)})$ with probability

$$\binom{n+l}{n}^{-1}\binom{h_{1,j}-1}{k_{1,j}-1}\left[\prod_{l=1}^{r_j-1}\binom{h_{l+1,j}-h_{l,j}-1}{k_{l+1,j}-k_{l,j}-1}^{I_{\{k_{l+1,j}>k_{l,j}\}}}\right]\binom{n+l-h_{r_j,j}}{n-k_{r_j,j}}$$

for all $(h_{1,j},\ldots,h_{r_j,j})$ such that $1\leq h_{1,j}\leq\cdots\leq h_{r_j,j}\leq n+l$ and

$$\begin{gathered}k_{1,j}\leq h_{1,j}, k_{2,j}-k_{1,j}\leq h_{2,j}-h_{1,j},\ldots,k_{r_j,j}-k_{r_j-1,j}\leq h_{r_j,j}-h_{r_j-1,j}, h_{r_j,j}\leq k_{r_j,j}+l,\\ I_{\{k_{2,j}>k_{1,j}\}}=I_{\{h_{2,j}>h_{1,j}\}},\ldots,I_{\{k_{r_j,j}>k_{r_j-1,j}\}}=I_{\{h_{r_j,j}>h_{r_j-1,j}\}}.\end{gathered}$$

With the independence of $X_j^{(1)},\ldots,X_j^{(n+l)}$ for different j, the required result readily follows. □

From Theorem 2.3, the relationship between multi-state survival signatures of multi-state systems of sizes n and $n+l$ can be established as follows.

Theorem 2.4. *Let* $\boldsymbol{S}=(\boldsymbol{S}^{(0)},\ldots,\boldsymbol{S}^{(M)})$, *where* $\boldsymbol{S}^{(i)}=(S^{(i)}_{i_1,\ldots,i_M}, 0\leq i_M\leq\cdots\leq i_1\leq n)$ $(i=0,1,\ldots,M)$, *be the multi-state survival signature of a multi-state coherent or mixed system with* n *i.i.d. multi-state components and a state space* $\Omega=\{0,\ldots,M\}$ *for both the system and the components. Suppose the component lifetimes* $X_j^{(1)},\ldots,X_j^{(n)}$ $(j=1,\ldots,M)$ *are i.i.d. with a common absolutely continuous distribution* $F_j(x)$, $x\geq 0$, *and are independent for different* j. *Then, its equivalent system of size* $n+l$ *has its multi-state survival signature as*

$$\boldsymbol{S}^{[l]*}=(\boldsymbol{S}^{[l]*(0)},\ldots,\boldsymbol{S}^{[l]*(M)})=\sum_{\boldsymbol{k}\in\mathscr{K}}s_{\boldsymbol{k}}\boldsymbol{S}^{[l]*}_{\boldsymbol{k}:n},$$

where $s_{\boldsymbol{k}}, \boldsymbol{k} \in \mathscr{K}$, are as in Theorem 2.2 and

$$\boldsymbol{S}_{\boldsymbol{k}:n}^{[l]*} = (\boldsymbol{S}_{n+l}^{(0)}, \boldsymbol{S}_{\boldsymbol{k}:n}^{[l]*(1)}, \ldots, \boldsymbol{S}_{\boldsymbol{k}:n}^{[l]*(M)})$$

is the multi-state survival signature of the equivalent system of size l of a multi-state $\boldsymbol{k}$-out-of-n:G system given by $\boldsymbol{S}_{\boldsymbol{k}:n}^{[l]*(i)} = (S_{\boldsymbol{k};i_1,\ldots,i_M}^{[l]*(i)}, 0 \le i_M \le \cdots \le i_1 \le n+l)$ $(i = 1, \ldots, M)$ with r_j $(j = 1, \ldots, M)$ being the number of zeros in $k_{1,j}, \ldots, k_{M,j}$ and

$$S_{\boldsymbol{k};i_1,\ldots,i_M}^{[l]*(i)} = \sum_{\boldsymbol{h} \in \mathscr{H}_{\boldsymbol{k}}} \prod_{j=1}^{M} \left\{ \binom{n+l}{n}^{-1} \binom{h_{r_j+1,j}-1}{k_{r_j+1,j}-1} \left[\prod_{l=r_j+1}^{M-1} \binom{h_{l+1,j}-h_{l,j}-1}{k_{l+1,j}-k_{l,j}-1}^{I_{\{k_{l+1,j}>k_{l,j}\}}} \right] \binom{n+l-h_{M,j}}{n-k_{M,j}} \right\}^{I_{\{r_j<M\}}}$$
$$\times I_{\{i_1 \ge h_{i,1},\ldots,i_M \ge h_{i,M}\}}, \; 0 \le i_M \le \cdots \le i_1 \le n+l,$$

and

$$\mathscr{H}_{\boldsymbol{k}} = \{(h_{i,j}, j = 1, \ldots, M, i = r_j+1, \ldots, M) : 1 \le h_{r_j+1,j} \le \cdots \le h_{M,j} \le n+l,$$
$$k_{r_j+1,j} \le h_{r_j+1,j}, k_{r_j+2,j} - k_{r_j+1,j} \le h_{r_j+2,j} - h_{r_j+1,j}, \ldots, k_{M,j} - k_{M-1,j} \le h_{M,j} - h_{M-1,j},$$
$$h_{M,j} \le k_{M,j} + l, I_{\{k_{r_j+2,j}>k_{r_j+1,j}\}} = I_{\{h_{r_j+2,j}>h_{r_j+1,j}\}}, \ldots, I_{\{k_{M,j}>k_{M-1,j}\}} = I_{\{h_{M,j}>h_{M-1,j}\}} \text{ for all } j\}.$$

Proof. Note that, for $0 \le k_{1,j} \le \cdots \le k_{M,j} \le n$ $(j = 1, \ldots, M)$, as before, if the lifetimes of a multi-state $\boldsymbol{k}$-out-of-n:G system are denoted by

$$T_i = \min(X_1^{(n+1-k_{i,1}:n)}, \ldots, X_M^{(n+1-k_{i,M}:n)}), \; i = 1, \ldots, M,$$

then the lifetimes of its equivalent system of size $n+l$ can be denoted as

$$T_i = \min(X_1^{(n+l+1-h_{i,1}:n+l)}, \ldots, X_M^{(n+l+1-h_{i,M}:n+l)}), \; i = 1, \ldots, M,$$

with probability

$$\prod_{j=1}^{M} \left\{ \binom{n+l}{n}^{-1} \binom{h_{r_j+1,j}-1}{k_{r_j+1,j}-1} \left[\prod_{l=r_j+1}^{M-1} \binom{h_{l+1,j}-h_{l,j}-1}{k_{l+1,j}-k_{l,j}-1}^{I_{\{k_{l+1,j}>k_{l,j}\}}} \right] \binom{n+l-h_{M,j}}{n-k_{M,j}} \right\}^{I_{\{r_j<M\}}}$$

for all $\boldsymbol{h} \in \mathscr{H}_{\boldsymbol{k}}$. The rest of the proof proceeds similar to that of Theorem 2.2, and is therefore omitted here for brevity. □

Remark 2.4. Specifically, for $M = 2$, $n = 4$ and $l = 2$, we have $\boldsymbol{S}_{\boldsymbol{k}:4}^{[2]*(i)} = (S_{\boldsymbol{k};i_1,i_2}^{[2]*(i)}, \; 0 \le i_2 \le i_1 \le 6)$ $(i = 1, 2)$, where

$$S_{\boldsymbol{k},i_1,i_2}^{[2]*(i)} = \sum_{\boldsymbol{h} \in \mathscr{H}_{\boldsymbol{k}}} \prod_{j=1}^{2} \left\{ \frac{1}{15} \binom{h_{1,j}-1}{k_{1,j}-1}^{I_{\{k_{1,j}>0\}}} \binom{h_{2,j}-h_{1,j}-1}{k_{2,j}-k_{1,j}-1}^{I_{\{k_{2,j}>k_{1,j}\}}} \binom{6-h_{2,j}}{4-k_{2,j}} \right\}^{I_{\{k_{2,j}>0\}}} I_{\{i_1 \ge h_{i,1}, i_2 \ge h_{i,2}\}},$$

with $0 \le i_2 \le i_1 \le 6$ and

$$\mathscr{H}_{\boldsymbol{k}} = \{(h_{1,1}, h_{1,2}; h_{2,1}, h_{2,2}) : 0 \le h_{1,j} \le h_{2,j} \le 6, k_{1,j} \le h_{1,j}, k_{2,j} - k_{1,j} \le h_{2,j} - h_{1,j},$$
$$h_{2,j} \le k_{2,j} + 2, I_{\{k_{2,j}>k_{1,j}\}} = I_{\{h_{2,j}>h_{1,j}\}} \text{ for all } j = 1, 2\}.$$

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3. Illustrative examples

For illustrating the results established in the last section, let us consider a wireless sensor system with four *i.i.d.* sensors and a multi-state linear consecutive $(2, 1)$-out-of-4 : G structure, which works (perfectly or imperfectly) if and only if there are at least two consecutive sensors working (perfectly or imperfectly), works perfectly if and only if there is also at least one sensor working perfectly, and fails if it does not work. According to Yi *et al.* [27], the multi-state survival signature of such a system is given by

$$\begin{aligned}\boldsymbol{S} &= (\boldsymbol{S}^{(0)}, \boldsymbol{S}^{(1)}, \boldsymbol{S}^{(2)})\\ &= \begin{pmatrix} S^{(0)}_{0,0} & S^{(0)}_{1,0} & S^{(0)}_{2,0} & S^{(0)}_{3,0} & S^{(0)}_{4,0} & S^{(0)}_{1,1} & S^{(0)}_{2,1} & S^{(0)}_{3,1} & S^{(0)}_{4,1} & S^{(0)}_{2,2} & S^{(0)}_{3,2} & S^{(0)}_{4,2} & S^{(0)}_{3,3} & S^{(0)}_{4,3} & S^{(0)}_{4,4}\\ S^{(1)}_{0,0} & S^{(1)}_{1,0} & S^{(1)}_{2,0} & S^{(1)}_{3,0} & S^{(1)}_{4,0} & S^{(1)}_{1,1} & S^{(1)}_{2,1} & S^{(1)}_{3,1} & S^{(1)}_{4,1} & S^{(1)}_{2,2} & S^{(1)}_{3,2} & S^{(1)}_{4,2} & S^{(1)}_{3,3} & S^{(1)}_{4,3} & S^{(1)}_{4,4}\\ S^{(2)}_{0,0} & S^{(2)}_{1,0} & S^{(2)}_{2,0} & S^{(2)}_{3,0} & S^{(2)}_{4,0} & S^{(2)}_{1,1} & S^{(2)}_{2,1} & S^{(2)}_{3,1} & S^{(2)}_{4,1} & S^{(2)}_{2,2} & S^{(2)}_{3,2} & S^{(2)}_{4,2} & S^{(2)}_{3,3} & S^{(2)}_{4,3} & S^{(2)}_{4,4} \end{pmatrix}^T\\ &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 0 & 1/2 & 1 & 1 & 0 & 1/2 & 1 & 1 & 1/2 & 1 & 1 & 1 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1 & 1 & 1/2 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T.\end{aligned}$$

Suppose the sensor lifetimes are independent for different state levels. Then, the multi-state survival signatures of its equivalent systems of sizes 5 and 6 are presented in Examples 3.1 and 3.2, respectively, by the use of Theorem 2.2, and the latter is also worked out in Example 3.3 by the use of Theorem 2.4.

Example 3.1. For such a multi-state linear consecutive $(2, 1)$-out-of-4 : G wireless sensor system, according to Remark 2.3, the multi-state survival signature of its equivalent system of size 5 can be given as $\boldsymbol{S}^* = \sum_{\boldsymbol{k}\in\mathscr{K}} s_{\boldsymbol{k}} \boldsymbol{S}^*_{\boldsymbol{k}:4}$, where $s_{\boldsymbol{k}}, \boldsymbol{k} \in \mathscr{K}$, with

$$\mathscr{K} = \{(k_{i,j}, i = 1, 2, j = 1, 2) : 0 \le k_{1,2} \le k_{1,1}, k_{2,2} \le k_{2,1} \le 4\},$$

are given by the following marginal distributions:

$$\boldsymbol{s}^{(1)} = \boldsymbol{M}^{-1}\boldsymbol{S}^{(1)} = \left(0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right)^T,$$

$$\boldsymbol{s}^{(2)} = \boldsymbol{M}^{-1}\boldsymbol{S}^{(2)} = \left(0, 0, 0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0\right)^T.$$

These imply $s^{(1)}_{2,0} = s^{(1)}_{3,0} = 1/2$, $s^{(1)}_{k_1,k_2} = 0$ for other $0 \le k_2 \le k_1 \le 4$, and $s^{(2)}_{2,1} = s^{(2)}_{3,1} = 1/2$, $s^{(2)}_{k_1,k_2} = 0$ for other $0 \le k_2 \le k_1 \le 4$, which leads to the fact that $s_{\boldsymbol{k}} = 0$ for all $\boldsymbol{k} \in \mathscr{K}$ except $s_{2,0;2,1}, s_{2,0;3,1}, s_{3,0;3,1}$. Now, upon solving the set of equations

$$\begin{cases} s_{2,0;2,1} + s_{2,0;3,1} = s^{(1)}_{2,0} = 1/2,\\ s_{3,0;3,1} = s^{(1)}_{3,0} = 1/2,\\ s_{2,0;2,1} = s^{(2)}_{2,1} = 1/2,\\ s_{2,0;3,1} + s_{3,0;3,1} = s^{(2)}_{3,1} = 1/2, \end{cases}$$

we get $s_{2,0;2,1} = s_{3,0;3,1} = 1/2$ and $s_{2,0;3,1} = 0$. We then have

$$\boldsymbol{S}^* = \frac{1}{2}\boldsymbol{S}^*_{2,0;2,1:4} + \frac{1}{2}\boldsymbol{S}^*_{3,0;3,1:4},$$

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where $\boldsymbol{S}^*_{2,0;2,1:4}=(\boldsymbol{S}^{(0)}_5, \boldsymbol{S}^{*(1)}_{2,0:4}, \boldsymbol{S}^{*(2)}_{2,1:4})^T$ and $\boldsymbol{S}^*_{3,0;3,1:4}=(\boldsymbol{S}^{(0)}_5, \boldsymbol{S}^{*(1)}_{3,0:4}, \boldsymbol{S}^{*(2)}_{3,1:4})^T$, with

$$\begin{aligned}
\boldsymbol{S}^{*(1)}_{2,0:4} &= \frac{3\times 5}{25}\boldsymbol{S}^{(1)}_{2,0:5} + \frac{3\times 0}{25}\boldsymbol{S}^{(1)}_{2,1:5} + \frac{2\times 5}{25}\boldsymbol{S}^{(1)}_{3,0:5} + \frac{2\times 0}{25}\boldsymbol{S}^{(1)}_{3,1:5},\\
\boldsymbol{S}^{*(2)}_{2,1:4} &= \frac{3\times 4}{25}\boldsymbol{S}^{(2)}_{2,1:5} + \frac{3\times 1}{25}\boldsymbol{S}^{(2)}_{2,2:5} + \frac{2\times 4}{25}\boldsymbol{S}^{(2)}_{3,1:5} + \frac{2\times 1}{25}\boldsymbol{S}^{(2)}_{3,2:5},\\
\boldsymbol{S}^{*(1)}_{3,0:4} &= \frac{2\times 5}{25}\boldsymbol{S}^{(1)}_{3,0:5} + \frac{2\times 0}{25}\boldsymbol{S}^{(1)}_{3,1:5} + \frac{3\times 5}{25}\boldsymbol{S}^{(1)}_{4,0:5} + \frac{3\times 0}{25}\boldsymbol{S}^{(1)}_{4,1:5},\\
\boldsymbol{S}^{*(2)}_{3,1:4} &= \frac{2\times 4}{25}\boldsymbol{S}^{(2)}_{3,1:5} + \frac{2\times 1}{25}\boldsymbol{S}^{(2)}_{3,2:5} + \frac{3\times 4}{25}\boldsymbol{S}^{(2)}_{4,1:5} + \frac{3\times 1}{25}\boldsymbol{S}^{(2)}_{4,2:5}.
\end{aligned}$$

Then, we clearly have

$$\begin{aligned}
\boldsymbol{S}^{*(1)} &= \frac{1}{2}\boldsymbol{S}^{*(1)}_{2,0:4} + \frac{1}{2}\boldsymbol{S}^{*(1)}_{3,0:4} = \frac{1}{50}(15\boldsymbol{S}^{(1)}_{2,0:5} + 20\boldsymbol{S}^{(1)}_{3,0:5} + 15\boldsymbol{S}^{(1)}_{4,0:5}),\\
\boldsymbol{S}^{*(2)} &= \frac{1}{2}\boldsymbol{S}^{*(2)}_{2,1:4} + \frac{1}{2}\boldsymbol{S}^{*(2)}_{3,1:4} = \frac{1}{50}(12\boldsymbol{S}^{(2)}_{2,1:5} + 3\boldsymbol{S}^{(2)}_{2,2:5} + 16\boldsymbol{S}^{(2)}_{3,1:5} + 4\boldsymbol{S}^{(2)}_{3,2:5} + 12\boldsymbol{S}^{(2)}_{4,1:5} + 3\boldsymbol{S}^{(2)}_{4,2:5});
\end{aligned}$$

in other words, the equivalent system of size 5 of a multi-state linear consecutive $(2, 1)$-out-of-4:G wireless sensor system has its multi-state survival signature as

$$\begin{aligned}
\boldsymbol{S}^* &= (\boldsymbol{S}^{(0)}_5, \boldsymbol{S}^{*(1)}, \boldsymbol{S}^{*(2)})\\
&= \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
S^{*(1)}_{0,0} & S^{*(1)}_{1,0} & S^{*(1)}_{2,0} & S^{*(1)}_{3,0} & S^{*(1)}_{4,0} & S^{*(1)}_{5,0} & S^{*(1)}_{1,1} & S^{*(1)}_{2,1} & S^{*(1)}_{3,1} & S^{*(1)}_{4,1} & S^{*(1)}_{5,1} & S^{*(1)}_{2,2} & S^{*(1)}_{3,2} & S^{*(1)}_{4,2} & S^{*(1)}_{5,2}\\
S^{*(2)}_{0,0} & S^{*(2)}_{1,0} & S^{*(2)}_{2,0} & S^{*(2)}_{3,0} & S^{*(2)}_{4,0} & S^{*(2)}_{5,0} & S^{*(2)}_{1,1} & S^{*(2)}_{2,1} & S^{*(2)}_{3,1} & S^{*(2)}_{4,1} & S^{*(2)}_{5,1} & S^{*(2)}_{2,2} & S^{*(2)}_{3,2} & S^{*(2)}_{4,2} & S^{*(2)}_{5,2}
\end{pmatrix.}\\
&\qquad \begin{matrix}
1 & 1 & 1 & 1 & 1 & 1\\
S^{*(1)}_{3,3} & S^{*(1)}_{4,3} & S^{*(1)}_{5,3} & S^{*(1)}_{4,4} & S^{*(1)}_{5,4} & S^{*(1)}_{5,5}\\
S^{*(2)}_{3,3} & S^{*(2)}_{4,3} & S^{*(2)}_{5,3} & S^{*(2)}_{4,4} & S^{*(2)}_{5,4} & S^{*(2)}_{5,5}
\end{matrix}\Bigg)^T\\
&= \frac{1}{50}\begin{pmatrix}
50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50\\
0 & 0 & 15 & 35 & 50 & 50 & 0 & 15 & 35 & 50 & 50 & 15 & 35 & 50 & 50 & 35 & 50 & 50 & 50 & 50 & 50\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 28 & 40 & 40 & 15 & 35 & 50 & 50 & 35 & 50 & 50 & 50 & 50 & 50
\end{pmatrix}^T.
\end{aligned}$$

Example 3.2. For the equivalent system in Example 3.1 with its multi-state survival signature as given above, namely,

$$\begin{aligned}
\boldsymbol{S} &= (\boldsymbol{S}^{(0)}, \boldsymbol{S}^{(1)}, \boldsymbol{S}^{(2)})\\
&= \frac{1}{50}\begin{pmatrix}
50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50\\
0 & 0 & 15 & 35 & 50 & 50 & 0 & 15 & 35 & 50 & 50 & 15 & 35 & 50 & 50 & 35 & 50 & 50 & 50 & 50 & 50\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 28 & 40 & 40 & 15 & 35 & 50 & 50 & 35 & 50 & 50 & 50 & 50 & 50
\end{pmatrix}^T,
\end{aligned}$$

according to Remark 2.3, the multi-state survival signature of its equivalent system of size 6 can be given as $\boldsymbol{S}^* = \sum_{\boldsymbol{k}\in\mathscr{K}} s_{\boldsymbol{k}}\boldsymbol{S}^*_{\boldsymbol{k}:5}$, where $s_{\boldsymbol{k}}, \boldsymbol{k} \in \mathscr{K}$, with

$$\mathscr{K} = \{(k_{i,j}, i = 1, 2, j = 1, 2) : 0 \le k_{1,2} \le k_{1,1}, k_{2,2} \le k_{2,1} \le 5\},$$

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are given by the marginal distributions as follows:

$$\boldsymbol{s}^{(1)} = \boldsymbol{M}^{-1}\boldsymbol{S}^{(1)} = \left(0, 0, \frac{3}{10}, \frac{2}{5}, \frac{3}{10}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right)^T,$$

$$\boldsymbol{s}^{(2)} = \boldsymbol{M}^{-1}\boldsymbol{S}^{(2)} = \left(0, 0, 0, 0, 0, 0, 0, \frac{6}{25}, \frac{8}{25}, \frac{6}{25}, 0, \frac{3}{50}, \frac{2}{25}, \frac{3}{50}, 0, 0, 0, 0, 0, 0, 0\right)^T.$$

These imply $s^{(1)}_{2,0} = 3/10$, $s^{(1)}_{3,0} = 2/5$, $s^{(1)}_{4,0} = 3/10$, $s^{(1)}_{k_1,k_2} = 0$ for other $0 \le k_2 \le k_1 \le 5$, and

$$s^{(2)}_{2,1} = \frac{6}{25}, s^{(2)}_{3,1} = \frac{8}{25}, s^{(2)}_{4,1} = \frac{6}{25}, s^{(2)}_{2,2} = \frac{3}{50}, s^{(2)}_{3,2} = \frac{2}{25}, s^{(2)}_{4,2} = \frac{3}{50},$$

$s^{(2)}_{k_1,k_2} = 0$ for other $0 \le k_2 \le k_1 \le 5$. Now, upon solving the set of equations

$$\begin{cases} s_{2,0;2,1} + s_{2,0;3,1} + s_{2,0;4,1} + s_{2,0;2,2} + s_{2,0;3,2} + s_{2,0;4,2} = s^{(1)}_{2,0} = 3/10, \\ s_{3,0;3,1} + s_{3,0;4,1} + s_{3,0;3,2} + s_{3,0;4,2} = s^{(1)}_{3,0} = 2/5, \\ s_{4,0;4,1} + s_{4,0;4,2} = s^{(1)}_{4,0} = 3/10, \\ s_{2,0;2,1} = s^{(2)}_{2,1} = 6/25, \\ s_{2,0;3,1} + s_{3,0;3,1} = s^{(2)}_{3,1} = 8/25, \\ s_{2,0;4,1} + s_{3,0;4,1} + s_{4,0;4,1} = s^{(2)}_{4,1} = 6/25, \\ s_{2,0;2,2} = s^{(2)}_{2,2} = 3/50, \\ s_{2,0;3,2} + s_{3,0;3,2} = s^{(2)}_{3,2} = 2/25, \\ s_{2,0;4,2} + s_{3,0;4,2} + s_{4,0;4,2} = s^{(2)}_{4,2} = 3/50, \end{cases}$$

we get

$$s_{2,0;2,1} = \frac{6}{25}, s_{3,0;3,1} = \frac{8}{25}, s_{4,0;4,1} = \frac{6}{25}, s_{2,0;2,2} = \frac{3}{50}, s_{3,0;3,2} = \frac{2}{25}, s_{4,0;4,2} = \frac{3}{50},$$

and $s_{\boldsymbol{k}} = 0$ for other $\boldsymbol{k} \in \mathscr{K}$. We then have

$$\boldsymbol{S}^* = \frac{6}{25}\boldsymbol{S}^*_{2,0;2,1:5} + \frac{8}{25}\boldsymbol{S}^*_{3,0;3,1:5} + \frac{6}{25}\boldsymbol{S}^*_{4,0;4,1:5} + \frac{3}{50}\boldsymbol{S}^*_{2,0;2,2:5} + \frac{2}{25}\boldsymbol{S}^*_{3,0;3,2:5} + \frac{3}{50}\boldsymbol{S}^*_{4,0;4,2:5},$$

where $\boldsymbol{S}^*_{\boldsymbol{k}:5} = (\boldsymbol{S}^{(0)}_6, \boldsymbol{S}^{*(1)}_{k_{1,1},k_{1,2}:5}, \boldsymbol{S}^{*(2)}_{k_{2,1},k_{2,2}:5})^T$, with

$$\begin{aligned} S^{*(i)}_{\boldsymbol{k}_i,i_1,i_2} = {} & \frac{(6-k_{i,1})(6-k_{i,2})}{36} I_{\{i_1 \ge k_{i,1}, i_2 \ge k_{i,2}\}} + \frac{(6-k_{i,1})k_{i,2}}{36} I_{\{i_1 \ge k_{i,1}, i_2 \ge k_{i,2}+1\}} \\ & + \frac{k_{i,1}(6-k_{i,2})}{36} I_{\{i_1 \ge k_{i,1}+1, i_2 \ge k_{i,2}\}} + \frac{k_{i,1}k_{i,2}}{36} I_{\{i_1 \ge k_{i,1}+1, i_2 \ge k_{i,2}+1\}}, \\ & 0 \le i_2 \le i_1 \le 6, \ i = 1, 2. \end{aligned}$$

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Then, we clearly have

$$\begin{aligned}
\boldsymbol{S}^{*(1)} &= \frac{3}{10}\boldsymbol{S}^{*(1)}_{2,0:5} + \frac{2}{5}\boldsymbol{S}^{*(1)}_{3,0:5} + \frac{3}{10}\boldsymbol{S}^{*(1)}_{4,0:5} = \frac{1}{5}\boldsymbol{S}^{(1)}_{2,0:5} + \frac{3}{10}\boldsymbol{S}^{(1)}_{3,0:5} + \frac{3}{10}\boldsymbol{S}^{(1)}_{4,0:5} + \frac{1}{5}\boldsymbol{S}^{(1)}_{5,0:5},\\
\boldsymbol{S}^{*(2)} &= \frac{6}{25}\boldsymbol{S}^{*(2)}_{2,1:5} + \frac{8}{25}\boldsymbol{S}^{*(2)}_{3,1:5} + \frac{6}{25}\boldsymbol{S}^{*(2)}_{4,1:5} + \frac{3}{50}\boldsymbol{S}^{*(2)}_{2,2:5} + \frac{2}{25}\boldsymbol{S}^{*(2)}_{3,2:5} + \frac{3}{50}\boldsymbol{S}^{*(2)}_{4,2:5}\\
&= \frac{2}{15}\boldsymbol{S}^{(2)}_{2,1:5} + \frac{4}{75}\boldsymbol{S}^{(2)}_{2,2:5} + \frac{1}{5}\boldsymbol{S}^{(2)}_{3,1:5} + \frac{2}{25}\boldsymbol{S}^{(2)}_{3,2:5} + \frac{1}{5}\boldsymbol{S}^{(2)}_{4,1:5} + \frac{2}{25}\boldsymbol{S}^{(2)}_{4,2:5} + \frac{2}{15}\boldsymbol{S}^{(2)}_{5,1:5}\\
&\quad + \frac{4}{75}\boldsymbol{S}^{(2)}_{5,2:5} + \frac{1}{75}\boldsymbol{S}^{(2)}_{2,3:5} + \frac{1}{50}\boldsymbol{S}^{(2)}_{3,3:5} + \frac{1}{50}\boldsymbol{S}^{(2)}_{4,3:5} + \frac{1}{75}\boldsymbol{S}^{(2)}_{5,3:5};
\end{aligned}$$

that is, the equivalent system of size 6 of a multi-state linear consecutive $(2,1)$-out-of-4:G wireless sensor system has its multi-state survival signature as

$$\begin{aligned}
\boldsymbol{S}^{*} &= (\boldsymbol{S}^{(0)}_{6}, \boldsymbol{S}^{*(1)}, \boldsymbol{S}^{*(2)})\\
&= \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & \frac{1}{5} & \frac{1}{2} & \frac{4}{5} & 1 & 1 & 0 & \frac{1}{5} & \frac{1}{2} & \frac{4}{5} & 1 & 1 & \frac{1}{5} & \frac{1}{2} & \frac{4}{5} & 1 & 1 & \frac{1}{2} & \frac{4}{5} & 1 & 1 & \frac{4}{5} & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{15} & \frac{1}{3} & \frac{8}{15} & \frac{2}{3} & \frac{2}{3} & \frac{14}{75} & \frac{7}{15} & \frac{56}{75} & \frac{14}{15} & \frac{14}{15} & \frac{1}{2} & \frac{4}{5} & 1 & 1 & \frac{4}{5} & 1 & 1 & 1 & 1 & 1
\end{pmatrix}^{T}.
\end{aligned}$$

Example 3.3. For the multi-state linear consecutive $(2,1)$-out-of-4:G wireless sensor system in Example 3.1, according to Theorem 2.4, the multi-state survival signature of its equivalent system of size 6 can be directly given as $\boldsymbol{S}^{[2]*} = \sum_{\boldsymbol{k}\in\mathbf{K}} s_{\boldsymbol{k}}\boldsymbol{S}^{[2]*}_{\boldsymbol{k}:4}$, with $s_{\boldsymbol{k}}, \boldsymbol{k} \in \mathscr{K}$, being the same as in Example 3.1. Then, we have

$$\boldsymbol{S}^{[2]*} = \frac{1}{2}\boldsymbol{S}^{[2]*}_{2,0;2,1:4} + \frac{1}{2}\boldsymbol{S}^{[2]*}_{3,0;3,1:4},$$

where $\boldsymbol{S}^{[2]*}_{2,0;2,1:4} = (\boldsymbol{S}^{(0)}_{6}, \boldsymbol{S}^{[2]*(1)}_{2,0;2,1:4}, \boldsymbol{S}^{[2]*(2)}_{2,0;2,1:4})^{T}$ and $\boldsymbol{S}^{[2]*}_{3,0;3,1:4} = (\boldsymbol{S}^{(0)}_{6}, \boldsymbol{S}^{[2]*(1)}_{3,0;3,1:4}, \boldsymbol{S}^{[2]*(2)}_{3,1;3,1:4})^{T}$ are

$$\begin{aligned}
\boldsymbol{S}^{[2]*}_{2,0;2,1:4} &= \frac{6\times 10}{225}\boldsymbol{S}_{2,0;2,1:6} + \frac{6\times 4}{225}\boldsymbol{S}_{2,0;2,2:6} + \frac{6\times 1}{225}\boldsymbol{S}_{2,0;2,3:6}\\
&\quad + \frac{6\times 10}{225}\boldsymbol{S}_{3,0;3,1:6} + \frac{6\times 4}{225}\boldsymbol{S}_{3,0;3,2:6} + \frac{6\times 1}{225}\boldsymbol{S}_{3,0;3,3:6}\\
&\quad + \frac{3\times 10}{225}\boldsymbol{S}_{4,0;4,1:6} + \frac{3\times 4}{225}\boldsymbol{S}_{4,0;4,2:6} + \frac{3\times 1}{225}\boldsymbol{S}_{4,0;4,3:6},\\
\boldsymbol{S}^{[2]*}_{3,0;3,1:4} &= \frac{3\times 10}{225}\boldsymbol{S}_{3,0;3,1:6} + \frac{3\times 4}{225}\boldsymbol{S}_{3,0;3,2:6} + \frac{3\times 1}{225}\boldsymbol{S}_{3,0;3,3:6}\\
&\quad + \frac{6\times 10}{225}\boldsymbol{S}_{4,0;4,1:6} + \frac{6\times 4}{225}\boldsymbol{S}_{4,0;4,2:6} + \frac{6\times 1}{225}\boldsymbol{S}_{4,0;4,3:6}\\
&\quad + \frac{6\times 10}{225}\boldsymbol{S}_{5,0;5,1:6} + \frac{6\times 4}{225}\boldsymbol{S}_{5,0;5,2:6} + \frac{6\times 1}{225}\boldsymbol{S}_{5,0;5,3:6}.
\end{aligned}$$

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Note that $\boldsymbol{S}^{(1)}_{2,0;2,3:6} = \boldsymbol{S}^{(1)}_{2,0;3,3:6}$. Then, we clearly have

$$\begin{aligned}
\boldsymbol{S}^{[2]*(1)} &= \frac{1}{2}\boldsymbol{S}^{[2]*(1)}_{2,0;2,1:4} + \frac{1}{2}\boldsymbol{S}^{[2]*(1)}_{3,0;3,1:4} = \frac{1}{5}\boldsymbol{S}^{(1)}_{2,0:6} + \frac{3}{10}\boldsymbol{S}^{(1)}_{3,0:6} + \frac{3}{10}\boldsymbol{S}^{(1)}_{4,0:6} + \frac{1}{5}\boldsymbol{S}^{(1)}_{5,0:6}, \\
\boldsymbol{S}^{[2]*(2)} &= \frac{1}{2}\boldsymbol{S}^{[2]*(2)}_{2,0;2,1:4} + \frac{1}{2}\boldsymbol{S}^{[2]*(2)}_{3,0;3,1:4} \\
&= \frac{2}{15}\boldsymbol{S}^{(1)}_{2,1:6} + \frac{4}{75}\boldsymbol{S}^{(1)}_{2,2:6} + \frac{1}{75}\boldsymbol{S}^{(1)}_{2,3:6} + \frac{1}{5}\boldsymbol{S}^{(1)}_{3,1:6} + \frac{2}{25}\boldsymbol{S}^{(1)}_{3,2:6} + \frac{1}{50}\boldsymbol{S}^{(1)}_{3,3:6} \\
&\quad + \frac{1}{5}\boldsymbol{S}^{(1)}_{4,1:6} + \frac{2}{25}\boldsymbol{S}^{(1)}_{4,2:6} + \frac{1}{50}\boldsymbol{S}^{(1)}_{4,3:6} + \frac{2}{15}\boldsymbol{S}^{(1)}_{5,1:6} + \frac{4}{75}\boldsymbol{S}^{(1)}_{5,2:6} + \frac{1}{75}\boldsymbol{S}^{(1)}_{5,3:6};
\end{aligned}$$

that is, the equivalent system of size 6 of a multi-state linear consecutive $(2,1)$-out-of-4:G wireless sensor system has its multi-state survival signature as

$$\begin{aligned}
\boldsymbol{S}^{[2]*} &= (\boldsymbol{S}^{(0)}_6, \boldsymbol{S}^{[2]*(1)}, \boldsymbol{S}^{[2]*(2)}) \\
&= \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & \frac{1}{5} & \frac{1}{2} & \frac{4}{5} & 1 & 1 & 0 & \frac{1}{5} & \frac{1}{2} & \frac{4}{5} & 1 & 1 & \frac{1}{5} & \frac{1}{2} & \frac{4}{5} & 1 & 1 & \frac{1}{2} & \frac{4}{5} & 1 & 1 & \frac{4}{5} & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{15} & \frac{1}{3} & \frac{8}{15} & \frac{2}{3} & \frac{2}{3} & \frac{14}{75} & \frac{7}{15} & \frac{56}{75} & \frac{14}{15} & \frac{14}{15} & \frac{1}{2} & \frac{4}{5} & 1 & 1 & \frac{4}{5} & 1 & 1 & 1 & 1 & 1
\end{pmatrix}^T,
\end{aligned}$$

which is exactly the same as $\boldsymbol{S}^*$ obtained earlier in Example 3.2, as we would expect.

4. Comparison of systems of different sizes

In the last section, we considered a wireless sensor system with four sensors and provided multi-state survival signatures of it and its equivalent systems of sizes 5 and 6. In this section, two different wireless sensor system structures of sizes 5 and 6 will be discussed in Examples 4.1 and 4.2, respectively, for demonstrating the use of previously established results in comparing multi-state coherent or mixed systems with different numbers of *i.i.d.* multi-state components.

Example 4.1. Consider a wireless sensor system with five *i.i.d.* sensors and a multi-state linear consecutive $(3,2)$-out-of-5:G structure with sparse $(0,1)$, which works (perfectly or imperfectly) if and only if there are at least three consecutive sensors work (perfectly or imperfectly), works perfectly if and only if there are also at least two consecutive sensors with sparse 1 working perfectly, and fails if it does not work. As in Example 3 of Yi *et al.* [27], the multi-state survival signature of such a system can be shown to be

$$\begin{aligned}
\boldsymbol{S} &= (\boldsymbol{S}^{(0)}_5, \boldsymbol{S}^{(1)}, \boldsymbol{S}^{(2)}) \\
&= \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
S^{(1)}_{0,0} & S^{(1)}_{1,0} & S^{(1)}_{2,0} & S^{(1)}_{3,0} & S^{(1)}_{4,0} & S^{(1)}_{5,0} & S^{(1)}_{1,1} & S^{(1)}_{2,1} & S^{(1)}_{3,1} & S^{(1)}_{4,1} & S^{(1)}_{5,1} & S^{(1)}_{2,2} & S^{(1)}_{3,2} & S^{(1)}_{4,2} & S^{(1)}_{5,2} & S^{(1)}_{3,3} & S^{(1)}_{4,3} & S^{(1)}_{5,3} & S^{(1)}_{4,4} & S^{(1)}_{5,4} & S^{(1)}_{5,5} \\
S^{(2)}_{0,0} & S^{(2)}_{1,0} & S^{(2)}_{2,0} & S^{(2)}_{3,0} & S^{(2)}_{4,0} & S^{(2)}_{5,0} & S^{(2)}_{1,1} & S^{(2)}_{2,1} & S^{(2)}_{3,1} & S^{(2)}_{4,1} & S^{(2)}_{5,1} & S^{(2)}_{2,2} & S^{(2)}_{3,2} & S^{(2)}_{4,2} & S^{(2)}_{5,2} & S^{(2)}_{3,3} & S^{(2)}_{4,3} & S^{(2)}_{5,3} & S^{(2)}_{4,4} & S^{(2)}_{5,4} & S^{(2)}_{5,5}
\end{pmatrix}^T \\
&= \frac{1}{10}\begin{pmatrix}
10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\
0 & 0 & 0 & 3 & 8 & 10 & 0 & 0 & 3 & 8 & 10 & 0 & 3 & 8 & 10 & 3 & 8 & 10 & 8 & 10 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 6 & 7 & 3 & 8 & 10 & 8 & 10 & 10
\end{pmatrix}^T \\
&< \frac{1}{50}\begin{pmatrix}
50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 & 50 \\
0 & 0 & 15 & 35 & 50 & 50 & 0 & 15 & 35 & 50 & 50 & 15 & 35 & 50 & 50 & 35 & 50 & 50 & 50 & 50 & 50 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 28 & 40 & 40 & 15 & 35 & 50 & 50 & 35 & 50 & 50 & 50 & 50 & 50
\end{pmatrix}^T,
\end{aligned}$$

which implies that its system structure is not as good as that of a multi-state linear consecutive $(2,1)$-out-of-4:G wireless sensor system whose equivalent system of size 5 has been discussed in Example 3.1. This means that a multi-state linear consecutive $(2,1)$-out-of-4:G wireless sensor system tends to have better performance than a multi-state linear consecutive $(3,2)$-out-of-5:G wireless sensor system with sparse $(0,1)$ if their *i.i.d.* multi-state sensors have the same lifetime distributions.

Example 4.2. Consider a wireless sensor system with six *i.i.d.* sensors and a multi-state linear consecutive $(3,2)$-out-of-6:G structure with sparse $(0,1)$, as defined in Example 4.1. As in Example 3 of Yi *et al.* [24], the multi-state survival signature of such a system can be shown to be

$$\begin{aligned}
\boldsymbol{S} &= (\boldsymbol{S}_6^{(0)}, \boldsymbol{S}^{(1)}, \boldsymbol{S}^{(2)}) \\
&= \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & \frac{1}{5} & \frac{3}{5} & 1 & 1 & 0 & 0 & \frac{1}{5} & \frac{3}{5} & 1 & 1 & 0 & \frac{1}{5} & \frac{3}{5} & 1 & 1 & \frac{1}{5} & \frac{3}{5} & 1 & 1 & \frac{3}{5} & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{37}{90} & \frac{3}{5} & \frac{3}{5} & \frac{1}{5} & \frac{3}{5} & 1 & 1 & \frac{3}{5} & 1 & 1 & 1 & 1 & 1
\end{pmatrix}^T \\
&< \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & \frac{1}{5} & \frac{1}{2} & \frac{4}{5} & 1 & 1 & 0 & \frac{1}{5} & \frac{1}{2} & \frac{4}{5} & 1 & 1 & \frac{1}{5} & \frac{1}{2} & \frac{4}{5} & 1 & 1 & \frac{1}{2} & \frac{4}{5} & 1 & 1 & \frac{4}{5} & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{15} & \frac{1}{3} & \frac{8}{15} & \frac{2}{3} & \frac{2}{3} & \frac{14}{75} & \frac{7}{15} & \frac{56}{75} & \frac{14}{15} & \frac{14}{15} & \frac{1}{2} & \frac{4}{5} & 1 & 1 & \frac{4}{5} & 1 & 1 & 1 & 1 & 1
\end{pmatrix}^T,
\end{aligned}$$

which implies that its system structure is not as good as that of a multi-state linear consecutive $(2,1)$-out-of-4:G wireless sensor system whose equivalent system of size 6 has been discussed in Examples 3.2 and 3.3. This means that a multi-state linear consecutive $(2,1)$-out-of-4:G wireless sensor system tends to have better performance than a multi-state linear consecutive $(3,2)$-out-of-6:G wireless sensor system with sparse $(0,1)$ if their *i.i.d.* multi-state sensors have the same lifetime distributions.

5. Concluding remarks

In this work, we have first redefined the concept of multi-state survival signature of multi-state coherent or mixed systems with *i.i.d.* multi-state components. For two multi-state survival signatures of different sizes, transformation formulas have been established that facilitate transforming one with smaller size to one with larger size, which would thus enable stochastic comparison of two corresponding multi-state coherent or mixed systems of different sizes. Specific examples have been presented for illustrating the transformation formulas established here, and also their use in comparing systems of different sizes. Theoretical results derived here could prove useful in system design and management decision making for practical multi-state systems such as telecommunication systems, wireless sensor systems, power systems, radar station and many others. These transformation formulas are derived under the independence of component lifetimes at different state levels. We are currently working on developing similar results under some weaker assumptions and hope to report the corresponding findings in a future paper.

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