A PROPERTY OF ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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(received May 9, 1966)

We prove the following

THEOREM 1. *Let* $f_1(z)$, $f_2(z)$ *be entire functions of exponential type* τ_1, τ_2 *respectively. Suppose that for certain constants* K_1, K_2,

$$f_1(x) = O(|x|^{K_1}), \qquad f_2(x) = O(|x|^{K_2})$$

on the real line. Then for every $\tau > \tau_1 + \tau_2$,

$$(1) \qquad \underset{-\infty<x<\infty}{\text{l.u.b.}} \left| c_1 e^{-i\tau x} + c_2 e^{i\tau x} - f_1(x)/f_2(x) \right| \geq (|c_1|^2 + |c_2|^2)^{\frac{1}{2}},$$

where c_1, c_2 *are arbitrary constants. It is understood that* $f_1(z)$, $f_2(z)$ *are not both identically zero.*

Proof. If the function $f_2(z)$ is identically zero the result is obvious. So we assume that $f_2(z) \not\equiv 0$. Let us first choose an integer K and then a real number δ such that the entire functions

$$F_1(z) = f_1(z)(\delta z)^{-K}(\sin \delta z)^K, \quad F_2(z) = f_2(z)(\delta z)^{-K}(\sin \delta z)^K,$$

which are clearly of exponential type $T_1 = \tau_1 + \delta K$, $T_2 = \tau_2 + \delta K$

Canad. Math. Bull. vol. 9, no. 4, 1966

https://doi.org/10.4153/CMB-1966-057-2 Published online by Cambridge University Press

respectively, belong to L^2 on the real line, and $T_1 + T_2 < \tau$. By the Paley-Wiener theorem [1, p. 103] we have

$$F_1(z) = \int_{-T_1}^{T_1} e^{izt}\, \phi_1(t)dt, \quad F_2(z) = \int_{-T_2}^{T_2} e^{izt}\, \phi_2(t)dt,$$

where

$$\phi_1(t) \in L^2(-T_1, T_1), \quad \phi_2(t) \in L^2(-T_2, T_2).$$

If the theorem is false, then

$$|c_1 e^{-i\tau x} + c_2 e^{i\tau x} - f_1(x)/f_2(x)| < (|c_1|^2 + |c_2|^2)^{1/2}$$

for $-\infty < x < \infty$. Since the left hand side of this inequality is the same as $|c_1 e^{-i\tau x} + c_2 e^{i\tau x} - F_1(x)/F_2(x)|$ we get

$$|c_1 e^{-i\tau x} + c_2 e^{i\tau x} - F_1(x)/F_2(x)| < (|c_1|^2 + |c_2|^2)^{1/2}$$

for all real x. Thus

$$(2) \quad |(c_1 e^{-i\tau x} + c_2 e^{i\tau x}) F_2(x) - F_1(x)| < (|c_1|^2 + |c_2|^2)^{\frac{1}{2}} |F_2(x)|,$$

except for those values of x for which $F_2(x)$ vanishes. However, this exceptional set is countable.

It is clear that

$$(c_1 e^{-i\tau z} + c_2 e^{i\tau z}) F_2(z) - F_1(z) = \int_{-(\tau+T_2)}^{\tau+T_2} e^{izt}\, \phi(t)dt,$$

where $\phi(t)$ coincides with $c_1 \phi_2(t+\tau)$, $-\phi_1(t)$, $c_2 \phi_2(t-\tau)$ in the intervals $-(\tau + T_2) \le t < -(\tau - T_2)$, $-T_1 \le t \le T_1$, $(\tau - T_2) < t \le (\tau + T_2)$ respectively, and is zero everywhere else in the range of integration. From (2) it follows that

$$\int_{-(\tau+T_2)}^{\tau+T_2} |\phi(t)|^2 \, dt < (|c_1|^2 + |c_2|^2) \int_{-T_2}^{T_2} |\phi_2(t)|^2 \, dt,$$

or

$$|c_1|^2 \int_{-(\tau+T_2)}^{-(\tau-T_2)} |\phi_2(t+\tau)|^2 \, dt + \int_{-T_1}^{T_1} |\phi_1(t)^2 \, dt + |c_2|^2$$

$$\int_{\tau-T_2}^{\tau+T_2} |\phi_2(t-\tau)|^2 \, dt < (|c_1|^2 + |c_2|^2) \int_{-T_2}^{T_2} |\phi_2(t)|^2 \, dt .$$

But this is the same as

$$\int_{-T_1}^{T_1} |\phi_1(t)|^2 \, dt < 0 .$$

Hence our assumption that the theorem is false leads to a contradiction. This proves the theorem.

Remark. If the function $f_2(z)$ of the theorem is such that $h_{f_2}(\pi/2) = b$ where $h_{f_2}(\theta)$ is its indicator function [1, p. 66], then

$$F_2(z) = \int_{-b-\delta K}^{T_2} e^{izt} \phi_2(t) \, dt,$$

and from the above proof it is clear that for every $\tau > \tau_1 + b$,

(3) $$\underset{-\infty<x<\infty}{\text{l.u.b.}} |ce^{i\tau x} - f_1(x)/f_2(x)| \geq |c| .$$

It is easy to see that for $\tau > \tau_1 + b$ this inequality is true also if

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$$f_1(x) = \int_{-\infty}^{\tau_1} e^{ixt}\, \phi_1(t)\, dt, \qquad \phi_1 \in L^2(-\infty, \tau_1)$$

and

$$f_2(x) = \int_{-b}^{\infty} e^{ixt}\, \phi_2(t)\, dt, \qquad \phi_2 \in L^2(-b, \infty),$$

i.e. $f_1(x)$ is the Fourier transform of a function $\phi_1(t)$ belonging to $L^2(-\infty, \infty)$ and vanishing a.e. in (τ_1, ∞), whereas $f_2(x)$ is the Fourier transform of a function $\phi_2(t)$ which belongs to $L^2(-\infty, \infty)$ and vanishes a.e. in $(-\infty, -b)$. In analogy with this we prove the following

THEOREM 2. If the function $f_1(z)$ is analytic everywhere in $|z| \geq 1$ except at the point at infinity, where it has a pole of order m, and $f_2(z)$ is analytic in $|z| \leq 1$, then, for every $n > m$,

$$\max_{|z|=1} |cz^n - f_1(z)/f_2(z)| \geq |c|. \tag{4}$$

Proof. We may assume that $f_1(z)$ and $f_2(z)$ do not have any common zeros on $|z| = 1$. Now, if $f_2(z)$ has a zero on the unit circle the result is obvious. So let us suppose that $f_2(z) \neq 0$ for $|z| = 1$. If the result is false, then

$$|cz^n f_2(z) - f_1(z)| < |c|\, |f_1(z)|$$

for $|z| = 1$. Let $f_2(z)$ have the power series expansion $\sum_{j=0}^{\infty} b_j z^j$ valid on and inside the unit circle. If $f_1(z^{-1}) =$

$\sum_{j=-m}^{\infty} a_j z^j$ for $0 < |z| \leq 1$, then

$$\left|ce^{in\theta} \sum_{j=0}^{\infty} b_j e^{ij\theta} - \sum_{j=-m}^{\infty} a_j e^{-ij\theta}\right| < |c| \left|\sum_{j=0}^{\infty} b_j e^{ij\theta}\right|$$

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for $0 \leq \theta < 2\pi$. On squaring the two sides and integrating with respect to θ from 0 to 2π we get

$$|c|^2 \sum_{j=0}^{\infty} |b_j|^2 + \sum_{j=-m}^{\infty} |a_j|^2 < |c|^2 \sum_{j=0}^{\infty} |b_j|^2 .$$

This gives a contradiction and the result is proved.

The following result is of the same general nature and, in fact, generalizes Theorem 2.

THEOREM 3. If $f_1(z)$ is represented in $\operatorname{Im} z \leq 0$ by the absolutely convergent Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-iz\alpha_n}, \quad -\infty < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \alpha_{n+1} < \ldots, \lim_{n\to\infty} \alpha_n = \infty,$$

and if $f_2(z)$ is a function defined in $\operatorname{Im} z \geq 0$ by the absolutely convergent Dirichlet series

$$\sum_{n=1}^{\infty} b_n e^{iz\beta_n}, \quad 0 < \beta_1 < \beta_2 < \ldots < \beta_n < \beta_{n+1} < \ldots, \lim_{n\to\infty} \beta_n = \infty,$$

then, for $\tau > -\alpha_1 - \beta_1$,

$$\underset{-\infty<x<\infty}{\text{l.u.b.}} |ce^{i\tau x} - f_1(x)/f_2(x)| \geq |c| .$$

There is equality only if $f_1(z) \equiv 0$.

Proof. Let $f_1(z) \not\equiv 0$. If the theorem is false then

$$|ce^{i\tau x} f_2(x) - f_1(x)| \leq |c| \, |f_2(x)|$$

a.e. on the real line. Hence

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |ce^{i\tau x} f_2(x) - f_1(x)|^2 \, dx \leq |c|^2 \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f_2(x)|^2 \, dx,$$

i.e.

$$|c|^2 \sum_{n=1}^{\infty} |b_n|^2 + \sum_{n=1}^{\infty} |a_n|^2 \leq |c|^2 \sum_{n=1}^{\infty} |b_n|^2 ,$$

- a contradiction.

https://doi.org/10.4153/CMB-1966-057-2 Published online by Cambridge University Press

A COROLLARY OF THEOREM 1. A function of the form

$$\frac{a_0 x^n + a_1 x^{n-1} + \dots + a_n}{b_0 x^n + b_1 x^{n-1} + \dots + b_n}$$

where the denominator does not vanish identically is called a rational function of x of degree n. Noting that, if $p(z)$ is a polynomial of degree n then $p(\cos z)$ is an entire function of exponential type n, we conclude the following result from Theorem 1 with $c_1 = c_2 = 1/2$.

COROLLARY. _If the degree_ m _of the Tchebycheff polynomial_ $\cos(m \cos^{-1} x)$ _is at least_ $2n+1$, _then, on the interval_ $[-1, 1]$, _it cannot be uniformly approximated more closely than_ $\frac{1}{\sqrt{2}}$, _by rational functions of degree_ n.

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https://doi.org/10.4153/CMB-1966-057-2 Published online by Cambridge University Press