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PSEUDO-EINSTEIN REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS

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In this paper we give a complete classification of $\mathfrak{D}^\perp$-*invariant* or *Hopf pseudo-Einstein* real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$.

0. INTRODUCTION

In the geometry of real hypersurfaces in complex space forms or in quaternionic space forms it can be easily checked that there do not exist any real hypersurfaces with parallel shape operator A by virtue of the equation of Codazzi.

From this point of a view many differential geometers have considered new notions weaker than such a parallel second fundamental form, that is, $\nabla A = 0$. In particular, Kimura and Maeda [6] have proved that a real hypersurface M in a complex projective space $\mathbb{C}P^m$ satisfying $\nabla_\xi A = 0$ is locally congruent to a real hypersurface of type A_1, A_2, that is, a tube over a totally geodesic complex submanifold $\mathbb{C}P^k$ with radius $0 < r < \pi/2$. The structure vector field ξ mentioned above is defined by $\xi = -JN$, where J denotes a Kähler structure of $\mathbb{C}P^m$ and N a local unit normal field of M in $\mathbb{C}P^m$. Moreover, in a class of Hopf hypersurfaces Kimura [5] has asserted that there do not exist any real hypersurfaces with parallel Ricci tensor, that is $\nabla S = 0$, where S denotes the Ricci tensor of a real hypersurface M in $\mathbb{C}P^m$.

On the other hand, in a quaternionic projective space $\mathbb{H}P^m$ Pérez [7] has considered the notion of $\nabla_{\xi_i} A = 0$, $i = 1, 2, 3$, for real hypersurfaces in $\mathbb{H}P^m$ and classified that M is locally congruent to of A_1, A_2-type, that is, a tube over $\mathbb{H}P^k$ with radius $0 < r < \pi/4$. The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_i = -J_i N$, $i = 1, 2, 3$, where J_i denotes a quaternionic Kähler structure of $\mathbb{H}P^m$ and N a unit normal field of M in $\mathbb{H}P^m$. Moreover, Pérez and the present author [8] have considered the notion of $\nabla_{\xi_i} R = 0$, $i = 1, 2, 3$, where R denotes the curvature tensor of a real hypersurface M in $\mathbb{H}P^m$, and proved that M is locally congruent to a tube of radius $\pi/4$ over $\mathbb{H}P^k$.

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Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Then the situation mentioned above is not so simple if we consider a real hypersurface in such a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$.

In this paper we study the analogous question in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space has a remarkable geometrical structure. It is the unique compact irreducible Riemannian manifold being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperkähler manifold. So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometrical conditions for real hypersurfaces that $[\xi] = \text{Span}\ \{\xi\}$ or $\mathfrak{D}^\perp = \text{Span}\ \{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. From such a view point Berndt and the present author [**2**] have proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(1) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(2) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

When the structure vector ξ of M in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, M is said to be a *Hopf* hypersurface. In such a case the integral curve of the structure vector field ξ is geodesic (See Berndt and Suh [**3**]). Moreover, the flow generated by integral curves of structure vector field ξ of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be a geodesic *Reeb* flow.

In the proof of Theorem A we have proved that the one-dimensional distribution $[\xi]$ is contained in either the 3-dimensional distribution $\mathfrak{D}^\perp$ or in the orthogonal complement $\mathfrak{D}$ such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$. The case (1) in Theorem A is just the case that the one dimensional distribution $[\xi]$ is contained in $\mathfrak{D}^\perp$.

A real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be *pseudo-Einstein* if the Ricci tensor S of M satisfies

$$(*) \qquad SX = aX + b\eta(X)\xi + c\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu$$

for any tangent vector field X on M, where b and c are nonvanishing constants and 1-forms η and η_ν are defined by $\eta(X) = g(\xi, X)$ and $\eta_\nu(X) = g(\xi_\nu, X)$ respectively. Moreover, a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be *Einstein* if the constants b and c identically both vanish.

Then the derivative of the Ricci tensor S of a pseudo-Einstein real hypersurface in $G_2(\mathbb{C}^{m+2})$ is given by

$$(**)\quad (\nabla_Y S)X = b(\nabla_Y \eta)(X)\xi + b\eta(X)\nabla_Y \xi + c\sum_{\nu=1}^{3}(\nabla_Y \eta_\nu)(X)\xi_\nu + c\sum_{\nu=1}^{3}\eta_\nu(X)\nabla_Y \xi_\nu$$

for nonvanishing constants b and c on M.

It is not difficult to check that any real hypersurfaces given in Theorem A is pseudo-Einstein. Then it must be a natural question to know whether pseudo-Einstein real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ can be classified or not ?

Related to such a problem the main result of this paper is to give a complete classification of $\mathfrak{D}^\perp$-invariant pseudo-Einstein real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ as follows:

THEOREM 1. *Let M be a $\mathfrak{D}^\perp$-invariant real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying the formula (**). Then M is congruent to*

(A) *a tube of radius r over $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ or*
(B) *a tube of radius r over $\mathbb{H}P^m$, $m = 2n$, in $G_2(\mathbb{C}^{m+2})$.*

The formula (**) mentioned in Theorem 1 is just covariant derivative of the Ricci tensor of pseudo-Einstein real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. From such a view point we give a $\mathfrak{D}^\perp$-invariant pseudo-Einstein real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ as follows:

THEOREM 2. *Let M be a $\mathfrak{D}^\perp$-invariant pseudo-Einstein real hypersurface in $G_2(\mathbb{C}^{m+2})$. Then M is congruent to*

(a) *a tube of radius r, $\cot^2\sqrt{2}r = (m-1)/2$, over $G_2(\mathbb{C}^{m+1})$, where $a = 4m+8$, $b+c = -2(m+1)$,*
(b) *a tube of radius r, $\cot^2 r = \left(2n\pm\sqrt{4n-1}\right)/\left(2(n-1)\right)$, over $\mathbb{H}P^m$, $m = 2n$, where $a = 8n+6$, $b = -16n+2$, $c = -2$.*

When M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, we assert the following:

THEOREM 3. *Let M be a Hopf pseudo-Einstein real hypersurface in $G_2(\mathbb{C}^{m+2})$. Then M is congruent to*

(a) *a tube of radius r, $\cot^2\sqrt{2}r = (m-1)/2$, over $G_2(\mathbb{C}^{m+1})$, where $a = 4m+8$, $b+c = -2(m+1)$, provided with $c \neq -4$,*
(b) *a tube of radius r, $\cot^2 r = \left(2n\pm\sqrt{4n-1}\right)/\left(2(n-1)\right)$, over $\mathbb{H}P^m$, $m = 2n$, where $a = 8n+6$, $b = -16n+2$, $c = -2$.*

In Section 2 we recall Riemannian geometry of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and in Section 3 we shall show some fundamental properties of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. The formula for the Ricci tensor S and its covariant derivative ∇S will be shown explicitly in this section.

In Section 4 (respectively, Section 5) we shall give a complete proof of Theorem 2 (respectively, Theorem 3) when M is a $\mathfrak{D}^\perp$-invariant (respectively, Hopf) pseudo-Einstein real hypersurface in $G_2(\mathbb{C}^{m+2})$.

1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarise basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [**1, 2, 3**]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabiliser isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalise g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the centre of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the centre $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $tr(JJ_1) = 0$. This fact will be used frequently throughout this paper.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken module three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\overline{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\overline{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\overline{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

(1.2)
$$\begin{aligned}\overline{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\ + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we derive some basic formulae from the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$.

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g and ∇ denotes the Riemannian connection on M. Let N be a local unit normal field of M and A the shape operator of M with respect to N.

For any local vector field X on a neighbourhood of a point x in M the transformation under the Kähler structure J of $G_2(\mathbb{C}^{m+2})$ can be given by

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where ϕ defines a skew-symmetric transformations of the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on a neighbourhood in M, respectively. Then it is seen that $g(\xi, X) = \eta(X)$. In such a case the set of tensors (ϕ, ξ, η, g) is said to be an *almost contact metric structure* on M. They satisfy the following

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1$$

for any tangent vector field X on M.

On the other hand, let us denote by $\{J_1, J_2, J_3\}$ a canonical local basis of $\mathfrak{J}$, which are said to be a *quaternionic Kähler structure* of $G_2(\mathbb{C}^{m+2})$. Then the transformation of the tangent vector field X on M under the quaternionic Kähler structure $\{J_1, J_2, J_3\}$ can be given by

$$J_\nu X = \phi_\nu X + \eta_\nu(X)N, \quad J_\nu N = -\xi_\nu$$

for any $\nu = 1, 2, 3$, where $\phi_\nu X$ denotes the tangent component of $J_\nu X$ and $\eta_\nu(X) = g(X, \xi_\nu)$. Then $J_\nu^2 = -I$ induces an *almost contact metric structure* $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M defined in such a way that

$$\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \phi_\nu\xi_\nu = 0, \eta_\nu(\phi_\nu X) = 0, \eta_\nu(\xi_\nu) = 1$$

for any tangent vector field X on M and any $\nu = 1, 2, 3$.

Using the above expression (1.2) for $\overline{R}$, the Gauss and the Codazzi equations are respectively given by

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\nu=1}^{3}\{g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z\} \\
&+\sum_{\nu=1}^{3}\{g(\phi_\nu \phi Y, Z)\phi_\nu \phi X - g(\phi_\nu \phi X, Z)\phi_\nu \phi Y\} \\
&-\sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu \phi X - \eta(X)\eta_\nu(Z)\phi_\nu \phi Y\} \\
&-\sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu \phi Y, Z) - \eta(Y)g(\phi_\nu \phi X, Z)\}\xi_\nu \\
&+ g(AY, Z)AX - g(AX, Z)AY
\end{aligned}
$$

and

$$
\begin{aligned}
(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi& \\
+\sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\}& \\
+\sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu \phi Y - \eta_\nu(\phi Y)\phi_\nu \phi X\}& \\
+\sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu&,
\end{aligned}
$$

where R denotes the curvature tensor of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

From $J_\nu J_{\nu+1} = -J_{\nu+1}J_\nu = J_{\nu+2}$ and $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, the following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$
\begin{aligned}
\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, &\qquad \phi_\nu \xi_{\nu+1} = \xi_{\nu+2}, \\
\phi\xi_\nu = \phi_\nu \xi, &\qquad \eta_\nu(\phi X) = \eta(\phi_\nu X), \\
\phi_\nu \phi_{\nu+1} X = \phi_{\nu+2}X &+ \eta_{\nu+1}(X)\xi_\nu, \\
\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X &+ \eta_\nu(X)\xi_{\nu+1}.
\end{aligned} \tag{2.1}
$$

Then from this and the formula (1.1) we have that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX, \tag{2.2}$$

$$\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.3}$$

$$(\nabla_X \phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX, Y)\xi_\nu. \tag{2.4}$$

Summing up these formulas, we find the following

$$
\begin{aligned}
\nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu) \\
&= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu) \\
&= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX \\
&\qquad - g(AX,\xi)\xi_\nu + \eta(\xi_\nu)AX.
\end{aligned} \tag{2.5}
$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$
\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{2.6}
$$

3. Proof of Main Theorem

Now let us contract Y and Z in the equation of Gauss in Section 2. Then the Ricci tensor S of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is given by

$$
\begin{aligned}
SX &= \sum_{i=1}^{4m-1} R(X,e_i)e_i \\
&= (4m+10)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\
&\quad + \sum_{\nu=1}^{3}\{(\operatorname{Tr}\phi_\nu\phi)\phi_\nu\phi X - (\phi_\nu\phi)^2X\} \\
&\quad - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(X)\phi_\nu\phi\xi_\nu\} \\
&\quad - \sum_{\nu=1}^{3}\{(\operatorname{Tr}\ \phi_\nu\phi)\eta(X) - \eta(\phi_\nu\phi X)\}\xi_\nu + hAX - A^2X,
\end{aligned} \tag{3.1}
$$

where h denotes the trace of the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. From the formula $JJ_\nu = J_\nu J$, $\operatorname{Tr}\ JJ_\nu = 0$, $\nu = 1,2,3$ we calculate the following for any basis $\{e_1,\dots,e_{4m-1},N\}$ of the tangent space of $G_2(\mathbb{C}^{m+2})$

$$
\begin{aligned}
0 &= \operatorname{Tr}\ JJ_\nu \\
&= \sum_{k=1}^{4m-1} g(JJ_\nu e_k, e_k) + g(JJ_\nu N, N) \\
&= \operatorname{Tr}\ \phi\phi_\nu - \eta_\nu(\xi) - g(J_\nu N, JN) \\
&= \operatorname{Tr}\ \phi\phi_\nu - 2\eta_\nu(\xi)
\end{aligned} \tag{3.2}
$$

and

$$
\begin{aligned}
(\phi_\nu\phi)^2X &= \phi_\nu\phi(\phi\phi_\nu X - \eta_\nu(X)\xi + \eta(X)\xi_\nu) \\
&= \phi_\nu\{-\phi_\nu X + \eta(\phi_\nu X)\xi\} + \eta(X)\phi_\nu{}^2\xi \\
&= X - \eta_\nu(X)\xi_\nu + \eta(\phi_\nu X)\phi_\nu\xi + \eta(X)\{-\xi + \eta_\nu(\xi)\xi\}.
\end{aligned} \tag{3.3}
$$

Substituting (3.2) and (3.3) into (3.1), we have

$$
\begin{aligned}
SX &= (4m+10)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\
&\quad + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi\} \\
&\quad + hAX - A^2X \\
&= (4m+7)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\
&\quad + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi\} \\
&\quad + hAX - A^2X.
\end{aligned} \tag{3.4}
$$

Now its covariant derivative of (3.4) becomes

$$
\begin{aligned}
(\nabla_Y S)X &= -3(\nabla_Y\eta)X - 3\eta(X)\nabla_Y\xi - 3\sum_{\nu=1}^{3}(\nabla_Y\eta_\nu)(X)\xi_\nu \\
&\quad - 3\sum_{\nu=1}^{3}(\nabla_Y\eta_\nu)(X)\xi_\nu - 3\sum_{\nu=1}^{3}\eta_\nu(X)\nabla_Y\xi_\nu \\
&\quad + \sum_{\nu=1}^{3}\Big[Y\big(\eta_\nu(\xi)\big)\phi_\nu\phi X + \eta_\nu(\xi)(\nabla_Y\phi_\nu)\phi X + \eta_\nu(\xi)\phi_\nu(\nabla_Y\phi)X \\
&\quad - (\nabla_Y\eta)(\phi_\nu X)\phi_\nu\xi - \eta((\nabla_Y\phi_\nu)X)\phi_\nu\xi - \eta(\phi_\nu X)\nabla_Y(\phi_\nu\xi) \\
&\quad - (\nabla_Y\eta)(X)\eta_\nu(\xi)\xi_\nu - \eta(X)\nabla_Y\big(\eta_\nu(\xi)\big)\xi_\nu - \eta(X)\eta_\nu(\xi)\nabla_Y\xi_\nu\Big] \\
&\quad + (Yh)AX + h(\nabla_Y A)X - (\nabla_Y A^2)X.
\end{aligned} \tag{3.5}
$$

Then from (3.5), together with the formulas in Section 2, we have

$$
\begin{aligned}
(\nabla_Y S)X &= -3g(\phi AY, X)\xi - 3\eta(X)\phi AY \\
&\quad - 3\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(X) - q_{\nu+1}(Y)\eta_{\nu+2}(X) + g(\phi_\nu AY, X)\}\xi_\nu \\
&\quad - 3\sum_{\nu=1}^{3}\eta_\nu(X)\{q_{\nu+2}(Y)\xi_{\nu+1} - q_{\nu+1}\xi_{\nu+2} + \phi_\nu AY\} \\
&\quad + \sum_{\nu=1}^{3}\Big[Y\big(\eta_\nu(\xi)\big)\phi_\nu\phi X + \eta_\nu(\xi)\{-q_{\nu+1}(Y)\phi_{\nu+2}\phi X \\
&\quad + q_{\nu+2}(Y)\phi_{\nu+1}\phi X + \eta_\nu(\phi X)AY - g(AY, \phi X)\xi_\nu\} \\
&\quad + \eta_\nu(\xi)\{\eta(X)\phi_\nu AY - g(AY, X)\phi_\nu\xi\} - g(\phi AY, \phi_\nu X)\phi_\nu\xi
\end{aligned}
$$

$$
\begin{aligned}
(3.6)\qquad & + \{q_{\nu+1}(Y)\eta(\phi_{\nu+2}X) - q_{\nu+2}(Y)\eta(\phi_{\nu+1}X) - \eta_\nu(X)\eta(AY) \\
& + \eta(\xi_\nu)g(AY,X)\}\phi_\nu\xi \\
& - \eta(\phi_\nu X)\{q_{\nu+2}(Y)\phi_{\nu+1}\xi - q_{\nu+1}(Y)\phi_{\nu+2}\xi + \phi_\nu\phi AY \\
& - \eta(AY)\xi_\nu + \eta(\xi_\nu)AY\} \\
& - g(\phi AY, X)\eta_\nu(\xi)\xi_\nu - \eta(X)Y(\eta_\nu(\xi))\xi_\nu - \eta(X)\eta_\nu(\xi)\nabla_Y\xi_\nu\Big] \\
& + (Yh)AX + h(\nabla_Y A)X - (\nabla_Y A^2)X.
\end{aligned}
$$

Let M be a pseudo-Einstein real hypersurface in $G_2(\mathbb{C}^{m+2})$. That is, M satisfies the formula (*) in the intoduction. Then by the formulas in Section 2 the derivative of the Ricci tensor S satisfies

$$
\begin{aligned}
& (\nabla_Y S)X = \nabla_Y(SX) - S(\nabla_Y X) \\
(**)\qquad & = b(\nabla_Y\eta)(X)\xi + b\eta(X)\nabla_Y\xi + c\sum_{\nu=1}^{3}(\nabla_Y\eta_\nu)(X)\xi_\nu + c\sum_{\nu=1}^{3}\eta_\nu(X)\nabla_Y\xi_\nu.
\end{aligned}
$$

From this, together with (3.6), we have

(3.7)

$$
\begin{aligned}
& (b+3)\{g(\phi AY,X)\xi + \eta(X)\phi AY\} \\
& + (c+3)\Big[\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(X) - q_{\nu+1}(Y)\eta_{\nu+2}(X) + g(\phi_\nu AY, X)\}\xi_\nu \\
& + \sum_{\nu=1}^{3}\eta_\nu(X)\{q_{\nu+2}(Y)\xi_{\nu+1} - q_{\nu+1}(Y)\xi_{\nu+2} + \phi_\nu AY\}\Big] \\
& - \sum_{\nu=1}^{3}\Big[\{g(q_{\nu+2}(Y)\phi_{\nu+1}\xi - q_{\nu+1}(Y)\phi_{\nu+2}\xi + \phi_\nu\phi AY, X)\phi_\nu\xi \\
& - g(\eta(AY)\xi_\nu - \eta(\xi_\nu)AY, X)\phi_\nu\xi \\
& + g(\phi_\nu\xi, X)\{q_{\nu+2}(Y)\phi_{\nu+1}\xi - q_{\nu+1}(Y)\phi_{\nu+2}\xi \\
& + \phi_\nu\phi AY - \eta(AY)\xi_\nu + \eta(\xi_\nu)AY\}\Big] \\
& - \sum_{\nu=1}^{3}\Big[Y(\eta_\nu(\xi))\phi_\nu\phi X + \eta_\nu(\xi)\{-q_{\nu+1}(Y)\phi_{\nu+2}\phi X + q_{\nu+2}(Y)\phi_{\nu+1}\phi X \\
& + \eta_\nu(\phi X)AY - g(AY,\phi X)\xi_\nu\} \\
& + \eta_\nu(\xi)\{\eta(X)\phi_\nu AY - g(AY,X)\phi_\nu\xi\} \\
& - g(\phi AY,X)\eta_\nu(\xi)\xi_\nu - \eta(X)Y(\eta_\nu(\xi))\xi_\nu - \eta(X)\eta_\nu(\xi)\nabla_Y\xi_\nu\Big] \\
& - (Yh)AX - (hI - A)(\nabla_Y A)X + (\nabla_Y A)AX = 0.
\end{aligned}
$$

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Note that pseudo-Einstein real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfies the formula (3.7). Now contracting X and W in (3.7) and using the formulas mentioned below

$$Y(\eta_\nu(\xi)) = Y(g(\xi_\nu,\xi)) = g(\nabla_Y\xi_\nu,\xi) + g(\xi_\nu,\nabla_Y\xi)$$
$$= q_{\nu+2}(Y)\eta(\xi_{\nu+1}) - q_{\nu+1}(Y)\eta(\xi_{\nu+2}) - 2g(AY,\phi_\nu\xi),$$
$$\sum_{\nu=1}^{3}\Big\{q_{\nu+2}(Y)\eta(\xi_{\nu+1})\operatorname{Tr}\ \phi_\nu\phi - q_{\nu+1}(Y)\eta(\xi_{\nu+2})\operatorname{Tr}\ \phi_\nu\phi$$
$$- q_{\nu+1}(Y)\eta_\nu(\xi)\operatorname{Tr}\ \phi_{\nu+2}\phi + q_{\nu+2}(Y)\eta_\nu(\xi)\operatorname{Tr}\ \phi_{\nu+1}\phi\Big\} = 0,$$

and

$$2\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta(\xi_{\nu+1}) - q_{\nu+1}(Y)\eta(\xi_{\nu+2})\} = 0,$$

then we have the following

$$-(Yh)h + \text{trace }(\nabla_Y A)(2A - hI) = 0. \tag{3.8}$$

Now let us take an inner product (3.7) with any vector field W and use the equation of Codazzi for the final terms of the left side of (3.7). From this, contracting Y and W, we have

(3.9)
$$(b+3)g(\phi A\xi, X)$$
$$+ (c+3)\sum_{\nu=1}^{3}\Big[\{q_{\nu+2}(\xi_\nu)\eta_{\nu+1}(X) - q_{\nu+1}(\xi_\nu)\eta_{\nu+2}(X) + g(\phi_\nu A\xi_\nu, X)\}$$
$$+ \eta_\nu(X)\{q_{\nu+2}(\xi_{\nu+1}) - q_{\nu+1}(\xi_{\nu+2})\}\Big]$$
$$- \sum_{\nu=1}^{3}\Big[\{q_{\nu+2}(\phi_\nu\xi)g(\phi_{\nu+1}\xi, X) - q_{\nu+2}(\phi_\nu\xi)g(\phi_{\nu+2}\xi, X)$$
$$+ g(\phi_\nu\phi A\phi_\nu\xi, X) - \eta(A\phi_\nu\xi)\eta_\nu(X)\}$$
$$+ g(\phi_\nu\xi, X)\{q_{\nu+2}(\phi_{\nu+1}\xi) - q_{\nu+1}(\phi_{\nu+2}\xi) - \eta(A\xi_\nu) + \eta(\xi_\nu)\operatorname{Tr} A\}\Big]$$
$$- \sum_{\nu=1}^{3}\Big[(\phi_\nu\phi X)(\eta_\nu(\xi)) + \eta_\nu(\xi)\{-q_{\nu+1}(\phi_{\nu+2}\phi X)$$
$$+ q_{\nu+2}(\phi_{\nu+1}\phi X) + \eta_\nu(\phi X)\operatorname{Tr} A\} + \eta_\nu(\xi)\eta(X)\operatorname{Tr}\phi_\nu A - \eta(X)\xi_\nu(\eta_\nu(\xi))$$
$$- \eta(X)\eta_\nu(\xi)\{q_{\nu+2}(\xi_{\nu+1}) - q_{\nu+1}(\xi_{\nu+2})\}\Big]$$
$$+ 3g(A\xi, \phi X) + 3\sum_{\nu=1}^{3}g(A\phi_\nu X, \xi_\nu)$$

$$
\begin{aligned}
&- \sum_{\nu=1}^{3} \Big\{ g\big((A-h)\phi\xi_\nu, \phi_\nu\phi X\big) - h\eta_\nu(\phi X) \Big\} \\
&+ \sum_{\nu=1}^{3} \Big\{ g\big((A-h)\xi, \xi_\nu\big)\eta_\nu(\phi X) + \eta(X) g(A\phi\xi_\nu, \xi_\nu) \Big\} \\
&- \sum_{\nu=1}^{3} \{ g(\phi_\nu\phi AX, \phi\xi_\nu) + \eta_\nu(\phi AX) \operatorname{Tr} \phi_\nu\phi - \eta(\xi_\nu)\eta_\nu(\phi AX) \} = 0.
\end{aligned}
$$

Now let us here calculate more explicitly the following terms in (3.9) as follows:

$$
\begin{aligned}
\phi_\nu\phi X(\eta_\nu(\xi)) &= g(\nabla_{\phi_\nu\phi X}\xi, \xi_\nu) + g(\xi, \nabla_{\phi_\nu\phi X}\xi_\nu) \\
&= 2g(\phi A\phi_\nu\phi X, \xi_\nu) + \eta(\xi_{\nu+1}) q_{\nu+2}(\phi_\nu\phi X) \\
&\qquad - \eta(\xi_{\nu+2}) q_{\nu+1}(\phi_\nu\phi X),
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
\eta(X)\xi_\nu(\eta_\nu(\xi)) &= \eta(X)\{ g(\nabla_{\xi_\nu}\xi, \xi_\nu) + g(\xi, \nabla_{\xi_\nu}\xi_\nu) \} \\
&= 2\eta(X) g(\phi A\xi_\nu, \xi_\nu) + \eta(X)\{ \eta(\xi_{\nu+1}) q_{\nu+2}(\xi_\nu) \\
&\qquad - \eta(\xi_{\nu+2}) q_{\nu+1}(\xi_\nu) \}.
\end{aligned} \tag{3.11}
$$

Now let us substitute (3.10) and (3.11) into (3.9). From this, together with the formulas

$$
g(\phi_\nu\phi A\phi_\nu\xi, X) = -g(\xi_\nu, \phi A\phi_\nu\phi X) + \eta_\nu(X) g(A\phi_\nu\xi, \xi) - \eta(X) g(A\phi_\nu\xi, \xi_\nu)
$$

and

$$
-\sum_{\nu=1}^{3} g\big((A - hI)\phi\xi_\nu, \phi_\nu\phi X\big) = \sum_{\nu=1}^{3} g(\phi A\phi_\nu X, \xi_\nu) - h\sum_{\nu=1}^{3} \eta_\nu(\phi X)\eta_\nu(\xi),
$$

we have

$$
\begin{aligned}
&(b+3) g(\phi A\xi, X) + c\sum_{\nu=1}^{3} g(\phi_\nu A\xi_\nu, X) - 2\sum_{\nu=1}^{3} \eta_\nu(\phi X)\eta_\nu(\xi) \operatorname{Tr} A \\
&\qquad - (AX)h + \operatorname{Tr}(A - hI)(\nabla_X A) \\
&\qquad\quad + 3g(A\phi X, \xi) + \operatorname{Tr} \nabla_{AX} A + h\sum_{\nu=1}^{3} \eta_\nu(\phi X) \operatorname{Tr} \phi_\nu\phi = 0,
\end{aligned} \tag{3.12}
$$

where we have used the following

$$\sum_{\nu=1}^{3}\Big[\{q_{\nu+2}(\xi_\nu)\eta_{\nu+1}(X)-q_{\nu+1}(\xi_\nu)\eta_{\nu+2}(X)\}$$
$$+\eta_\nu(X)\{q_{\nu+2}(\xi_{\nu+1})-q_{\nu+1}(\xi_{\nu+2})\}\Big]=0,$$

$$\sum_{\nu=1}^{3}\Big[\{q_{\nu+2}(\phi_\nu\xi)g(\phi_{\nu+1}\xi,X)-q_{\nu+1}(\phi_\nu\xi)g(\phi_{\nu+2}\xi,X)\}$$
$$+g(\phi_\nu\xi,X)\{q_{\nu+2}(\phi_{\nu+1}\xi)-q_{\nu+1}(\phi_{\nu+2}\xi)\}\Big]=0,$$

$$\sum_{\nu=1}^{3}\Big[\{\eta(\xi_{\nu+1})q_{\nu+2}(\phi_\nu\phi X)-\eta(\xi_{\nu+2})q_{\nu+1}(\phi_\nu\phi X)\}$$
$$+\eta_\nu(\xi)\{-q_{\nu+1}(\phi_{\nu+2}\phi X)+q_{\nu+2}(\phi_{\nu+1}\phi X)\}\Big]=0,$$

and

$$\sum_{\nu=1}^{3}\eta(X)\Big[\{\eta(\xi_{\nu+1})q_{\nu+2}(\xi_\nu)-\eta(\xi_{\nu+2})q_{\nu+1}(\xi_\nu)\}$$
$$+\eta_\nu(\xi)\{q_{\nu+2}(\xi_{\nu+1})-q_{\nu+1}(\xi_{\nu+2})\}\Big]=0.$$

On the other hand, by (3.8) we know that

$$\mathrm{Tr}\ (\nabla_Y A)A=(Yh)h.$$

From this and using $\mathrm{Tr}\ \phi_\nu\phi=2\eta_\nu(\xi)$ in (3.12), we have

$$b\phi A\xi+c\sum_{\nu=1}^{3}\phi_\nu A\xi_\nu=0. \tag{3.13}$$

When $g(A\mathfrak{D},\mathfrak{D}^\perp)=0$, that is $A\xi_\nu=\beta_\nu\xi_\nu$, $\nu=1,2,3$, then by (3.13) the structure vector ξ is principal. Then by virtue of a theorem due to Berndt and the present author [**3**], we summarise the above arguments as follows:

THEOREM 3.1. *Let M be a $\mathfrak{D}^\perp$-invariant real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying (**). Then M is congruent to one of the following:*

(A) *a tube of radius r over $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*
(B) *a tube of radius r over $\mathbb{H}P^n$, $m=2n$, in $G_2(\mathbb{C}^{m+2})$.*

REMARK 3.1. A real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be *Einstein* if the constants b and c in the formula (*) identically both vanish. Then by the formula (**) its Ricci tensor should be parallel. Moreover, in such a case the Ricci tensor S commutes with the structure tensor ϕ. On the other hand, in a paper due to Pérez and the present author [**9**] we have proved that there do not exist any Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel and commuting Ricci tensor. So naturally we know that there do not exist any Höpf Einstein real hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

4. Proof of Theorem 2

In this section, let M be a $\mathfrak{D}^\perp$-invariant pseudo-Einstein real hypersurface in $G_2(\mathbb{C}^{m+2})$. Then the Ricci tensor S of M satisfies the formula (*). Then naturally it satisfies the formula (**). So by virtue of Theorem 3.1 M is congruent to of type (A) or of type (B). First we consider M is congruent to of type (A), that is, a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Now let us introduce a proposition in [**2**] concerned with a tube of type (A) as follows:

PROPOSITION A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D}\subset\mathfrak{D}$, $A\xi=\alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1\in\mathfrak{J}$ be the almost Hermitian structure such that $JN=J_1N$. Then M has three(if $r=\pi/2$) or four(otherwise) distinct constant principal curvatures*

$$\alpha=\sqrt{8}\cot\left(\sqrt{8}r\right),\ \beta=\sqrt{2}\cot\left(\sqrt{2}r\right),\ \lambda=-\sqrt{2}\tan\left(\sqrt{2}r\right),\ \mu=0 \tag{4.1}$$

with some $r\in(0,\pi/4)$. The corresponding multiplicities are

$$m(\alpha)=1\ ,\ m(\beta)=2\ ,\ m(\lambda)=2m-2=m(\mu),$$

and the corresponding eigenspaces we have

$$\begin{aligned}
T_\alpha&=\mathbb{R}\xi=\mathbb{R}JN,\\
T_\beta&=\mathbb{C}^\perp\xi=\mathbb{C}^\perp N=\{\xi_2,\xi_3\},\\
T_\lambda&=\{X\mid X\perp\mathbb{H}\xi, JX=J_1X\},\\
T_\mu&=\{X|X\perp\mathbb{H}\xi, JX=-J_1X\}.
\end{aligned}$$

Now let us check for a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. We put $X\in T_\mu$ in (3.4). Then by the formula (*) we know that

$$a=4m+8. \tag{4.2}$$

From this, together with (4.1), (3.4) gives for any $X \in T_\lambda$

$$\begin{aligned}\{(4m+7) - \sqrt{2}\tan\sqrt{2}rh - \left(\sqrt{2}\tan\sqrt{2}r\right)^2\}X + \phi_1\phi X \\ = \{4m+6 - \sqrt{2}\tan\sqrt{2}rh - \left(\sqrt{2}\tan\sqrt{2}r\right)^2\}X \\ = (4m+8)X.\end{aligned}$$

This gives $h = -\sqrt{2}(\tan\sqrt{2}r + \cot\sqrt{2}r)$. From this, together with the trace of h, which is given by $h = (2m-2)(-\sqrt{2}\tan\sqrt{2}r) + \sqrt{8}\cot\sqrt{8}r + 2\sqrt{2}\cot\sqrt{2}r$, we have $\cot^2\sqrt{2}r = (m-1)/2$.

On the other hand, by putting $X = \xi = \xi_1$ in (3.4) and the formula (*), we know

$$a + b + c = 4m + h\alpha - \alpha^2.$$

From this, together with (4.1) and (4.2), it follows that

$$\begin{aligned}b + c &= h\alpha - \alpha^2 - 8 \\ &= \sqrt{2}h\left(\cot\sqrt{2}r - \tan\sqrt{2}r\right) - 2\left(\cot\sqrt{2}r - \tan\sqrt{2}r\right)^2 - 8 \\ &= -2\left(\cot^2\sqrt{2}r - \tan^2\sqrt{2}r\right) - 2\left(\cot\sqrt{2}r - \tan\sqrt{2}r\right)^2 - 8 \\ &= -4\left(\cot^2\sqrt{2}r + 1\right) \\ &= -4\left(\frac{m-1}{2} + 1\right) \\ &= -2(m+1).\end{aligned}$$

Next we consider M is congruent to of type (B), that is, a tube of radius r over $\mathbb{H}P^m$, $m = 2n$ in $G_2(\mathbb{C}^{m+2})$. Moreover, for a tube of type (B) in Theorem A we introduce the following proposition due to [**2**] as follows:

Proposition B. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r)\ ,\ \beta = 2\cot(2r)\ ,\ \gamma = 0\ ,\ \lambda = \cot(r)\ ,\ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1\ ,\ m(\beta) = 3 = m(\gamma)\ ,\ m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi \ , \ T_\beta = \{\xi_1, \xi_2, \xi_3\}, \ T_\gamma = \{\phi_1\xi, \phi_2\xi, \phi_3\xi\}, \ T_\lambda \ , \ T_\mu \ ,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp \ , \ \mathfrak{J}T_\lambda = T_\lambda \ , \ \mathfrak{J}T_\mu = T_\mu \ , \ JT_\lambda = T_\mu \ .$$

By the formula (3.4) and (4.2) for the Ricci tensor of a pseudo-Einstein real hypersurface M in $G_2(\mathbb{C}^{m+2})$, we have respectively for $X \in T_\lambda$ and $X \in T_\beta$

$$a = 4m + 7 + h\cot r - \cot^2 r \tag{4.3}$$

and

$$a + c = 4m + 4 + 2\cot 2rh - 4\cot^2 2r. \tag{4.4}$$

Moreover, by Proposition B we can put h the trace of the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ as follows

$$h = 4(n-1)\{\cot r - \tan r\} + 6\cot 2r - 2\tan 2r. \tag{4.5}$$

On the other hand, for $X \in T_{\cot r}$ and $Y \in T_{-\tan r}$ we know that $SX = aX$ and $SY = aY$. Then by (3.4) we have $h\cot r - \cot^2 r = h(-\tan r) - \tan^2 r$. Then it follows that

$$h = \cot r - \tan r. \tag{4.6}$$

From this, together with (4.3), we assert that $a = 4m + 6 = 8n + 6$.

On the other hand, by comparing both two equations (4.5) and (4.6) mentioned above, we know that

$$(4n-2)(\cot r - \tan r) = \frac{4}{\cot r - \tan r}, \tag{4.7}$$

which gives $\cot^2 r = \left(2n \pm \sqrt{4n-1}\right)/(2n-1)$. Moreover, from (4.4) and (4.6) we know

$$c = -2 + (\cot r - \tan r)h - (\cot r - \tan r)^2 = -2.$$

Finally, we assert that $b = -16n + 2$.

In fact, by the formula (*), (3.4) and Proposition B for $X \in T_\alpha$ we have

$$a + b = 4m + 4 - \frac{4}{2\cot 2r}h - \frac{16}{4\cot^2 2r}.$$

From this, together with (4.3), it follows that

$$(4m+7) + h\cot r - \cot^2 r + b = (4m+4) - \frac{4}{\cot r - \tan r}h - \frac{16}{(\cot r - \tan r)^2}.$$

Then substituting (4.6) and (4.7) into this formula, we know that

$$(4m+7) + (\cot r - \tan r)\cot r - \cot^2 r + b = 4m - 8(2n-1),$$

which gives above assertion that $b = -16n + 2$. So summing up all the situations mentioned above, we give a complete proof of Theorem 2.

5. Proof of Theorem 3

In this section, we consider a Hopf pseudo-Einstein real hypersurface M in $G_2(\mathbb{C}^{m+2})$, provided with $c \neq -4$. Then, its Ricci tensor is given by

$$
\begin{aligned}
SX &= (4m+7)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\
&\quad + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi\} \\
&\quad + hAX - A^2X \\
&= aX + b\eta(X)\xi + c\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu
\end{aligned} \tag{5.1}
$$

for nonvanishing constants b and $c \neq -4$ on M. Then by putting $X = \xi$ into (5.1), we have

$$
4(m+1)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu + hA\xi - A^2\xi = (a+b)\xi + c\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu. \tag{5.2}
$$

On the other hand, by (5.1) we have

$$
\begin{aligned}
A^2X - hAX + \rho X = \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\} \\
- (3+b)\eta(X)\xi - (3+c)\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu,
\end{aligned} \tag{5.3}
$$

where $\rho = a - (4m+7)$. When M is a Hopf hypersurface, (5.2) gives the following

$$
\{4(m+1) + h\alpha - \alpha^2 - (a+b)\}\xi = (c+4)\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu. \tag{5.4}
$$

From this, applying ξ_μ to both sides, we have

$$
\{4(m+1) + h\alpha - \alpha^2 - (a+b) - (c+4)\}\eta_\nu(\xi) = 0. \tag{5.5}
$$

Now let us divide two cases. Then first we consider

Case I. $4(m+1) + h\alpha - \alpha^2 - (a+b) - (c+4) \neq 0$

Then from (5.5) we know that $\eta_\nu(\xi) = 0$. that is, $\xi \in \mathfrak{D}$. Now let us put $T = A^2 - hA$. Then by (5.3) we have the following

$$
\begin{aligned}
T\xi_1 &= (A^2 - hA)\xi_1 = \{\rho - (3+c)\}\xi_1, \\
T\xi_2 &= (A^2 - hA)\xi_1 = \{\rho - (3+c)\}\xi_2, \\
T\xi_3 &= (A^2 - hA)\xi_1 = \{\rho - (3+c)\}\xi_3,
\end{aligned}
$$

where we have put $\rho = a - (4m+7)$. From this, together with the fact $TA = AT$ and $g(T\mathfrak{D}, \mathfrak{D}^\perp) = 0$ we know that there exists a basis X_1, X_2, X_3 of $\mathfrak{D}^\perp$ that $AX_i = \lambda_i X_i$ and $TX_i = \lambda_i X_i$, $i = 1, 2, 3$, in such a way that

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = SO(3) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

where $SO(3)$ denotes the 3×3 special orthogonal matrix. Then

$$\text{Span } \{X_1, X_2, X_3\} = \text{Span } \{\xi_1, \xi_2, \xi_3\}.$$

Accordingly, we also assert that $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.

CASE II. $4(m+1) + h\alpha - \alpha^2 - (a+b) - (c+4) = 0$

By (5.4) and $c \neq -4$, we know that $\xi = \sum_{\nu=1}^{3} \eta_\nu(\xi)\xi_\nu \in \mathfrak{D}^\perp$. So we may put $\xi = \xi_1$. Then by (5.3) we have the following

$$A^2\xi_1 - hA\xi_1 = \{4m - (a+b+c)\}\xi_1, \tag{5.6}$$

$$A^2\xi_2 - hA\xi_2 = \{4m + 7 - a - (1+c)\}\xi_2, \tag{5.7}$$

$$A^2\xi_3 - hA\xi_3 = \{4m + 7 - a - (1+c)\}\xi_3. \tag{5.8}$$

Then we know that $g(T\mathfrak{D}, \mathfrak{D}^\perp) = 0$, where we have put $T = A^2 - hA$. Moreover as in Case I we know $TA = AT$. From this, together with the fact $g(T\mathfrak{D}, \mathfrak{D}^\perp) = 0$, by the same method as in Case I we assert that $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. Accordingly, summing up Cases I and II we know that M is $\mathfrak{D}^\perp$-invariant. Then by Theorem 3.1 and using the same method given in the proof of Theorem 2 we give a complete proof of Theorem 3.

REFERENCES

[1] J. Berndt, 'Riemannian geometry of complex two-plane Grassmannians', *Rend. Sem. Mat. Univ. Politec. Torino* **55** (1997), 19-83.

[2] J. Berndt and Y.J. Suh, 'Real hypersurfaces in complex two-plane Grassmannians', *Monatsh. Math.* **127** (1999), 1-14.

[3] J. Berndt and Y.J. Suh, 'Isometric flows on real hypersurfaces in complex two-plane Grassmannians', *Monatsh. Math.* **137** (2002), 87-98.

[4] T.E. Cecil and P.J. Ryan, 'Focal sets and real hypersurfaces in complex projective space', *Trans. Amer. Math. Soc.* **269** (1982), 481-499.

[5] M.Kimura, 'Real hypersurfaces of a complex projective space', *Bull. Austral. Math. Soc.* **33** (1986), 383-387.

[6] M. Kimura and S. Maeda, 'On real hypersurfaces of a complex projective space II', *Tsukuba J. Math.* **15** (1991), 547–561.

[7] J.D. Pérez, 'Real hypersurfaces of quaternionic projective space satisfying $\nabla_{U_i} A = 0$', *J. Geom.* **49** (1994), 166–177.

[8] J.D. Pérez and Y.J. Suh, 'Real hypersurfaces of quaternionic projective space satisfying $\nabla_{U_i} R = 0$', *Differential Geom. Appl.* **7** (1997), 211–217.

[9] J.D. Pérez and Y.J. Suh, 'Real hypersurfaces in complex two-plane Grassmannians with parallel and commuting Ricci tensor', (submitted).

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