Ergod. Th. & Dynam. Sys. (1984), **4**, 311-322
Printed in Great Britain

Ergodic rational maps with dense critical point forward orbit

MARY REES

Department of Pure Mathematics, University of Liverpool, P.O. Box 147, *Liverpool, L69 3BX, England*

(*Received* 2 *February* 1983 *and revised* 8 *June* 1983)

Abstract. Examples are constructed of ergodic rational maps with dense critical orbit.

In this note we consider rational maps of the extended complex plane S^2. These maps all preserve the equivalence class of Lebesgue measure, and some of them (though not, it is thought, many) are topologically transitive. The simplest topologically transitive examples are probably those coming from endomorphisms of the torus $\mathbb{C}/\mathbb{Z}+i\mathbb{Z}$, where the sphere is a ramified quotient of the torus via the equivalence $z \sim -z$. These are clearly ergodic. The next simplest topologically transitive examples are possibly those with eventually periodic (but not periodic) critical points, mentioned in [**4**], [**3**]. There are countably many such non-topologically conjugate examples of degree k, for each k. It is not hard to show (theorem 1) that if the periodic points in the forward critical point orbits of R are expanding periodic, then R is ergodic. Such a map R also has 'good' expanding properties on any closed set disjoint from the finite forward critical point orbits (made precise in lemma 2). However, ergodic rational maps without 'good' expanding properties can be constructed from a certain countable set A of rational maps of degree 2 (similarly of degree k, any k) for which all critical points are eventually periodic. It is shown (theorem 2) that $\bar{A}$ is perfect, and that residually many rational maps in A are ergodic with dense critical point forward orbits, that is, do not have 'good' expanding properties.

The ergodic maps found in $\bar{A}$ probably do not all have positive exponents. In a later paper it will be shown (among other things) that $\bar{A}$ has positive λ-measure and that for f in a positive measure subset of $\bar{A}$, f is ergodic and has positive exponents – but this is probably not a residual property in $\bar{A}$.

It is known [**4**] that a topologically transitive rational map f has no positive measure set B with B disjoint from $\{y: f^m y = f^n x, \quad \text{some } x \in B, y \neq x\}$. One can ask whether all such examples are ergodic, also whether all other known examples are ergodic, for instance those of Herman [**2**], to which the methods of this paper do not apply at all.

I should like to thank the referee for pointing out a fundamental error in my original proof of ergodicity being residual in $\bar{A}$. This work was carried out while the author was at the University of Minnesota.

https://doi.org/10.1017/S0143385700002467 Published online by Cambridge University Press

We consider rational maps for which all critical points are eventually mapped to expanding periodic points. The following lemmas show that such maps have 'good' expanding properties. The ideas are mostly adapted from those in the classical literature (as expounded in [1]) and in [4] applying to the case of Axiom A rational maps.

Let R be a fixed rational map of this type, and $\mathscr{C}$ the finite set of its critical points and forward orbits of critical points.

LEMMA 1. *If α is sufficiently small then there exist $K>0$, $0<C<1$ and $N>0$ such that:*

(1) if $x\in\mathscr{C}$ is periodic with period t and S is the local inverse of R^{nt} with $Sx=x$ defined on $B(x,\alpha)$, then $|S'|\le KC^n$; and

(2) if S is a local inverse of R^m, $m\ge N$ defined on any ball $B(x,\alpha)$ then $|S'|\le C$ on $B(x,\alpha/2)$; if S is a local inverse of any R^m on $B(x,\alpha)$, then $|S'|\le K$ on $B(x,\alpha/2)$.

Proof. (1) It can be assumed that $\overline{B(x,\alpha)}$ is the image under R^t of the open neighbourhood of x on which there exists analytic ψ with

$$\psi\circ R^t=\lambda\circ\psi \qquad \psi(x)=0, \qquad |\lambda|>1$$

So $\psi^{-1}\circ\lambda^{-n}\circ\psi=S$. Take $C>1/|\lambda|$, for λ corresponding to all periodic points in $\mathscr{C}$. (2) Any $S(B(x,\alpha))$ omits at least 3 points bounded away from each other by some $\delta>0$. So by Montel's theorem the family of S defined on the $\overline{B(x,3\alpha/4)}$ is equicontinuous, and all $|S'|\le K$ on $\overline{B(x,3\alpha/4)}$.

There must be a bound on the n with S a local inverse of R^n with diam $(S(B(x,3\alpha/4)))\ge\varepsilon$. Otherwise, taking limits there would be a disk of diameter $\ge\varepsilon/2$ on which the derivatives of R^{n_i} (for an infinite sequence $\{n_i\}$) would be bounded. This is impossible because the Julia set is S^2, so any disk of radius $\varepsilon/2$ contains expanding periodic points. Taking $\varepsilon=4C/\alpha$, if diam $(S(B(x,\alpha/4)))\le\varepsilon$ then $|S'|\le C$ on $B(x,\alpha/2)$. □

Now let A_1 be the complement of the (α/K_1)-neighbourhood of $\mathscr{C}$, for $K_1>1$ to be chosen. Let A_2 be the complement of the α-neighbourhood of $\mathscr{C}$.

If $x\in A_1$, all local inverses of all R^n are well-defined on $B(x,\alpha/K_1)$.

If $x\in\mathscr{C}$, some local inverses of the R^n are well-defined on $B(x,\alpha)$. Any connected component of some $R^{-n}(B(x,\alpha))$ on which R^n is not a homeomorphism is a topological disk of the form

$$S_2R^{-p}S_1B(x,\alpha),$$

with $S_1=$ identity unless x is periodic, $S_1(x)=x$ and S_1 is a local inverse of some R^{nt} with t the period of x, $p\le r$, for some r depending on R,

$$R^{-p}S_1B(x,\alpha)\subseteq A_2,$$

and S_2 is a local inverse of some R^m (possibly $S_2=$ identity).

LEMMA 2. *If S is a local inverse of some R^m defined on $B(x,\alpha/K_1)$, $x\in A_2$, for suitable K_1, then $|S'|\le K_2C_2^m$, for some $C_2<1$.*

Proof. By suitable choice of K_1, S can be written $U_m\circ U_{m-1}\circ\cdots\circ U_1$, where:

(a) U_i is a local inverse of R^{k_i} with successive images all in A_1 (for i odd), and all in S^2-A_2 (for i even).

https://doi.org/10.1017/S0143385700002467 Published online by Cambridge University Press

(b) for i odd, the first and last images in the sequence corresponding to U_i lie in A_2, and no string of M_1 successive images lies in $A_1 \setminus A_2$, (M_1 depending on K_1).
(c) for i even, $k_i \geq M$, where $K^2C^M = C_1 < 1$, (see lemma 1).
By lemma 1, $|U'_{2i-1}| \leq K$ and $|U'_{2i}| \leq KC^M$, so $|U'_{2i}U'_{2i-1}| < C_1$. If $k_{2i-1} < M_1N$ then

$$k_{2i-1} + k_{2i} < \left(\frac{M_1N}{M} + 1\right)k_{2i}.$$

So if $k_{2i-1} \leq M_1$,

$$|U'_{2i}U'_{2i-1}| < C_1^{(1/(M+1)\cdot M/(M_1N+1))(k_{2i-1}+k_{2i})}.$$

If $k_{2i-1} > M_1N$,

$$|U'_{2i}U'_{2i-1}| < C_1^{k_{2i-1}/(M_1N+1)}C_1^{k_{2i}/(M+1)}.$$

If m is odd, $|U'_m| \leq K_2$, for suitable K_2, if $k_m \leq M_1N$; $|U'_m| \leq C^{k_m/(M_1N+1)}$, if $k_m > M_1N$. So take $C_2 = C_1^{1/(M_1N+M+2)}$. □

LEMMA 3. *There exists K_3 such that if S is a local inverse of some R^n defined on $B(x, \alpha/K_1)$, $x \in A_2$, or is the inverse of some R^{nt} with $Sx = x$ defined on $B(x, \alpha)$ where $x \in \mathscr{C}$ is expanding periodic, of period t, then*

$$\big|\log|S'(y)| - \log|S'(z)|\big| \leq K_3$$

for all y, z in the stated domain.

Proof. The result for expanding periodic $x \in \mathscr{C}$ follows immediately from the local linearization of R^t. Otherwise, write $S = U_m \circ \cdots \circ U_1$, as in lemma 2. Write $B = B(x, \alpha/K_1)$. Write ε_i for the diameter of $U_iU_{i-1}\cdots U_1B$.

$$\big|\log|S'y| - \log|S'z|\big| \leq \sum_{i=1}^{m} \big|\log|U'_i(U_{i-1}\cdots U_1y)| - \log|U'_i(U_{i-1}\cdots U_1z)|\big|.$$

If i is odd then

$$\big|\log|U'_iw| - \log|U'_iv|\big| \leq K_4\varepsilon_{i-1}$$

for $w, v \in U_{i-1}\cdots U_1B$, (because if k_i is large, the successive images corresponding to U_i decrease geometrically, and first and second derivatives of all local inverses of R are bounded away from 0 and ∞ on A_1).

If i is even, then up to conjugation by an analytic function ψ, (with bounds on the first and second derivatives of ψ), U_i is the inverse of a map:

$$z \mapsto \mu z^p, \qquad |\mu| > 1, p \text{ bounded, depending only on } R,$$

the inverse being given by:

$$z \mapsto \left(\frac{z}{\mu}\right)^{1/p}$$

on a set of diameter $\leq \varepsilon_{i-1}$, with $|z| = O(\alpha)$. The second derivative is thus $O(|\mu|^{-1/p}|\alpha|^{1/(p-2)})$. So, again,

$$\big|\log|U'_i(w)| - \log|U'_i(v)|\big| \leq K_4\varepsilon_{i-1}.$$

The result follows, since $\sum \varepsilon_i < \infty$. □

LEMMA 4. *If D is a connected component of $R^{-n}B(x, \alpha)$ for $x \in \mathscr{C}$ and D_1 is the corresponding component of $R^{-n}B(x, \alpha) - R^{-n}B(x, \alpha/2)$ then* meas $(D_1) \geq K_5$ meas (D)

https://doi.org/10.1017/S0143385700002467 Published online by Cambridge University Press

and

$$|\log|(R^n)'(y)|-\log|(R^n)'(z)||\le K_5,$$

$y, z\in D_1$, for K_5 independent of n, D.

Proof. Let $D=S_2R^{-p}S_1B(x,\alpha)$ where S_i is a local inverse of R^{n_i}, p bounded. We already have bounds on the variation of S_1', and also of S_2', since $S_1(B(x,\alpha))$ lies in A_2, and can be covered by finitely many α/K_1-balls.

$$\begin{aligned} B(S_1x,\beta/K_7)\setminus B(S_1x,\beta/K_8) &\subseteq S_1B_1B(x,\alpha)-S_1B(x,\alpha/2)\\ &\subseteq B(S_1x,\beta)-B(S_1x,\beta/K_6)\end{aligned}$$

for K_6, K_7, K_8 independent of S_1 (but β depending on S_1).

R^p is given locally by $z\mapsto\mu z^q$, (q bounded), from a neighbourhood of $x_1\in\mathscr{C}\cap R^{-p}(x)$ onto a neighbourhood of x. A bound on the variation of $(R^p)'$ on $R^{-p}S_1(B(x,\alpha))$ follows, as does a lower bound on

$$\frac{\text{meas}\,(R^{-p}S_1(B(x,\alpha)-B(x,\alpha/2)))}{\text{meas}\,(R^{-p}S_1B(x,\alpha))}. \qquad \square$$

THEOREM 1. *R is ergodic with respect to Lebesgue measure.*

Proof. The lemmas show there exists L such that for any local inverse S on $B(x,\alpha/K_1)$,

(a) $\dfrac{1}{L}\le\left|\dfrac{S'(y)}{S'(z)}\right|\le L,\quad y,z\in B(x,\alpha/K_1),\ x\in A_2,$

or $\quad y,z\in B(x,\alpha),\ x\in\mathscr{C}$.

(b) $|S'|$ is small if S is a local inverse of R^n, n large.

(c) If $D_1\subseteq D$, D, D_1 are connected components of $R^{-n}(B(x,\alpha))$, $R^{-n}(B(x,\alpha)-B(x,\alpha/2))$ respectively, $x\in\mathscr{C}$, then D is small for n large, and

$$\frac{\text{meas}\,(D_1)}{\text{meas}\,(D)}\ge\frac{1}{L}$$

$$\frac{1}{L}\le\left|\frac{(R^n)'(y)}{(R^n)'(z)}\right|\le L,\qquad y,z\in D_1.$$

Now let A be an arbitrary set of positive measure. To show R is ergodic, it suffices to show that the R-orbit of A covers almost all of some $B(x,\alpha/K_1)$, $(x\in A_2)$, or $B(x,\alpha)\setminus B(x,\alpha/2)$, $(x\in\mathscr{C})$. For R-forward images of one of these open sets cover S^2. Let $\mathscr{U}$ be some finite cover of S^2 by balls $B(x,\alpha/K_1)$, $(x\in A_2)$, and $B(x,\alpha)$, $(x\in\mathscr{C})$. If $\mathscr{U}$ is of index q then so is $R^{-n}\mathscr{U}$, if this denotes the cover by connected components of $R^{-n}U$, $U\in\mathscr{U}$. Because the elements of $R^{-n}\mathscr{U}$ are small for n large, for some connected component D of some R^{-n},

$$\frac{\text{meas}\,(D\cap A)}{\text{meas}\,(D)}>1-\varepsilon.$$

If D is not the image of U under a local inverse of R^n, let D_1 be the appropriate annulus with

$$\frac{\text{meas}\,(D_1\cap A)}{\text{meas}\,(D_1)}>1-L\varepsilon.$$

https://doi.org/10.1017/S0143385700002467 Published online by Cambridge University Press

In any case apply R^n to get

$$\frac{\text{meas}\,(R^nD\cap R^nA)}{\text{meas}\,(R^nD)}>1-L^2\varepsilon \qquad \text{for } D \text{ a local inverse image,}$$

$$\frac{\text{meas}\,(R^nD_1\cap R^nA)}{\text{meas}\,(R^nD_1)}>1-L^3\varepsilon \qquad \text{otherwise.}$$

The result follows since $R^nD=B(x,\alpha/K_1)$, $(x\in A_2)$, or $B(x,\alpha)$, $(x\in \mathscr{C})$, and $R^nD_1=B(x,\alpha)\setminus B(x,\alpha/2)$, $(x\in\mathscr{C})$. □

From now on, for simplicity, we consider only the family of maps

$$f_\lambda: z\mapsto \lambda\left(1-\frac{2}{z}\right)^2 \qquad \lambda\neq 0,\ z\in\mathbb{C}.$$

All these maps have critical points at $z=0, 2$, with

$$f_\lambda(2)=0,\quad f_\lambda(0)=\infty,\quad f_\lambda(\infty)=\lambda.$$

Write

$$f(\lambda,z)=f_\lambda(z)=f_1(\lambda,z).$$
$$f_n(\lambda,z)=f(\lambda,f_{n-1}(\lambda,z))=f^n_\lambda(z).$$

Let A be the set of all λ such that:

(P1) There exists a least n such that $f_n(\lambda,2)=f_{n+1}(\lambda,2)$.

(P2) $|(\partial f/\partial z)(\lambda, f_n(\lambda,2))|>1$.

(P3) $f_n(\lambda,2)$ has a preimage under f_λ which is not periodic, nor ∞.

(P4) $(\partial f_n/\partial\lambda)(\lambda,2)(\partial f/\partial z)(\lambda,f_n(\lambda,2))-1)+(\partial f/\partial\lambda)(\lambda,f_n(\lambda,2))\neq 0$.

Our aim is to prove:

THEOREM 2. *$\bar{A}$ is perfect. There exists A_0 residual in $\bar{A}$ such that for $\lambda\in A_0, f_\lambda$ is topologically transitive, ergodic, and the forward orbit of the critical points is dense.*

The first step is:

LEMMA 5. $A\neq\varnothing$.

Proof. $\lambda=1$ (for which $f_4(\lambda,2)=f_3(\lambda,2)=1$) will not do, as $1,\infty$ are the only preimages of $f_3(\lambda,2)$, and P3 does not hold. So we consider those λ for which $f_4(\lambda,2)=f_5(\lambda,2)$, $f_3(\lambda,2)\neq f_4(\lambda,2)$; that is, $\lambda\neq 1$.

Now $f_3(\lambda,2)=\lambda$, so $f_4(\lambda,2)=(\lambda-2)^2/\lambda$. If $f_4(\lambda,2)=f_5(\lambda,2)$ then

$$\lambda\left(1-\frac{2}{f_4(\lambda,2)}\right)^2=\lambda\left(1-\frac{2}{f_3(\lambda\ 2)}\right)^2.$$

So

$$f_4(\lambda,2)f_3(\lambda,2)=f_3(\lambda,2)+f_4(\lambda,2)$$
$$(\lambda-2)^2(\lambda-1)-\lambda^2=0$$
$$\lambda^3-6\lambda^2+8\lambda-4=0$$

This is irreducible with roots $\alpha,\bar{\alpha},\beta$ with β real. P3 is satisfied by $\alpha,\bar{\alpha},\beta$, since λ is a preimage of $f_4(\lambda,2)$ for all λ and $f_4(\lambda,2)\neq\lambda$ for $\lambda\neq 1$.

https://doi.org/10.1017/S0143385700002467 Published online by Cambridge University Press

P2 is satisfied by β. For if not, then

$$\frac{\partial f}{\partial z}(\beta, f_4(\beta, 2)) = \pm 1.$$

But since $2, \infty$ are eventually periodic, f_β is topologically transitive, and has no attractive or parabolic fixed points. This rules out the possibilities that $|(\partial f/\partial z)(\beta, f_4(\beta, 2))| < 1$ or $(\partial f/\partial z)(\beta, f_4(\beta, 2)) = \pm 1$.

To verify that P4 holds for $\alpha, \bar{\alpha}, \beta$, we start by computing $(\partial f/\partial z)(\lambda, f_4(\lambda, 2))$, $(\partial f/\partial \lambda)(\lambda, f_4(\lambda, 2))$, and $(\partial f_4/\partial z)(\lambda, 2)$ under the assumption

$$f_4(\lambda, 2) f_3(\lambda, 2) = f_4(\lambda, 2) + f_3(\lambda, 2),$$

which implies $f_4(\lambda, 2) = f_5(\lambda, 2)$. Thus $f_4 = f_3/(f_3 - 1)$, so $(\lambda - 2)^2/\lambda = \lambda/(\lambda - 1)$.

$$\frac{\partial f}{\partial z}(\lambda, f_4(\lambda, 2)) = \frac{4\lambda}{(f_4(\lambda, 2))^2}\left(1 - \frac{2}{f_4(\lambda, 2)}\right) = \frac{-2f_5(\lambda, 2)}{f_4(\lambda, 2)} + \frac{2\lambda}{f_4(\lambda, 2)}$$

$$= \frac{2\lambda}{f_4(\lambda, 2)} - 2 = 2(\lambda - 1) - 2 = 2\lambda - 4.$$

$$\frac{\partial f}{\partial \lambda}(\lambda, f_4(\lambda, 2)) = \frac{1}{\lambda} f_5(\lambda, 2) = \frac{1}{\lambda} f_4(\lambda, 2) = \frac{1}{\lambda - 1}$$

$$\frac{\partial f_4}{\partial \lambda} = \frac{(\lambda - 2)(\lambda + 2)}{\lambda^2} = \frac{\lambda + 2}{(\lambda - 2)(\lambda - 1)}.$$

So if the expression in P4 is equal to zero, for $\lambda = \alpha, \bar{\alpha}$ or β,

$$\frac{(\lambda + 2)(2\lambda - 5)}{(\lambda - 2)(\lambda - 1)} + \frac{1}{\lambda - 1} = 0, \quad \text{i.e.} \quad (\lambda + 2)(2\lambda - 5) + \lambda - 2 = 0.$$

This cannot happen since $\alpha, \bar{\alpha}, \beta$ are the roots of an irreducible polynomial. So $\beta \in A$. □

We claim that ergodicity is a residual property for $\lambda \in \bar{A}$.

Fix a conformal metric on S^2 and let m denote the corresponding finite measure on S^2. For f analytic, $|f'|$ denotes the conformal derivative with respect to this metric. Let g be a continuous strictly positive function on S^2. To prove residuality it suffices to prove:

$$\frac{S_n^\lambda(h)}{S_n^\lambda(g)}(x) \to \frac{\int h\, dm}{\int g\, dm} \quad \text{uniformly in } x \text{ as } n \to \infty, \text{ for } \lambda \in A, \tag{I}$$

for any continuous function h on S^2, where

$$S_n^\lambda(h)(x) = \sum_{i=0}^{n-1} \sum_{\substack{S \text{ inverse} \\ \text{of } f_\lambda^i}} |S'(x)|^2 h(Sx),$$

($S_n^\lambda(h)$ is not defined on the critical point forward orbit), and

$$\min S_n^\lambda(g) \geq Cn \int g\, dm \tag{I'}$$

https://doi.org/10.1017/S0143385700002467 Published online by Cambridge University Press

For if (I) and (I′) are true then for a residual set of λ there exists a sequence k_n (depending on λ) such that for a dense set h_n of continuous functions,

$$\frac{S^\lambda_{k_n}(h_p)}{S^\lambda_{k_n}(g)} \to \frac{\int h_p\,dm}{\int g\,dm} \quad \text{uniformly in } x \text{ as } n\to\infty, \quad \min S^\lambda_{k_n}(g) \geq Ck_n \int g\,dm,$$

and for any bounded measurable h,

$$\int \left| \frac{S_{k_n}(h)}{S_{k_n}(g)} - \frac{\int h\,dm}{\int g\,dm} \right| dm \to 0 \qquad \text{as } n\to\infty.$$

Then f_λ must be ergodic, for if φ is measurable, f_λ-invariant and bounded, $S_n(\varphi g)/S_n(g) = \varphi$ for all n. Since the integral of $S^\lambda_n(h)/S^\lambda_n(g)$ against the measure $S^\lambda_n(g)\,dm$ – which has mass $n\int g\,dm$ – is $n\int h\,dm$, the function takes the value $\int h\,dm/\int g\,dm$ at some point (for continuous h). So to prove (I) uniformly for $x \in S^2$ it suffices to prove, for fixed h, $\lambda \in A$, $x, y \in S^2$:

$$(1-\varepsilon_n)\frac{S^\lambda_n(h)}{S^\lambda_n(g)}(x) \leq \frac{S^\lambda_n(h)}{S^\lambda_n(g)}(y) \leq (1+\varepsilon_n)\frac{S^\lambda_n(h)}{S^\lambda_n(g)}(x), \tag{II}$$

where $\varepsilon_n \to 0$ as $n\to\infty$. Let B be a neighbourhood of $\{f^n_\lambda(0)\colon n \geq 0\}$, (recall $0 = f_\lambda(2)$ and 2 are the critical points).

$$S^\lambda_n(h) = S^\lambda_m(h) + \sum_{\substack{S \text{ inverse}\\ \text{of } f^m_\lambda}} |S'|^2 S^\lambda_{n-m}(h)\circ S \qquad \text{for any } m \leq n. \tag{III}$$

Now on $S^2\setminus B$, by lemma 2, $|S'| \to 0$ uniformly as $m\to\infty$. So $d(Sx, Sy) \to 0$ as $m\to\infty$ for x, y in the domain of definition of S. So if it can be shown that:

$$S^\lambda_n(g) \text{ varies by a bounded proportion on } S^2\setminus B, \tag{IVa}$$

$$\{S^\lambda_n(h)/S^\lambda_n(g)\colon n \geq 0\} \text{ is equicontinuous on } S^2\setminus B, \tag{IVb}$$

then on $S^2\setminus B$, (II) will be proved: for (IVa) implies $S^\lambda_n(g)\to\infty$ uniformly on $S^2\setminus B$ (in fact on S^2 since $S^\lambda_n(g)$ is even larger there) since, by theorem 1, f is ergodic and

$$\int_{S^2\setminus B} S_n(g) = \sum_i \int_{f^{-i}_\lambda(S^2\setminus B)} g\,dm,$$

and in the expression (III), the terms $S^\lambda_{n-m}(h)\circ S$ will be nearly constant if m is fairly large, and $S^\lambda_m(h)$ will be negligible compared with $S^\lambda_n(g)$ for $n \gg m$. The $|S'|^2$ terms will simply cancel. But (IVa), (IVb) follow from equicontinuity of $\{\log|S'|\colon S$ is an inverse of f^m_λ, $m \geq 0\}$ (these inverses only really make sense on balls of radius α, but this does not really matter), and that diameter $(SB') \to 0$ as $m\to\infty$, if B' is a ball on which S is defined. For then

$$|g(Sx)|S'(x)|^2 - g(Sy)|S'(y)|^2| \leq |S'(x)|^2 g(Sx)\varepsilon$$

if $d(x,y) < \delta$, given ε, similarly for h (but with g still on the righthand side).

https://doi.org/10.1017/S0143385700002467 Published online by Cambridge University Press

Equicontinuity is proved in the same way as lemma 3: the constant K_3 in the statement of lemma 3 depends on the radius of the domain of inverses, and clearly decreases with this.

Now consider $x \in B$, $x \neq f_\lambda^n(0)$, $n \geqslant 0$. Let R_i: $i \geq 1$ be the inverses of maps $f_\lambda^{p_i}$ composed of inverses of f_λ mapping the image of x always into B, except that $R_i x \in S^2 \setminus B$. Let $R_i = T_i S_i$ for T_i an inverse of f_λ. Possibly more than one R_i (but only finitely many) will have the same S_i. Then

$$S_n^\lambda(h)(x) = \sum |R_i'(x)|^2 S_{n-p_i}^\lambda(h) \circ R_i(x) + \sum |S_i'(x)|^2 h \circ S_i(x),$$

where the summation is over R_i, S_i of length $\leq n$ and each S_i is included only once. $S_{n-p_i}^\lambda(h) \circ R_i / S_{n-p_i}^\lambda(g) \circ R_i$ is almost constant (uniformly in x) except for finitely many i. We claim the remaining terms are negligible, even though those R_i', S_i' with $R_i x$, $S_i x$ near 2, 0 are arbitrarily large. If $R_i x$ is near 2 then R_i is an inverse of $z \mapsto C_i \mu^{p_i} z^4$ with $C^{-1} \leq C_i \leq C$, $\mu > 1$, so that $|R_i'(w)| \sim |\mu|^{-p_i/4} w^{-3/4}$. There is exactly one such inverse for each positive integral value of p_i, so that for $p_i \geq n - N$ (and $n - p_i \leq N$) the terms are dominated by those for smaller values of p_i. Similarly those terms involving S_i for $S_i x$ near 0 are dominated by terms involving R_i (using also that $S_{n-p_i}^\lambda(g) R_i$ is large for $n - p_i$ large), and the remaining terms involving S_i are summable. So the remaining terms are negligible as claimed, and (I) holds. To prove (I′), note that from integrating the expression given for $S_n^\lambda(h)$ on B, with $h = g$,

$$\int_B S_n^\lambda(g) \geq C_2 + C_1 \max_{x \in S^2 \setminus B} S_n^\lambda(g)(x).$$

Therefore, to prove theorem 2, it suffices to prove:

PROPOSITION. *Given $\lambda_0 \in A$, U open in S^2 and $\varepsilon > 0$, if N is the least integer with $f_{N+1}(\lambda_0, 2) = f_N(\lambda_0, 2)$, there exists $\lambda_1 \in A$ with $|\lambda_1 - \lambda_0| < \varepsilon$ and $N < R < T$ with $f_R(\lambda_1, 2) \in U$ and $f_{T+1}(\lambda_1, 2) = f_T(\lambda_1, 2)$.*

Note. This is sufficient to show that $\bar{A}$ is perfect and that having dense critical point forward orbit is a residual property.

LEMMA 6. *Let $x_0 = f_{\lambda_0}^N(2)$ be the expanding fixed point of λ_0 whose existence is given by* P1, P2. *Then given $\eta > 0$ there exist x_1, r, t, with $|x_0 - x_1| < \eta$, $f_{\lambda_0}^r(x_1) \in U$, $f_{\lambda_0}^t(x_1) = x_0$, $0 < r < t$, with $f_{\lambda_0}^s(x_1) \neq 2$ or 0 for any $s \geq 0$, $f_{\lambda_0}^s(x_1) \neq x_0$ for $s < t$.*

Proof. By P3, x_0 has preimages $\neq x_0$ which do not have 2, 0 in the forward orbit. This is an easy consequence of the fact that preimages of x_0 under f_{λ_0} are dense in S^2 (because the Julia set of f_{λ_0} is S^2) [1]. □

LEMMA 7. *Given $\delta > 0$ sufficiently small, there exists $K = K(\delta)$ (with $K(\delta) \to 0$ as $\delta \to 0$) such that for each $n \geq 3$,*

$$e^{-K} \left| \frac{\partial f_n}{\partial \lambda}(\lambda_0, 2) \right| \leq \left| \frac{\partial f_n}{\partial \lambda}(\lambda, 2) \right| \leq e^K \left| \frac{\partial f_n}{\partial \lambda}(\lambda_0, 2) \right|,$$

for all λ with $|f_i(\lambda, 2) - f_i(\lambda_0, 2)| \leq \delta$, $3 \leq i \leq n$, $|\lambda - \lambda_0| \leq \delta$.
(Recall $f_2(\lambda, 2) = \infty$.)

https://doi.org/10.1017/S0143385700002467 Published online by Cambridge University Press

Proof. The proof will be by induction. First, let N be the least integer with $f_s(\lambda_0, 2) = f_N(\lambda_0, 2)$, $s \geq N$. Write

$$C_i(\lambda) = \prod_{j=N}^{i} \frac{\partial f}{\partial z}(\lambda, f_j(\lambda, 2)), \quad c = \frac{\partial f}{\partial z}(\lambda_0, f_N(\lambda_0, 2)) \quad \text{and} \quad d = \frac{1}{|c|}.$$

Then $d < 1$, $C_i(\lambda_0) = c^{i+1-N}$, and

$$\frac{\partial f_m}{\partial \lambda}(\lambda, 2) = C_{m-1}(\lambda)\left(\frac{\partial f_N}{\partial \lambda}(\lambda, 2) + \sum_{i=N}^{m-1} \frac{1}{C_i(\lambda)} \frac{\partial f}{\partial \lambda}(\lambda, f_i(\lambda, 2))\right) \tag{1}$$

$$\frac{\partial f_m}{\partial \lambda}(\lambda_0, 2) = \frac{c^{m-N}}{c-1}\left(\frac{\partial f_N}{\partial \lambda}(\lambda_0, 2)(c-1) + \frac{\partial f}{\partial \lambda}(\lambda_0, f_N(\lambda_0, 2))\left(1 - \left(\frac{1}{c}\right)^{m-N}\right)\right). \tag{2}$$

By the inductive hypothesis,

$$e^{-K}\left|\frac{\partial f_m}{\partial \lambda}(\lambda_0, 2)\right| \leq \left|\frac{\partial f_m}{\partial \lambda}(\lambda, 2)\right| \leq e^{K}\left|\frac{\partial f_m}{\partial \lambda}(\lambda_0, 2)\right|, \qquad m \leq n-1 \tag{3}$$

for all λ with $|f_i(\lambda, 2) - f_i(\lambda_0, 2)| \leq \delta$, $3 \leq i \leq n-1$, $|\lambda - \lambda_0| \leq \delta$. Write A_n for this set of λ. Because P4 holds for λ_0, formula (2) for $(\partial f_m/\partial\lambda)(\lambda_0, 2)$ implies $(\partial f_m/\partial\lambda)(\lambda_0, 2)$ is $O(d^{N-m})$. So for suitable C_1, C_2, C_3, C_4 (depending on N, K but not on m, n) and $\lambda \in A_n$, by (3),

$$C_1 d^{-m} \leq \left|\frac{\partial f_m}{\partial \lambda}(\lambda, 2)\right| \leq C_2 d^{-m} \qquad (m \leq n-1) \tag{4}$$

So

$$|\lambda - \lambda_0| \leq C_3 d^{n-1}\delta, \tag{5}$$

$$|f_m(\lambda, 2) - f_m(\lambda_0, 2)| \leq C_3 d^{n-1-m}\delta \tag{6}$$

$$\left|\frac{\partial f}{\partial \lambda}(\lambda, f_m(\lambda, 2)) - \frac{\partial f}{\partial \lambda}(\lambda_0, f_m(\lambda_0, 2))\right| \leq C_4 d^{n-1-m}\delta. \tag{7}$$

We conclude from (6), if $m \leq n-1$ and $\lambda \in A_n$,

$$\left|\log \prod_{j=N}^{m}\left|\frac{\partial f}{\partial z}(\lambda, f_j(\lambda, 2))\right| - \log \prod_{j=N}^{m}\left|\frac{\partial f}{\partial z}(\lambda, f_j(\lambda_0, z))\right|\right|$$
$$\leq C_5\delta \sum_{j=N}^{m} d^{n-1-j} = C_6\delta d^{n-m}. \tag{8}$$

We conclude from (5)

$$\left|\log \prod_{j=N}^{m}\left|\frac{\partial f}{\partial z}(\lambda, f_j(\lambda_0, 2))\right| - \log \prod_{j=N}^{m}\left|\frac{\partial f}{\partial z}(\lambda_0, f_j(\lambda_0, 2))\right|\right|$$
$$\leq C_7\delta d^n(m - N + 1). \tag{9}$$

From (8), (9) we conclude

$$|\log C_m(\lambda) - \log C_m(\lambda_0)| \leq C_8\delta. \tag{10}$$

Write

$$\frac{\partial f_n}{\partial \lambda}(\lambda, 2) = C_{n-1}(\lambda)(S_i(\lambda) + R_{i,n}(\lambda)), \tag{11}$$

https://doi.org/10.1017/S0143385700002467 Published online by Cambridge University Press

where $R_{i,n}$ denotes the last $n-1-i$ terms in the series. $S_i(\lambda_0)$ converges to a non-zero number as $i\to\infty$ and $R_{i,n}(\lambda_0)\to 0$ independently of n. So for $\eta_1>0$ we can find s independently of n so that:

$$|R_{s,n}(\lambda_0)|\le \sum_{i=s+1}^{n-1} d^i\left|\frac{\partial f}{\partial\lambda}(\lambda_0, f_i(\lambda_0,2)\right|\le \eta_1|S_s(\lambda_0)|. \tag{12}$$

From (7), (10) we see that given $\eta_2>1$, if δ is sufficiently small:

$$|R_{s,n}(\lambda)|<\eta_2\sum_{i=s+1}^{n} d^i\left|\frac{\partial f}{\partial\lambda}(\lambda_0, f_i(\lambda_0,2))\right|\le \eta_2\eta_1|S_s(\lambda_0)|. \tag{13}$$

Also, if δ is sufficiently small,

$$\frac{1}{\eta_2}|S_s(\lambda_0)|\le|S_s(\lambda)|\le\eta_2|S_s(\lambda_0)|. \tag{14}$$

So

$$\frac{\dfrac{1}{\eta_2}-\eta_2\eta_1}{1+\eta_1}|S_s(\lambda_0)+R_{s,n}(\lambda_0)|\le|S_s(\lambda)+R_{s,n}(\lambda)|$$

$$\le\frac{\eta_2+\eta_2\eta_1}{1+\eta_1}|S_s(\lambda_0)+R_{s,n}(\lambda_0)|. \tag{15}$$

From (10), if δ is sufficiently small,

$$\frac{1}{\eta_2}|C_{n-1}(\lambda_0)|\le|C_{n-1}(\lambda)|\le\eta_2|C_{n-1}(\lambda_0)|. \tag{16}$$

From (11), (15), (16) with $s=i$, we obtain the required bound on $(\partial f_n/\partial\lambda)(\lambda,2)$, for suitable η_1, η_2, that is, for suitable δ. □

LEMMA 8. *If d, K are as in the previous lemma and $|\lambda-\lambda_0|\le d^n e^{-K}\delta/2$, then*

$$a_1 d^{-m}\le\left|\frac{\partial f_m}{\partial\lambda}(\lambda,2)\right|\le a_2 d^{-m},$$

$$\left|\frac{\partial^2 f_m}{\partial\lambda^2}(\lambda,2)\right|\le a_3 d^{-2m},\qquad m\le n$$

for constants a_1, a_2, a_3.

Proof. The first statement is a restatement of the previous lemma, since $|(\partial f_m/\partial\lambda)(\lambda_0,2)|=O(d^{-2})$. The second statement follows from Cauchy's integral formula. □

LEMMA 9. *If η in lemma 6 is $\le e^{-K}a_1\delta/4$ then there exists $\lambda_2=\lambda_2(n)$ with $|\lambda_2-\lambda_0|<d^n e^{-K}\delta/4$, with $f_n(\lambda_2,2)=x_1$, using the notation of previous lemmas.*

Proof. Immediate from the previous lemmas. □

Proof of proposition. We take $\lambda_2=\lambda_2(n)$ as in lemma 9 and take λ_2 as the first approximation to a solution λ_1 of

$$f_{n+t+1}(\lambda_1,2)=f_{n+t}(\lambda_1,2).$$

https://doi.org/10.1017/S0143385700002467 Published online by Cambridge University Press

This works for n sufficiently large. The proposition is then proved for $R=n+r$, $T=n+t$, (r, t as in lemma 6). Define

$$F(\lambda)=f_{n+t}(\lambda,2)-f_{n+t+1}(\lambda,2).$$

Then

$$\begin{aligned}F(\lambda_2)&=f_{t+1}(\lambda_2,f_n(\lambda_2,2))-f_t(\lambda_2 f_n(\lambda_2,2))\\&=f_{t+1}(\lambda_2,x_1)-f_t(\lambda_2,x_1).\end{aligned}$$

Since $|\lambda_2-\lambda_0|\le d^n e^{-K}\delta/4$,

$$|F(\lambda_2)|\le D_1 d^n, \tag{1'}$$

where D_1 depends on t on but not on n. Inductively define $\lambda_{i+1}=\lambda_i-(F'(\lambda_i))^{-1}F(\lambda_i)\ge 2$. Then inductively it can be shown that

$$|\lambda_i-\lambda_0|\le d^n e^{-K}\delta/2 \tag{2'}$$

so that, since

$$F'(\lambda)=\frac{\partial f_{n+t}}{\partial\lambda}(\lambda,2)\left(\frac{\partial f}{\partial z}(\lambda,f_{n+t}(\lambda,2))-1\right)+\frac{\partial f}{\partial\lambda}(\lambda,f_{n+t}(\lambda,2)),$$

$$|F'(\lambda_i)|^{-1}\le D_2 d^n \tag{3'}$$

for D_2 depending on t but not on n, provided only that n is sufficiently large,

$$|\lambda_{i+1}-\lambda_i|\le D_2 d^n|F(\lambda_i)|, \tag{4'}$$

$$|F(\lambda_{i+1})|\le D_4|F(\lambda_i)|^2, \tag{5'}$$

where D_4 depends on t but not on n. (5′) can be proved because $\sup_{|\lambda-\lambda_0|\le d^n e^{-K}\delta/2}|F''(\lambda)|\le D_3 d^{-2n}$, where D_3 depends on t but not on n, (this follows from the bound on $(\partial^2 f_n/\partial\lambda^2)(\lambda,2)$ in lemma 8), and

$$|F(\lambda_{i+1})|\le\sup_{\lambda}\tfrac{1}{2}|F''(\lambda)|\cdot|F'(\lambda_i)|^{-2}|F(\lambda_i)|^2.$$

Take $\lambda_1=\lim_{i\to\infty}\lambda_i$. Then

$$|\lambda_1-\lambda_0|\le d^n e^{-K}\delta/2,\qquad |\lambda_1-\lambda_2|\le D_5 d^{2n}. \tag{6'}$$

$$f_{n+t+1}(\lambda_1,2)=f_{n+t}(\lambda_1,2),\qquad \text{(P1)}. \tag{7'}$$

$$|f_n(\lambda_1,2)-x_1|=|f_n(\lambda_1,2)-f_n(\lambda_2,2)|\le D_6 d^n. \tag{8'}$$

So

$$|f_{n+t}(\lambda_1,2)-x_0|=|f_t(\lambda_1,f_n(\lambda_1,2))-f_t(\lambda_0,f_n(\lambda_2,2)|\le D_7 d^n. \tag{9'}$$

$T=n+t$, so

$$\left|\frac{\partial f}{\partial z}(\lambda_1,f_T(\lambda_1,2)-\frac{\partial f}{\partial z}(\lambda_0,x_0)\right|<D_8 d^n,$$

and

$$\left|\frac{\partial f}{\partial z}(\lambda_1,f_T(\lambda_1,2))\right|>1,\qquad \text{(P2)}. \tag{10'}$$

Clearly P3 holds for λ_1.

$$\frac{\partial f_T}{\partial\lambda}(\lambda,2)=\frac{\partial f_t}{\partial z}(\lambda,f_n(\lambda,2))\frac{\partial f_n}{\partial\lambda}(\lambda,2)+\frac{\partial f_t}{\partial\lambda}(\lambda,f_n(\lambda,2)).$$

https://doi.org/10.1017/S0143385700002467 Published online by Cambridge University Press

$(\partial f_t/\partial z)(\lambda_1, f_n(\lambda_1, 2))$ and $(\partial f_t/\partial \lambda)(\lambda_1, f_n(\lambda_1, 2))$ are bounded independently of n, so

$$\left|\frac{\partial f_T}{\partial \lambda}(\lambda, 2)\right| \geq D_9 d^n. \tag{11'}$$

From (10), (11), since $(\partial f/\partial \lambda)(\lambda_1, f_T(\lambda_1, 2))$ is bounded independently of n,

$$\frac{\partial f_T}{\partial \lambda}(\lambda_1, 2)\left(\frac{\partial f}{\partial z}(\lambda_1, f_T(\lambda_1, 2)) - 1\right) + \frac{\partial f}{\partial \lambda}(\lambda_1, f_T(\lambda_1, 2)) \neq 0, \tag{12'}$$

that is, P4 holds for λ_1. Since, by taking n arbitrarily large, λ_1 can be taken arbitrarily close to λ_0, the proposition is proved. □

REFERENCES

[1] H. Brolin. Invariant sets under iteration of rational functions. *Arkiv För Matematik*, **6** (1965), 103-144.
[2] M. R. Herman. Exemples de fractions rationelles ayant une orbit dense. To appear.
[3] R. Mane, P. Sad & D. Sullivan. On the dynamics of rational maps. To appear.
[4] D. Sullivan. Conformal dynamical systems. To appear.

https://doi.org/10.1017/S0143385700002467 Published online by Cambridge University Press