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GENERALIZED SPECTRAL THEORY IN COMPLEX BANACH ALGEBRAS

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Introduction. Let A be an element of a complex Banach algebra $\mathscr{B}$ with identity I. The ordinary spectrum of A, $\mathrm{sp}(A)$, consists of those points z in the complex plane such that $A - zI$ has no inverse in $\mathscr{B}$. If Q is any other element of $\mathscr{B}$, we define $\mathrm{sp}_Q(A)$, the *spectrum of* A *relative to* Q, or Q-*spectrum of* A, as those points z such that $A - zI - \bar{z}Q$ has no inverse in $\mathscr{B}$. Thus if $Q = 0$ the Q-spectrum of A is the same as the ordinary spectrum of A.

The generalized notion of spectrum, $\mathrm{sp}_Q(A)$, retains many of the properties of the ordinary spectrum, particularly when A and Q commute and the ordinary spectrum of Q does not meet the unit circle. Under these conditions the Q-spectrum of A is a nonempty compact subset of the plane, and if both $\mathrm{sp}(A)$ and $\mathrm{sp}(Q)$ are finite (or countable), so is $\mathrm{sp}_Q(A)$. The ordinary functional calculus has an analogue in the case of the Q-spectrum, and one may integrate the "Q-resolvent", $(A - zI - \bar{z}Q)^{-1}$, around suitable contours and thereby generate idempotents which commute with both A and Q. One obtains in the usual way a Laurent series expansion of the Q-resolvent around an isolated singularity.

We also obtain an analogue of the "Riesz Decomposition Theorem" (see [**9**], Chapter 2) concerning the generation, by contour integration, of nontrivial idempotents whenever $\mathrm{sp}_Q(A)$ is not connected. In the case that A and Q are bounded operators on a Hilbert space we may conclude that both A and Q have nontrivial invariant subspaces whenever $\mathrm{sp}_Q(A)$ is not connected. In the last section we give an example of commuting operators A and Q on a Hilbert space such that $\mathrm{sp}(A)$ and $\mathrm{sp}(Q)$ are connected, but $\mathrm{sp}_Q(A)$ is not connected. Therefore it might appear that these results could have applications to the problem of determining invariant subspaces of operators.

This paper is somewhat analogous to an earlier paper of the authors [**3**] regarding a generalized notion of the spectrum in real Banach algebras. There an analogue of ordinary spectral theory was obtained for real Banach algebras without resorting to the usual procedure of "complexifying" the algebra. This is possible provided there is at least one element in

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the algebra with empty real spectrum. A generalized functional calculus was also developed in that paper, without the assumption of complex elements in the algebra, and several of the techniques used there have been adapted in this paper for the complex algebra.

Some of our techniques involving the functional calculus are adaptations of those of the first author in [**4**, **5**]. In those papers a function theory is presented for first order elliptic systems of partial differential equations with constant coefficients, and this theory parallels to some extent the development in Sections 2 and 3.

1. Definition and basic notions. $\mathscr{B}$ will denote a complex Banach algebra with identity element I. $\mathbf{C}$ denotes the complex plane, and T the unit circle in the complex plane. The letter z denotes a point in $\mathbf{C}$, and $\bar{z}$ is the complex conjugate of z.

Definition 1. Let $A \in \mathscr{B}$ and $Q \in \mathscr{B}$. The spectrum of A relative to Q, called the *Q-spectrum of A*, is the subset of $\mathbf{C}$ defined by

$$\mathrm{sp}_Q(A) = \{z \in \mathbf{C}\colon A - zI - \bar{z}Q \text{ is not invertible in } \mathscr{B}\}.$$

The *Q-resolvent of A* is defined as the complement in $\mathbf{C}$ of $\mathrm{sp}_Q(A)$, that is

$$\mathrm{res}_Q(A) = \{z \in \mathbf{C}\colon [A - zI - \bar{z}Q]^{-1} \text{ exists in } \mathscr{B}\}.$$

(We denote by $\mathrm{sp}\, A$, $\mathrm{res}\, A$, the ordinary spectrum and ordinary resolvent of A, respectively).

We consider the question of when $\mathrm{sp}_Q(A)$ is nonempty. Some restrictions on Q are necessary as the following examples show.

Example 1. Let $\mathscr{B}$ be $\mathbf{C}^{2\times 2}$, the algebra of all 2×2 complex matrices. For

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

we have $\mathrm{sp}_Q(A) = \emptyset$. For later reference we note that

$$AQ \neq QA \quad \text{and} \quad \mathrm{sp}\, Q \cap T = \emptyset.$$

Example 2.

$$\mathscr{B} = \mathbf{C}^{2\times 2}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $\mathrm{sp}_Q(A) = \emptyset$ and in this case $AQ = QA$ and $\mathrm{sp}\, Q \subset T$.

THEOREM 1. *Let A and Q be elements of $\mathscr{B}$. If $AQ = QA$ and at least one point of* $\mathrm{sp}\, Q$ *lies off of the unit circle, then* $\mathrm{sp}_Q(A) \neq \emptyset$.

Proof. Let $z \in \operatorname{sp} Q$ with $z \notin T$. Let $\mathscr{B}^*$ be a maximal commutative subalgebra of $\mathscr{B}$ with A and Q in $\mathscr{B}^*$. Then the spectrum of each element of $\mathscr{B}^*$ is the same as in $\mathscr{B}$ [**10**]. By the Gelfand theory [**10**] there exists a nonzero homomorphism σ on $\mathscr{B}^*$ into $\mathbf{C}$ such that $\sigma(Q) = z$.

Then we would like to find $w \in \mathbf{C}$ such that

$$\sigma(A - wI - \bar{w}Q) = \sigma(A) - w - \bar{w}z = 0,$$

for this would mean $A - wI - \bar{w}Q$ was singular and therefore $w \in \operatorname{sp}_Q(A)$. Taking the complex conjugate of this equation and solving the two equations for w yields the solution

$$w = \frac{\sigma(A) - \overline{\sigma(A)}z}{1 - |z|^2}.$$

A few more examples clarify matters further concerning the boundedness of $\operatorname{sp}_Q(A)$.

Example 3.

$$\mathscr{B} = \mathbf{C}^{2\times 2}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\operatorname{sp}_Q(A) = \{z \in \mathbf{C}: z = 1 \text{ or } z = \frac{1}{2} + iy \text{ for } y \text{ real}\},$$

and in this case

$$AQ = QA \quad \text{and} \quad \operatorname{sp} Q = \{0, 1\}.$$

Example 4.

$$\mathscr{B} = \mathbf{C}^{2\times 2}, \quad A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then

$$\operatorname{sp}_Q(A) = \{z \in \mathbf{C}: z = x + iy \text{ with } 5x^2 + y^2 = 1\}$$

and in this case

$$AQ \neq QA \quad \text{and} \quad \operatorname{sp} Q = \left\{\frac{3 + \sqrt{5}}{2}, \frac{3 - \sqrt{5}}{2}\right\}.$$

Since we would like $\operatorname{sp}_Q(A)$ to be compact, Example 3 leads us to investigate Q where $\operatorname{sp} Q \cap T = \emptyset$. For such elements of $\mathscr{B}$ the following machinery is useful.

We set up a mapping between elements of $\mathbf{C}$ and certain elements in $\mathscr{B}$, namely

(1.1) $\phi(z) = zI + \bar{z}Q.$

For fixed $Q \in \mathscr{B}$ with sp $Q \cap T = \emptyset$ it is easily seen that $z \neq 0$ if and only if $\phi(z)$ is invertible in $\mathscr{B}$ and we shall use this fact throughout this paper.

We define the positive constants

$$(1.2)\quad \begin{aligned} \gamma_1 &= \sup_{|z|=1} \|\phi(z)\|, \quad \gamma_2 = \sup_{|z|=1} \|[\phi(z)]^{-1}\| \\ \gamma_3 &= \inf_{|z|=1} \|\phi(z)\|, \quad \gamma_4 = \inf_{|z|=1} \|[\phi(z)]^{-1}\| \end{aligned}$$

Here $|z|$ is the usual absolute value in $\mathbf{C}$ and $\| \ \|$ is the norm in $\mathscr{B}$.

Then for any $z \in \mathbf{C}$, $z \neq 0$ we have

$$\|\phi(z)\| = \|zI + \bar{z}Q\| = |z| \left\| \phi\left(\frac{z}{|z|}\right) \right\|,$$

so we have the inequality

$$(1.3)\quad \gamma_3|z| \leqq \|\phi(z)\| \leqq \gamma_1|z|.$$

In a similar manner we obtain

$$(1.4)\quad \gamma_4|z|^{-1} \leqq \|[\phi(z)]^{-1}\| \leqq \gamma_2|z|^{-1}, \quad z \neq 0.$$

THEOREM 2. *Let A and Q be elements of $\mathscr{B}$ with* sp $Q \cap T = \emptyset$. *Then*

$$|z| \leqq \gamma_2\|A\| \quad \textit{for each } z \in \mathrm{sp}_Q(A).$$

Proof. Suppose $z \in \mathbf{C}$. If $|z| > \gamma_2\|A\|$, then $[\phi(z)]^{-1}$ exists in $\mathscr{B}$ and

$$\|A[\phi(z)]^{-1}\| \leqq \|A\| \, \|[\phi(z)]^{-1}\| < \frac{|z|}{\gamma_2}\gamma_2|z|^{-1} = 1$$

(using (1.4)).

Therefore $A[\phi(z)]^{-1} - I$ is invertible in $\mathscr{B}$, which implies that

$$(A[\phi(z)]^{-1} - I)\phi(z) = A - \phi(z)$$

is invertible in $\mathscr{B}$. Therefore $z \notin \mathrm{sp}_Q(A)$. The above implies that if $z \in \mathrm{sp}_Q(A)$, then $|z| \leqq \gamma_2\|A\|$.

Definition 2. For A and Q in $\mathscr{B}$ we define the *Q-spectral radius of* A, $\rho_Q(A)$ by

$$\rho_Q(A) = \sup\{ |z| : z \in \mathrm{sp}_Q(A) \}.$$

Under the hypothesis of Theorem 2 we have the inequality

$$(1.5)\quad \rho_Q(A) \leqq \gamma_2\|A\|.$$

By analogy with the ordinary spectral radius, one might hope that we would have $\rho_Q(A) \leqq \|A\|$, but the following example shows that as a general upper bound for $\rho_Q(A)$ one cannot hope to do better than (1.5).

Example 5. Let $\mathscr{B} = \mathbf{C}$, $Q = \epsilon$ for some real ϵ, with $0 < \epsilon < 1$. In this case we calculate that

$$\gamma_2 = \frac{1}{1 - \epsilon}.$$

If we then set $A = i$ we get

$$\mathrm{sp}_Q(A) = \left\{\frac{i}{1 - \epsilon}\right\}$$

and

$$\rho_Q(A) = \frac{1}{1 - \epsilon} = \gamma_2 \|A\|.$$

THEOREM 3. *If A and Q are elements of $\mathscr{B}$, then* $\mathrm{sp}_Q(A)$ *is closed in* $\mathbf{C}$.

Proof. We show that $\mathrm{res}_Q(A)$ is open in $\mathbf{C}$. Let $z_0 \in \mathrm{res}_Q(A)$, then

$$A - \phi(z_0) = A - z_0 I - \bar{z}_0 Q$$

is invertible. Since the set of invertible elements in $\mathscr{B}$ is open, if z is sufficiently close to z_0 then $A - \phi(z)$ is close enough to $A - \phi(z_0)$ so that $A - \phi(z)$ is invertible. Hence $z \in \mathrm{res}_Q(A)$ and we have $\mathrm{res}_Q(A)$ is open in $\mathbf{C}$.

COROLLARY 1. *If A and Q are elements of $\mathscr{B}$ with* $\mathrm{sp}\, Q \cap T = \emptyset$, *then* $\mathrm{sp}_Q(A)$ *is a compact subset of* $\mathbf{C}$. *If moreover $AQ = QA$, then* $\mathrm{sp}_Q(A)$ *is nonempty.*

THEOREM 4. *Let A and Q be elements of $\mathscr{B}$ with*

$$AQ = QA \quad \text{and} \quad \mathrm{sp}\, Q \cap T = \emptyset.$$

Then

a) *if* $\mathrm{sp}\, A$ *and* $\mathrm{sp}\, Q$ *are both finite, so is* $\mathrm{sp}_Q(A)$;

b) *if* $\mathrm{sp}\, A$ *and* $\mathrm{sp}\, Q$ *are both countable, so is* $\mathrm{sp}_Q(A)$.

Proof. Let $\mathscr{B}^*$ be a maximal commutative subalgebra of $\mathscr{B}$ containing A and Q. If $z \in \mathrm{sp}_Q(A)$ then

$$\sigma(A - zI - \bar{z}Q) = \sigma(A) - z - \bar{z}\sigma(Q) = 0,$$

for some homomorphism σ of $\mathscr{B}^*$ onto $\mathbf{C}$. As in the proof of Theorem 1, it follows that

$$z = \frac{\sigma(A) - \overline{\sigma(A)}\sigma(Q)}{1 - |\sigma(Q)|^2}.$$

This implies that if $\mathrm{sp}\, A$ and $\mathrm{sp}\, Q$ are both finite, so is $\mathrm{sp}_Q(A)$ and also if $\mathrm{sp}\, A$ and $\mathrm{sp}\, Q$ are both countable, so must be $\mathrm{sp}_Q(A)$.

Note that Theorem 4 shows that if A and Q are commutative $n \times n$ matrices and sp Q contains no numbers of magnitude 1, then $\mathrm{sp}_Q(A)$ is a finite set in the plane.

COROLLARY 2. *Let A and Q be commutative bounded operators on a Banach space, such that A is compact and* sp Q *is finite with no numbers of magnitude* 1. *Then* $\mathrm{sp}_Q(A)$ *is countable, with the origin being the only possible accumulation point.*

Proof. Since A is compact, sp A is countable with the origin the only possible accumulation point [7]. By Theorem 4, $\mathrm{sp}_Q(A)$ is countable and by the proof of Theorem 4 if $z \in \mathrm{sp}_Q(A)$ then there exists $a \in \mathrm{sp}\, A$ and $q \in \mathrm{sp}\, Q$ such that

$$a = z + \bar{z}q, \quad z = \frac{a - \bar{a}q}{1 - |q|^2}.$$

If z_0 is an accumulation point of $\mathrm{sp}_Q A$, then there exist sequences

$$\{z_n\} \subset \mathrm{sp}_Q A, \quad \{a_n\} \subset \mathrm{sp}\, A, \quad \{q_n\} \subset \mathrm{sp}\, Q$$

such that $z_n \to z_0$, with the z_n's being distinct, and

$$a_n = z_n + \bar{z}_n q_n.$$

Since sp Q is finite, by passing to a subsequence if necessary we may assume $q_n = q$ for all n, where $q \in \mathrm{sp}\, Q$. Hence

$$a_n = z_n + \bar{z}_n q \to z_0 + \bar{z}_0 q.$$

We conclude $z_0 + \bar{z}_0 q = 0$, since $|q| \neq 1$ implies that the a_n's are also distinct, and 0 is the only accumulation point of sp A. Thus $z_0 = 0$.

2. Theory of Q-analytic functions. In order to prove analogous results for the Q-spectrum as for the usual spectrum we first develop a theory which in many ways parallels the theory for analytic functions in a Banach space.

Throughout this section we assume that Q is an element in a complex Banach algebra $\mathscr{B}$ with no points of magnitude 1 in its usual spectrum and we adopt the notation of (1.1) for points in $\mathbf{C}$.

In the following, f will denote a function from a domain Ω in $R^2 \equiv \mathbf{C}$ into our Banach algebra $\mathscr{B}$.

We will say that f is in $C^1(\Omega)$ if each of the strong limits

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

and

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

exist everywhere in Ω and the resulting functions f_x and f_y are continuous from Ω into $\mathscr{B}$.

In this situation we define functions

$$\frac{\partial}{\partial \bar{z}} f = f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$$

and

$$\frac{\partial}{\partial z} f = f_z = \frac{1}{2}(f_x - if_y).$$

Also for $z \in \mathbf{C}$, $z = x + iy$ we will use the common notation

$$\Delta z = \Delta x + i\Delta y$$

and

$$\Delta \bar{z} = \overline{\Delta z} = \Delta x - i\Delta y$$

when considering increments in $\mathbf{C}$.

Definition 3. f is *Q-analytic in* Ω if $f \in C^1(\Omega)$ and

(2.1) $f_{\bar{z}} = Qf_z$ holds in Ω.

Definition 4. f is *Q-differentiable* at a point $z \in \mathbf{C}$ if the limit

$$(2.2) \quad \frac{df(z)}{d\phi} \equiv f'(z) \equiv \lim_{\Delta z \to 0} [(\Delta z)I + (\Delta \bar{z})Q]^{-1}[f(z + \Delta z) - f(z)]$$

exists in $\mathscr{B}$.

THEOREM 5. *Let f be in $C^1(\Omega)$. Then f is Q-differentiable at a point z_0 in Ω if and only if*

$$f_{\bar{z}} = Qf_z \textit{ is satisfied at } z_0.$$

Furthermore $f'(z_0) = f_z(z_0)$ in this case.

Proof. Suppose $f'(z_0)$ exists. Letting $\Delta z = \Delta x$ and $z_0 = x_0 + iy_0$, we have

$$(2.3) \quad f'(z_0) = \lim_{\Delta x \to 0} [\Delta x I + \Delta x Q]^{-1}[f(x_0 + \Delta x, y_0) - f(x_0, y_0)]$$
$$= [I + Q]^{-1} f_x(z_0).$$

Letting $\Delta z = i\Delta y$, we obtain

$$(2.4) \quad f'(z_0) = \lim_{\Delta y \to 0} [i\Delta y I - i\Delta y Q]^{-1}[f(x_0, y_0 + \Delta y) - f(x_0, y_0)]$$
$$= [iI - iQ]^{-1} f_y(z_0) = [I - Q]^{-1}(-if_y(z_0)).$$

Equating (2.3) and (2.4) we have

(2.5) $[I + Q]^{-1}f_x(z_0) = [I - Q]^{-1}(if_y(z_0))$.

Simplifying (2.5) we obtain

$$f_{\bar{z}} = Qf_z \text{ at } z_0.$$

Conversely, suppose

$$f_{\bar{z}} = Qf_z \text{ at } z_0.$$

We first state that

$$[f(z_0 + \Delta z) - f(z_0)] - \Delta x f_x(z_0) - \Delta y f_y(z_0)$$
$$= o(|\Delta z|) \quad \text{as } |\Delta z| \to 0,$$

that is,

$$\text{(2.6)} \quad \frac{[f(z_0 + \Delta z) - f(z_0)] - \Delta x f_x(z_0) - \Delta y f_y(z_0)}{|\Delta z|} \to 0 \quad \text{as } |\Delta z| \to 0.$$

Since every complex Banach algebra is a real Banach algebra, (2.6) was shown in Theorem 5 of [**3**], so we will not repeat the arguments here.

Using (2.6) we have

$$\begin{aligned}
&[(\Delta z)I + (\Delta\bar{z})Q]^{-1}[f(z_0 + \Delta z) - f(z_0)] \\
&\quad = [(\Delta z)I + (\Delta\bar{z})Q]^{-1}[\Delta x f_x(z_0) + \Delta y f_y(z_0) + o(|\Delta z|)] \\
&\quad = [(\Delta z)I + (\Delta\bar{z})Q]^{-1}[\Delta z f_z(z_0) + \Delta\bar{z} f_{\bar{z}}(z_0) + o|\Delta z|] \\
&\quad = [(\Delta z)I + (\Delta\bar{z})Q]^{-1}[((\Delta z)I + (\Delta\bar{z})Q)f_z(z_0) + o|\Delta z|] \\
&\quad = [(\Delta z)I + (\Delta\bar{z})Q]^{-1}[(\Delta z)I + (\Delta\bar{z})Q] \\
&\qquad\qquad \times [f_z(z_0) + [(\Delta z)I + (\Delta\bar{z})Q]^{-1}o(|\Delta z|)] \\
&\quad = f_z(z_0) + \left(I + \frac{\Delta\bar{z}}{\Delta z}Q\right)^{-1}(\Delta z)^{-1}o(|\Delta z|) \to f_z(z_0)
\end{aligned}$$

as $|\Delta z| \to 0$, since

$$\left|\frac{\Delta\bar{z}}{\Delta z}\right| = 1$$

and $\mathrm{sp}(Q)$ is closed with no elements of magnitude 1 implies $\left\|\left(I + \frac{\Delta\bar{z}}{\Delta z}Q\right)^{-1}\right\|$ is bounded.

Note that the function

$$f(z) = \phi(z)$$

is Q-analytic on $\mathbf{C}$

and

$$\phi'(z) = I \quad \text{for all } z \in \mathbf{C}.$$

If we have two functions f and g in $C^1(\Omega)$ from Ω into $\mathscr{B}$ and f satisfies the commuting property

(2.7) $f(z)Q = Qf(z)$ for each $z \in \Omega$,

then

$$(2.8)\quad \begin{aligned} \frac{\partial}{\partial z}(fg) &= \frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z} = f_z g + f g_z \text{ and} \\ \frac{\partial}{\partial \bar{z}}(fg) &= \frac{\partial f}{\partial \bar{z}}g + f\frac{\partial g}{\partial \bar{z}} = f_{\bar{z}} g + f g_{\bar{z}}. \end{aligned}$$

Thus under these conditions the product fg is Q-analytic provided both f and g are Q-analytic.

Differentiation of (2.7) yields

$$f_x Q = Qf_x, \quad f_y Q = Qf_y, \quad f_z Q = Qf_z \quad \text{and} \quad f_{\bar{z}} Q = Qf_{\bar{z}}.$$

Moreover if f is invertible then f^{-1} also commutes with Q. And since

$$\frac{f^{-1}(x_0, y) - f^{-1}(x, y)}{x_0 - x} = -f^{-1}(x_0, y)\frac{f(x_0, y) - f(x, y)}{x_0 - x}f^{-1}(x, y)$$

and

$$\frac{f^{-1}(x, y_0) - f^{-1}(x, y)}{y_0 - y} = -f^{-1}(x, y_0)\frac{f(x, y_0) - f(x, y)}{y_0 - y}f^{-1}(x, y)$$

we see that if f is in $C^1(\Omega)$, then f^{-1} is in $C^1(\Omega)$ and

$$\frac{\partial}{\partial \bar{z}}(f^{-1}) = -f^{-1}f_z f^{-1}$$

and

$$\frac{\partial}{\partial z}(f^{-1}) = -f^{-1}f_{\bar{z}} f^{-1}$$

hold. Hence if f is Q-analytic and f commutes with Q, then f^{-1} is Q-analytic wherever it exists.

By the above and an easy induction argument we have that, except at $z = 0$ when n is negative, the function $[\phi(z)]^n$ is Q-analytic for any integer n. Moreover

$$\frac{d}{d\phi}[\phi(z)]^n = n[\phi(z)]^{n-1}.$$

Also all polynomials in $\phi(z)$ are Q-analytic.

3. Cauchy's formula and series expansions. A domain Ω in $\mathbf{C}$ will be called *regular* if it is bounded and its boundary Γ consists of a finite number of simple closed curves with piecewise continuous tangent. We employ the notation

$$d\phi = \mathrm{I}dz + Qd\bar{z}.$$

If $g:\Omega \to \mathscr{B}$ is in $C^1(\bar{\Omega})$ with Ω regular, we define

$$\int_\Gamma g(z)d\phi(z) = \int_\Gamma g(z)dz + g(z)Qd\bar{z},$$

$$\int_\Gamma d\phi(z)g(z) = \int_\Gamma g(z)dz + Qg(z)d\bar{z}.$$

Since our elements are in $\mathscr{B}$ and don't necessarily commute, the order of multiplication must be observed.

This integral exists by simple continuity considerations and the extension of the definition of line integrals to functions with values in $\mathscr{B}$ [**8**].

We define the constant $P \in \mathscr{B}$ by

$$P = \int_{|z|=1} [\phi(z)]^{-1}d\phi(z).$$

In the ordinary theory of analytic functions of a complex variable we have $P = 2\pi i$. In our Banach algebra setting for Q-analytic functions we have an analogous result concerning the spectrum of P:

LEMMA 1. *For given Q in $\mathscr{B}$ with no elements of* $\mathrm{sp}(Q)$ *of magnitude* 1 *the associated element P is invertible. In particular,*

$$\mathrm{sp}(P) \subset \{2\pi i, -2\pi i\}$$

and if $\mathrm{sp}(Q)$ *lies entirely inside T then* $\mathrm{sp}(P) = \{2\pi i\}$, *and if* $\mathrm{sp}(Q)$ *lies entirely outside T, then* $\mathrm{sp}(P) = \{-2\pi i\}$.

Proof. Let $\mathscr{B}^*$ be a maximal commutative subalgebra of $\mathscr{B}$ which contains Q. Let σ be a nonzero homomorphism from $\mathscr{B}^*$ onto $\mathbf{C}$. It is sufficient to show $\sigma(P) = \pm 2\pi i$. Since σ is linear and multiplicative, and $\sigma(I) = 1$, we have

$$\begin{aligned}\sigma(P) &= \int_{|z|=1} \sigma[(zI + \bar{z}Q)^{-1}(\mathrm{I}dz + Qd\bar{z})] \\ &= \int_{|z|=1} \frac{1}{z + \bar{z}\sigma(Q)}[dz + \sigma(Q)d\bar{z}] \qquad (3.1) \\ &= \int_{|z|=1} \frac{1}{z + \lambda\bar{z}}(dz + \lambda d\bar{z}),\end{aligned}$$

where $\sigma(Q) = \lambda$. We know $|\lambda| \neq 1$.

First assume $|\lambda| < 1$. Then the expression $z + \lambda\bar{z}$ never vanishes on $|z| = 1$, and as z traces a path around the unit circle in the positive direction the change in the argument of $z + \lambda\bar{z}$ is the same as the change in the argument of z, namely 2π. Thus, for an appropriate branch of the complex logarithm function we have

$$\sigma(P) = \int_{|z|=1} d[\log(z + \lambda\bar{z})] = 2\pi i.$$

On the other hand, if $|\lambda| > 1$, we make the change of variable $w = \bar{z}$ in (3.1) to obtain

$$\begin{aligned}\sigma(P) &= -\int_{|w|=1} \frac{1}{\bar{w} + \lambda w}[d\bar{w} + \lambda dw] \\ &= -\int_{|w|=1} \frac{1}{w + \lambda^{-1}\bar{w}}(dw + \lambda^{-1}d\bar{w}) = -2\pi i,\end{aligned}$$

since $|\lambda^{-1}| < 1$.

COROLLARY 3. *Under the hypotheses of Lemma* 1, *if* $\mathrm{sp}(Q) \subset \mathrm{int}\, T$ *then* $P = 2\pi i I + N$ *where* N *is a quasinilpotent element of* $\mathscr{B}$.

Proof. If σ is a homomorphism from $\mathscr{B}^*$ into $\mathbf{C}$, then

$$\sigma(P - 2\pi i I) = \sigma(P) - 2\pi i = 0$$

implies $\mathrm{sp}(P - 2\pi i I) = \{0\}$ and $P - 2\pi i I = N$ is quasinilpotent [**10**].

THEOREM 6 (Cauchy Theorem). *Let* f *be* Q*-analytic in* Ω, *and let* Γ *be the boundary of a regular subdomain whose closure is contained in* Ω. *Then*

$$\int_\Gamma d\phi(z)f(z) = 0. \qquad (3.2)$$

Proof.

$$\begin{aligned}\int_\Gamma d\phi(z)f(z) &= \int_\Gamma f(z)dz + \int_\Gamma Qf(z)d\bar{z} \qquad (3.3) \\ &= \int_\Gamma fdx + i\int_\Gamma fdy + \int_\Gamma Qfdx - i\int_\Gamma Qfdy.\end{aligned}$$

If ω is any continuous linear functional on $\mathscr{B}$, then by Green's theorem for scalar functions

$$\omega\left(\int_\Gamma f dx\right) = \int_\Gamma \omega(f)dx = -\int\int_\Omega (\omega \circ f)_y dxdy$$
$$= -\int\int_\Omega \omega(f_y)dxdy$$

and similarly

$$\omega\left(\int_\Gamma f dy\right) = \int\int_\Omega \omega(f_x)dxdy.$$

Therefore (3.3) gives

$$(3.4) \quad \omega\left(\int_\Gamma d\phi(z)f(z)\right) = -\int\int_\Omega \omega(f_y)dxdy + i\int\int_\Omega \omega(f_x)dxdy$$
$$-\int\int_\Omega \omega(Qf_y)dxdy - i\int\int_\Omega \omega(Qf_x)dxdy$$
$$= \int\int_\Omega \omega[i(f_x + if_y) - iQ(f_x - if_y)]dxdy$$
$$= 2i\int\int_\Omega \omega(f_{\bar{z}} - Qf_z)dxdy = 0$$
$$\text{since } f_{\bar{z}} = Qf_z.$$

Since (3.4) holds for all ω, Equation (3.2) is valid.

THEOREM 7 (Cauchy Integral Formula). *Let f be Q-analytic in Ω and in $C^1(\bar{\Omega})$ with Ω regular. Let Γ = boundary of Ω. Then for z in Ω*

$$(3.5) \quad f(z) = P^{-1}\int_\Gamma [\phi(w) - \phi(z)]^{-1}d\phi(w)f(w).$$

Note. All expressions involving ϕ commute in $\mathscr{B}$.

Proof. Because of continuity considerations it is sufficient to assume f is Q-analytic in a neighborhood of Ω. Fix z in Ω and let Ω_ϵ be the domain Ω with a small disk of radius ϵ about z deleted. Apply (3.2) with z replaced by w to the domain Ω_ϵ and with f replaced by $[\phi(w) - \phi(z)]^{-1}f(w)$. One obtains

$$(3.6) \quad \int_\Gamma [\phi(w) - \phi(z)]^{-1}d\phi(w)f(w) = \int_\Gamma d\phi(w)[\phi(w) - \phi(z)]^{-1}f(w)$$
$$= \int_{|w-z|=\epsilon} d\phi(w)[\phi(w) - \phi(z)]^{-1}f(w).$$

Since

$$\int_{|w-z|=\epsilon} d\phi(z)[\phi(w) - \phi(z)]^{-1} = \int_{|w-z|=\epsilon} [\phi(w) - \phi(z)]^{-1}d\phi(z)$$
$$= \int_{|w-z|=\epsilon} [\phi(w)]^{-1}d\phi(w) = P$$

we have

$$\left\| \int_{|w-z|=\epsilon} d\phi(w)[\phi(w) - \phi(z)]^{-1} f(w) - Pf(z) \right\|$$

$$\leqq \int_{|w-z|=\epsilon} \|[\phi(w) - \phi(z)]^{-1}\| \, \|d\phi(w)\| \, \|f(w) - f(z)\|$$

$$\leqq \gamma_2\gamma_1 \int_{|w-z|=\epsilon} |w - z|^{-1}|dw| \, \|f(w) - f(z)\|$$

$$\leqq \gamma_2\gamma_1(2\pi) \sup_{|w-z|=\epsilon} \|f(w) - f(z)\|.$$

Since f is $C^1(\bar{\Omega})$ this last estimate approaches zero as ϵ approaches zero and (3.5) follows from (3.6).

Remarks. (i) It follows in the standard way from the Cauchy Integral Formula that Q-analytic functions are of class C^∞.

(ii) From the Cauchy Integral Formula we also obtain Liouville's Theorem in the standard way. Morera's Theorem can also be proved.

The following results on series expansions follow from Cauchy's Integral Formula in the standard way, by using the following expansions of the kernel,

$$[\phi(w) - \phi(z)]^{-1} = \sum_{k=0}^{n} [\phi(w)]^{-k-1}[\phi(z)]^k$$
$$+ [\phi(w) - \phi(z)]^{-1}[\phi(w)]^{-n-1}[\phi(z)]^{n+1}$$

and

$$[\phi(w) - \phi(z)]^{-1} = -\sum_{k=-n}^{-1} [\phi(w)]^{-k-1}[\phi(z)]^k$$
$$+ [\phi(w) - \phi(z)]^{-1}[\phi(w)]^n[\phi(z)]^{-n-1}.$$

These theorems are proved in detail in [**1**] for the special case of matrix algebras, but those proofs carry over to the more general case with almost no change. For proofs therefore we refer the interested reader to that paper.

THEOREM 8 (Taylor Series). *Let f be Q-analytic in the disk described by $|z - z_0| < \rho_0$. Then the series*

$$\sum_{n=0}^{\infty} [\phi(z) - \phi(z_0)]^n \frac{f^{(n)}(z_0)}{n!}$$

converges to $f(z)$ *uniformly on compact subsets contained in*

$$|z - z_0| < \frac{\rho_0}{\gamma_1\gamma_2}.$$

THEOREM 9 (Laurent Series). *Let* f *be* Q*-analytic in the region* $0 < |z - z_0| < \rho_0$. *Then the series*

$$\sum_{n=-\infty}^{\infty} [\phi(z) - \phi(z_0)]^n A_n$$

converges to $f(z)$ *uniformly on compact subsets of*

$$0 < |z - z_0| < \frac{\rho_0}{\gamma_1\gamma_2},$$

where A_n *is given for* $0 < \rho_1 < \rho_0$ *by*

$$A_n = P^{-1} \int_{|w-z_0|=\rho_1} [\phi(w) - \phi(z_0)]^{-n-1} d\phi(w) f(w).$$

4. The Q-resolvent equation and the generation of idempotents. We would like to extend the ideas of the resolvent equation and the generation of idempotents in $\mathscr{B}$ to our Q-resolvent. Throughout this section we assume that $\mathscr{B}$ is a complex Banach algebra with identity I, that $A, Q \in \mathscr{B}$ with $AQ = QA$, and that $(\mathrm{sp}\, Q) \cap T = \emptyset$. For $z \in \mathrm{res}_Q(A)$ we define

$$R(z) \equiv [A - \phi(z)]^{-1} = [A - zI - \bar{z}Q]^{-1}.$$

We observe that for the case $Q = 0$ this definition of $R(z)$ reduces to the ordinary definition.

THEOREM 10. *For each* z *and* w *in* $\mathrm{res}_Q(A)$ *we have the* Q*-resolvent equation*

$$R(z) - R(w) = [\phi(z) - \phi(w)]R(z)R(w). \tag{4.1}$$

Proof. Since A and Q commute it follows that $\phi(z)$, $\phi(w)$, $A - \phi(z)$, $A - \phi(w)$, $[A - \phi(z)]^{-1}$ and $[A - \phi(w)]^{-1}$ all commute in $\mathscr{B}$. If we multiply both sides of (4.1) by

$$[A - \phi(z)][A - \phi(w)]$$

we see that (4.1) is equivalent to

$$[A - \phi(w)] - [A - \phi(z)] = \phi(z) - \phi(w),$$

which is clear.

THEOREM 11. *Let* C_0 *be a simple closed rectifiable curve lying in* $\mathrm{res}_Q(A)$ *and let* C_1 *be a simple closed rectifiable curve in* $\mathrm{res}_Q(A)$ *which is obtained from* C_0 *by an allowable deformation* (*i.e., one which remains within* $\mathrm{res}_Q(A)$).

Then

$$\int_{C_0} R(z)d\phi(z) = \int_{C_1} R(z)d\phi(z).$$

Proof. Since $R(z)$ is Q-analytic on and inside $\Gamma = C_0 \cup (-C_1)$, the result follows from Cauchy's theorem.

Recall that P is defined in Section 3 as the element

$$P = \int_{|z|=1} \phi(z)^{-1} d\phi(z).$$

LEMMA 2. *Let C be any simple closed rectifiable curve such that C and its exterior lie in* $\mathrm{res}_Q(A)$. *Then*

$$\int_C R(z)d\phi(z) = -P.$$

Proof. We may deform C into a circle about the origin of very large radius r because of Theorem 11. Then

$$P + \int_C R(z)d\phi(z) = \int_C \{ [\phi(z)]^{-1} + [A - \phi(z)]^{-1} \} d\phi(z)$$

$$= A \int_C [\phi(z)]^{-1}[A - \phi(z)]^{-1} d\phi(z).$$

But if $z = r \cos \theta + ir \sin \theta$, then by (1.2),

$$\| [\phi(z)]^{-1}[A - \phi(z)]^{-1} d\phi(z) \| \leqq \gamma_2 |z|^{-1} \| [A - \phi(z)]^{-1} \| \gamma_1 |dz|$$

$$= \gamma_1 \gamma_2 \| [A - \phi(z)]^{-1} \| d\theta.$$

Since $\| [A - \phi(z)]^{-1} \| \to 0$ uniformly as $r \to \infty$, the last integral above can be made arbitrarily small by making C large, and the theorem follows.

THEOREM 12. *Let C be a simple closed rectifiable curve which lies in* $\mathrm{res}_Q(A)$. *Let*

$$J = -P^{-1} \int_C R(z)d\phi(z).$$

Then J is an idempotent which commutes with A and Q and

$$[A - \phi(z)]P^{-1} \int_C R(w)[\phi(w) - \phi(z)]^{-1} d\phi(w) \tag{4.3}$$

$$= \begin{cases} I - J \text{ if } z \in \mathrm{int}\, C \\ -J \text{ if } z \in \mathrm{ext}\, C. \end{cases}$$

Moreover, $J = 0$ if and only if the interior of C belongs to the Q-resolvent set of A, and $J = I$ if and only if $\mathrm{sp}_Q(A)$ *lies entirely interior to C.*

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Proof. Construct a simple closed rectifiable curve C_1 which lies entirely inside C and which is obtained from C by a slight allowable (within $\mathrm{res}_Q(A)$) deformation.

By Cauchy's theorem,

$$\int_C R(z)d\phi(z) = \int_{C_1} R(w)d\phi(w).$$

Thus we have

$$\begin{aligned}
P^2J^2 &= \int_C R(z)d\phi(z) \int_{C_1} R(w)d\phi(w) \\
&= \int_C \int_{C_1} R(z)R(w)d\phi(w)d\phi(z) \\
&= \int_C \int_{C_1} [R(z) - R(w)][\phi(z) - \phi(w)]^{-1}d\phi(w)d\phi(z) \qquad \text{by (4.1)} \\
&= \int_C R(z) \int_{C_1} [\phi(z) - \phi(w)]^{-1}d\phi(w)d\phi(z) \\
&\quad - \int_{C_1} R(w) \int_C [\phi(z) - \phi(w)]^{-1}d\phi(z)d\phi(w) \\
&= 0 - \int_{C_1} R(w)\int_C [\phi(z) - \phi(w)]^{-1}d\phi(z)d\phi(w)
\end{aligned}$$

(since $\int_{C_1} [\phi(z) - \phi(w)]^{-1}d\phi(w) = 0$ for each $z \in C$ because $[\phi(z) - \phi(w)]^{-1}$ is Q-analytic inside and on C_1)

$$\begin{aligned}
&= - \int_{C_1} R(w)Pd\phi(w) = -P\int_{C_1} R(w)d\phi(w) \\
&= -P(-PJ) = P^2J.
\end{aligned}$$

Therefore $J = J^2$. The fact that J commutes with A and Q follows from the formulation.

Now if $z \notin C$, $w \in C$, then

$$[A - \phi(z)]R(w)[\phi(w) - \phi(z)]^{-1} = [\phi(w) - \phi(z)]^{-1} + R(w)$$

and

$$(4.4) \quad (A - \phi(z)) \int_C R(w)[\phi(w) - \phi(z)]^{-1}d\phi(w)$$

$$= \int_C [\phi(w) - \phi(z)]^{-1}d\phi(w) - PJ.$$

If $z \in \mathrm{ext}\ C$, the right side of (4.4) becomes $0 - PJ = -JP$, and if $z \in \mathrm{int}\ C$, it becomes $P - PJ = P(I - J)$. Thus (4.3) follows.

Suppose $J = 0$. If $z \in \mathrm{int}\ C$, then (4.3) shows $A - \phi(z)$ is invertible, and hence $z \in \mathrm{res}_Q(A)$. Conversely, if the interior of C belongs to

$\text{res}_Q(A)$, then $R(z)$ is Q-analytic inside and on C, so $J = 0$ by Cauchy's theorem.

Now suppose $J = I$. If $z \in \text{ext } C$, then (4.3) shows $A - \phi(z)$ is invertible, $z \in \text{res}_Q(A)$. Thus $\text{sp}_Q(A) \subset \text{int } C$. Conversely, if $\text{sp}_Q(A)$ lies in int C, then by Lemma 2, $J = -P^{-1}(-P) = I$.

We discuss briefly the implications of Theorem 12 for the case that $\mathscr{B}$ is a commutative Banach algebra. Since J is idempotent, we have the decomposition

$$\mathscr{B} = \mathscr{B}_1 \oplus \mathscr{B}_2$$

where

$$\mathscr{B}_1 = J\mathscr{B}, \quad \mathscr{B}_2 = (I - J)\mathscr{B}.$$

The identity elements on $\mathscr{B}_1$ and $\mathscr{B}_2$ are J and $(I - J)$, respectively. Thus formula (4.3) shows that the projection of $A - \phi(z)$ onto $\mathscr{B}_1$ is invertible if $z \in \text{ext } C$, and if $z \in \text{int } C$ the projection of $A - \phi(z)$ onto $\mathscr{B}_2$ is invertible. Moreover, (4.3) gives explicit integral formulas for the inverses of these projections in the algebras $\mathscr{B}_1$ and $\mathscr{B}_2$.

THEOREM 13. *Let $A \in \mathscr{B}(X)$, a bounded linear transformation on a Banach space X. Let $Q \in \mathscr{B}(X)$, $\text{sp } Q \cap T = \emptyset$ and A and Q commute. Let C be a simple closed rectifiable curve lying in $\text{res}_Q(A)$. Let J be the projection*

$$J = -P^{-1} \int_C R(z)d\phi(z).$$

Let $\{M, N\}$ be the pair of linear manifolds associated with J. (That is $M = J(X)$ and $N = (I - J)(X)$, so $X = M \oplus N$.) Let A' and A'' represent the restrictions of A to M and N respectively. Then $\text{sp}_Q(A')$ is precisely that subset of $\text{sp}_Q(A)$ which lies in the interior of C. Also, $\text{sp}_Q(A'')$ is precisely the subset of $\text{sp}_Q(A)$ which lies exterior of C.

Proof. The cases $J = 0$ and $J = I$ are covered by Theorem 12. We assume $J \neq 0$ and $J \neq I$. Since $JA = AJ$ we have $A(M) \subset M$ and $A(N) \subset N$. Likewise, $Q(M) \subset M$ and $Q(N) \subset N$. Let J' and J'' denote the restrictions of J to M and N respectively. Note that $J' = I'$ and $J'' = 0''$ where I' and $0''$ are the identity and zero transformations on M and N respectively.

Since $R(z)J = JR(z)$ for each $z \in \text{res}_Q(A)$ we may introduce the transformations $R'(z)$ and $R''(z)$, the restriction of $R(z)$ to M and N respectively.

It is easy to verify that if $z \in \text{res}_Q(A)$, then $z \in \text{res}_Q(A')$ and $z \in \text{res}_Q(A'')$.

We consider the converse situation. Let $z \in \text{res}_Q(A')$ and $z \in \text{res}_Q(A'')$. Let B' and B'' be the transformations which satisfy

$$B'(A' - \phi(z)) = (A' - \phi(z))B' = J$$

and

$$B''(A'' - \phi(z)) = (A'' - \phi(z))B'' = I - J.$$

(Note also since Q commutes with J we have

$$(A - \phi(z))M \subset M \quad \text{and} \quad (A - \phi(z))N \subset N,$$

so the above restrictions make sense as operations on M and N respectively.)

Now define $B \in \mathscr{B}(X)$ by

$$B(x) = B'J(x) + B''(I - J)(x)$$

and we have

$$B(A - \phi(z)) = I = (A - \phi(z))B,$$

so $z \in \mathrm{res}_Q(A)$ and by the uniqueness of B, $B = R(z)$. This shows that $B' = R'(z)$ and $B'' = R''(z)$.

The above conclusions imply that $z \in \mathrm{sp}_Q(A)$ if and only if $z \in \mathrm{sp}_Q(A')$ or $z \in \mathrm{sp}_Q(A'')$.

Now if z lies exterior of C, (4.3) shows that $A' - \phi(z)$ has an inverse on M, and if z lies interior of C, it shows that $A'' - \phi(z)$ has an inverse on N. Therefore $\mathrm{sp}_Q(A')$ is precisely that subset of $\mathrm{sp}_Q(A)$ which lies interior of C, and $\mathrm{sp}_Q(A'')$ is precisely that subset of $\mathrm{sp}_Q(A)$ which lies exterior of C.

5. Expansions of Q-resolvent around an isolated singularity. The following development follows that in **[6]** for the usual resolvent. Recall:

$$R(z) = [A - zI - \bar{z}Q]^{-1} = [A - \phi(z)]^{-1}.$$

Assume that $z = z_0$ is an isolated singularity of $R(z)$. Recall that $R(z)$ is Q-analytic wherever it exists. For simplicity we take $z_0 = 0$. According to Theorem 9, $R(z)$ has a Laurent expansion about $z = 0$, namely

$$(5.1) \quad R(z) = \sum_{n=-\infty}^{\infty} B_n[\phi(z)]^n,$$

where

$$(5.2) \quad B_n = P^{-1} \int_{|z|=\rho_1} [\phi(z)]^{-n-1} d\phi(z) R(z)$$

$$= P^{-1} \int_{|z|=\rho_1} [A - \phi(z)]^{-1}[\phi(z)]^{-n-1} d\phi(z)$$

since all the elements involved in this integral commute.

We assume $[A - \phi(z)]^{-1}$ exists in $0 < |z| < \rho_0$ and that $0 < \rho_1 < \rho_0$.

Thus (5.1) converges uniformly in compact subsets of

$$0 < |z| < \frac{\rho_0}{\gamma_1 \gamma_2}.$$

Now consider the product $B_m B_n$:

$$B_m B_n = P^{-2} \int_{|z|=\rho_1} [A - \phi(z)]^{-1} [\phi(z)]^{-m-1} d\phi(z)$$

$$\times \int_{|s|=\rho_2} [A - \phi(s)]^{-1} [\phi(s)]^{-n-1} d\phi(s)$$

(here $0 < \rho_1 < \rho_2 < \rho_0$)

$$= P^{-2} \int_{|z|=\rho_1} \int_{|s|=\rho_2} R(z) R(s) [\phi(z)]^{-m-1} \times [\phi(s)]^{-n-1} d\phi(s) d\phi(z)$$

$$= P^{-2} \int_{|z|=\rho_1} \int_{|s|=\rho_2} [\phi(z) - \phi(s)]^{-1} [R(z) - R(s)]$$

$$\times [\phi(z)]^{-m-1} [\phi(s)]^{-n-1} d\phi(s) d\phi(z)$$

$$= P^{-2} \int_{|z|=\rho_1} R(z) [\phi(z)]^{-m-1}$$

$$\underbrace{\times \int_{|s|=\rho_2} [\phi(z) - \phi(s)]^{-1} [\phi(s)]^{-m-1} d\phi(s) d\phi(z)}_{\alpha}$$

$$- P^{2} \int_{|s|=\rho_2} R(s) [\phi(s)]^{-n-1}$$

$$\underbrace{\times \int_{|z|=\rho_1} [\phi(z) - \phi(s)]^{-1} [\phi(z)]^{-m-1} d\phi(z) d\phi(s).}_{\beta}$$

Now, to compute α and β,

$$[\phi(z) - \phi(s)]^{-1} = -[\phi(s)]^{-1} [I - \phi(z)\phi(s)^{-1}]^{-1}$$

$$= -[\phi(s)]^{-1} \sum_{k=0}^{\infty} \phi(z)^k \phi(s)^{-k}$$

$$= -\sum_{k=0}^{\infty} \phi(z)^k \phi(s)^{-k-1}$$

provided that

$$\|\phi(z)\phi(s)^{-1}\| < 1.$$

We choose ρ_1 and ρ_2 so that

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$$\rho_1 < \frac{\rho_2}{\gamma_1\gamma_2}.$$

Then

$$\|\phi(z)\phi(s)^{-1}\| \leqq \|\phi(z)\| \, \|\phi(s)^{-1}\| \leqq \gamma_1\gamma_2|z|\,|s|^{-1}$$
$$= \gamma_1\gamma_2\rho_1\rho_2^{-1} < 1.$$

Therefore,

$$\alpha = -\sum_{k=0}^{\infty} \phi(z)^k \int_{|s|=\rho_2} \phi(s)^{-n-k-2} d\phi(s)$$

and

$$\beta = -\sum_{k=0}^{\infty} \phi(s)^{-k-1} \int_{|z|=\rho_1} \phi(z)^{-k-m-1} d\phi(z).$$

Recall:

$$\int_{|z|=\rho_0} [\phi(z)]^n d\phi(z) = \begin{cases} 0, & n \neq -1 \\ P, & n = -1. \end{cases}$$

Thus,

$$\alpha = \begin{cases} -\phi(z)^{-n-1}P, & n \leqq -1 \\ 0, & n \geqq 0, \end{cases}$$

$$\beta = \begin{cases} 0, & m \leqq -1 \\ -\phi(s)^{-m-1}P, & m \geqq 0. \end{cases}$$

We have 4 cases:

(1) $m, n \geqq 0$

$$B_mB_n = P^{-1} \int_{|s|=\rho_2} R(s)[\phi(s)]^{-m-n-2} d\phi(s) = B_{m+n+1},$$

(2) $m, n < 0$

$$B_mB_n = -P^{-1} \int_{|z|=\rho_1} R(z)[\phi(z)]^{-m-n-2} d\phi(z) = -B_{m+n+1},$$

(3) $m < 0, n \geqq 0$

$$B_mB_n = 0,$$

(4) $m \geqq 0, n < 0$

$$\begin{aligned} B_mB_n &= -P^{-1} \int_{|z|=\rho_1} R(z)[\phi(z)]^{-m-n-2} d\phi(z) \\ &\quad + P^{-1}\int_{|s|=\rho_2} R(s)[\phi(s)]^{-m-n-2} d\phi(s) \\ &= -B_{m+n+1} + B_{m+n+1} = 0. \end{aligned}$$

In conclusion,

$$B_mB_n = \begin{cases} 0 & \text{if } m < 0,\ n \geqq 0 \\ B_{m+n+1} & \text{if } m,\ n \geqq 0 \\ -B_{m+n+1} & \text{if } m,\ n < 0. \end{cases}$$

Let $B_0 = B$. Then

$$B_1 = B_0B_0 = B^2$$
$$B_2 = B_0B_1 = B^3$$
$$B_3 = B_0B_2 = B^4$$
$$\vdots$$
$$B_n = B^{n+1},\ n \geqq 0.$$

Let $-B_{-1} = E$. Then

$$E^2 = B_{-1}B_{-1} = -B_{-1} = E$$

and E is idempotent. (Note: We already know this since

$$E = -B_{-1} = -P^{-1}\int_{|z|=\rho_1} R(z)d\phi(z).)$$

Let $D = -B_{-2}$. Then

$$ED = B_{-1}B_{-2} = -B_{-2} = D.$$

Also

$$B_{-3} = -B_{-2}B_{-2} = -D^2$$
$$B_{-4} = -B_{-2}B_{-3} = D(-D^2) = -D^3$$
$$B_{-5} = -B_{-2}B_{-4} = D(-D^3) = -D^4$$
$$\vdots$$
$$B_{-n} = -D^{n-1},\ n \geqq 2.$$

Thus

$$B_{-1} = -E = -ED$$
$$B_{-n} = -ED^{n-1},\ n \geqq 1.$$

Now (5.1) becomes

$$R(z) = \sum_{n=1}^{\infty} B_{-n}[\phi(z)]^{-n} + \sum_{n=0}^{\infty} B_n[\phi(z)]^n$$
$$= -E\sum_{n=1}^{\infty} D^{n-1}[\phi(z)]^{-n} + \sum_{n=0}^{\infty} B^{n+1}[\phi(z)]^n,$$

where

$$EB = 0$$

$$ED = D$$

$$E^2 = E.$$

We would like to show D is quasinilpotent. We have

$$D = -B_{-2} = -P^{-1}\int_{|z|=\rho_1} [A - \phi(z)]^{-1}\phi(z)d\phi(z).$$

Let σ be a homomorphism of $\mathscr{B}^*$ onto $\mathbf{C}$, where as usual $\mathscr{B}^*$ is a maximal commutative subalgebra of $\mathscr{B}$ containing A and Q. Then

$$\sigma(D) = \frac{-1}{\sigma(P)}\int_{|z|=\rho_1} \frac{\sigma[\phi(z)]}{\sigma(A) - \sigma[\phi(z)]}(dz + \sigma(Q)d\bar{z}).$$

We need to show $\sigma(D) = 0$.

Let $\sigma(Q) = \lambda$, $\sigma(A) = \alpha$. Then

$$(5.3) \quad -\sigma(D)\sigma(P) = \int_{|z|=\rho_1} \frac{z + \bar{z}\sigma(Q)}{\sigma(A) - z - \bar{z}\sigma(Q)}(dz + \sigma(Q)d\bar{z})$$

$$= \int_{|z|=\rho_1} \frac{z + \bar{z}\lambda}{\alpha - z - \bar{z}\lambda}(dz + \lambda d\bar{z}).$$

Now assume $|\lambda| < 1$ and let $w = z + \bar{z}\lambda$ to get

$$(5.4) \quad = \int_C \frac{w}{\alpha - w}dw$$

where C turns out to be an ellipse with positive orientation about the origin.

Now since $A - \phi(z)$ was invertible for all $0 < |z| \leqq \rho_1$ it is true that $\alpha = 0$ or α is outside C.

Case 1. If $\alpha = 0$ then (5.4) becomes

$$-\sigma(D)\sigma(P) = -\int_C dw = 0.$$

Case 2. If α is outside of C then (5.4) becomes

$$-\sigma(D)\sigma(P) = \int_C \frac{w}{\alpha - w}dw = 0$$

since $\dfrac{w}{\alpha - w}$ is analytic in and on C.

So if $|\lambda| < 1$, then $\sigma(D) = 0$. $|\lambda| = 1$ is impossible since

$$\mathrm{sp}(Q) \cap T = \emptyset.$$

Now if $|\lambda| > 1$ we use the substitution $w = \bar{z}$ in (5.3) to obtain

$$\sigma(D)\sigma(P) = \int_{|w|=\rho_1} \frac{\bar{w} + w}{\alpha - \bar{w} - w\lambda}(d\bar{w} + \lambda dw)$$

$$= \lambda \int_{|w|=\rho_1} \frac{w + \frac{1}{\lambda}\bar{w}}{\frac{\alpha}{\lambda} - w - \lambda^{-1}\bar{w}}(dw + \lambda^{-1}d\bar{w})$$

which by the argument previously given equals 0 since $\left|\frac{1}{\lambda}\right| < 1$, and either

$$\alpha = 0 \quad \text{or} \quad \alpha - \lambda w - \bar{w} \neq 0 \quad \text{for } |w| \leqq \rho_1.$$

By the above $\sigma(D) = 0$, so D is quasinilpotent in $\mathcal{B}$.
We have proved the following:

THEOREM 14. *Let $z = 0$ be an isolated singularity of $R(z)$. Then $R(z)$ has the expansion*

$$R(z) = -E \sum_{n=1}^{\infty} D^{n-1}[\phi(z)]^{-n} + \sum_{n=0}^{\infty} B^{n+1}[\phi(z)]^n$$

where D is quasinilpotent, and

$$E^2 = E$$

$$EB = 0$$

$$ED = D.$$

The first series converges uniformly on compact subsets of $z \neq 0$ and the second series converges uniformly on compact subsets of

$$|z| < \frac{\rho_0}{\gamma_1\gamma_2}$$

where ρ_0 is the distance to the nearest singularity. (The statements on convergence follow from Theorems 8 and 9.) Explicitly, B, E and D are given by

$$B = P^{-1} \int_{|z|=\rho_1} R(z)[\phi(z)]^{-1} d\phi(z),$$

$$D = -P^{-1} \int_{|z|=\rho_1} R(z)\phi(z) d\phi(z),$$

$$E = -P^{-1} \int_{|z|=\rho_1} R(z)d\phi(z),$$

where $0 < \rho_1 < \rho_0$.

6. An example of a disconnected Q-spectrum. By Theorem 13 if A and Q are commuting bounded operators on a Banach space and $\text{sp}\, Q \cap T = \emptyset$, then if $\text{sp}_Q(A)$ is disconnected we can integrate on a simple closed curve C about one of the components of $\text{sp}_Q(A)$ to produce a nontrivial projection J where J commutes with both A and Q. This observation gives the following results:

THEOREM 15. *Let A and Q be bounded linear transformations on a Banach space X. If $AQ = QA$ and* $\text{sp}\, Q \cap T = \emptyset$ *with* $\text{sp}_Q(A)$ *disconnected, then there exists a nontrivial projection J on X which commutes with both A and Q.*

COROLLARY 4. *With the hypothesis of Theorem* 15 *and with* $\text{sp}_Q(A)$ *disconnected, A and Q have a nontrivial separating invariant subspace.*

Proof. The projection J of Theorem 15 is such that $J(X)$ and $(I - J)(X)$ are each nontrivial invariant subspaces of A and Q. This follows since J commutes with both A and Q.

The above results raise the question as to whether or when the Q-spectrum is disconnected. And since the usual theory of the spectrum does not give rise to nontrivial invariant subspaces of A or Q if they have connected usual spectra, we have the interesting question as to whether the Q-spectrum of A is ever disconnected when both A and Q have connected usual spectra.

The following example demonstrates that the spectra of A and Q can be connected, yet $\text{sp}_Q(A)$ is disconnected.

Note. Since in Example 6 both A and Q are normal operators, by other methods [**9**], we already know that each has a common nontrivial invariant subspace.

Example 6. Let T be the unit circle in the complex plane and let H be the Hilbert space $H = L^2(T)$, with ordinary arc length measure on T. We consider the Banach algebra $B = \mathscr{B}(H)$, the collection of bounded linear operators on H.

Let ϕ be the function defined on T by

$$\phi(z) = \frac{\arg z^2}{4\pi}$$

where we take the branch of arg such that

$$0 \leqq \arg z < 2\pi, \forall z \neq 0.$$

Then $\phi \in L^\infty(T)$. Hence the multiplication operator M_ϕ defined by

$$M_\phi f = \phi f, \quad \forall f \in L^2(T),$$

is a bounded linear operator on $L^2(T)$, i.e., $M_\phi \in B = \mathcal{B}(H)$. (See [**2**], problems 49-52, for relevant information on multipliers.) Also, we know that

$$\text{spectrum } M_\emptyset = \text{ess range } \phi = \left[0, \frac{1}{2}\right].$$

Hence the ordinary spectrum of M_ϕ is connected, and lies inside the unit disk.

We also consider the multiplier M_z (multiplication by z). Then

$$(M_z f)(z) = zf(z), \quad \forall f \in H, z \in T$$

and

$$\text{spectrum } M_z = \{z:|z| = 1\} = T.$$

Hence the ordinary spectrum of M_z also is connected.

Now consider the relative spectrum of M_z with respect to M_ϕ, i.e., $\text{sp}_{M_\phi}(M_z)$. We will show that this spectrum is disconnected. We want to find all complex λ such that the operator

$$M_z - \lambda I - \bar{\lambda} M_\phi$$

is not invertible in $\mathcal{B}(H)$. But we have

$$M_z - \lambda I - \bar{\lambda} M_\phi = M_{z-\lambda-\bar{\lambda}\phi}.$$

Thus we want to find all λ such that

$$0 \in \text{ess range}(z - \lambda - \bar{\lambda}\phi).$$

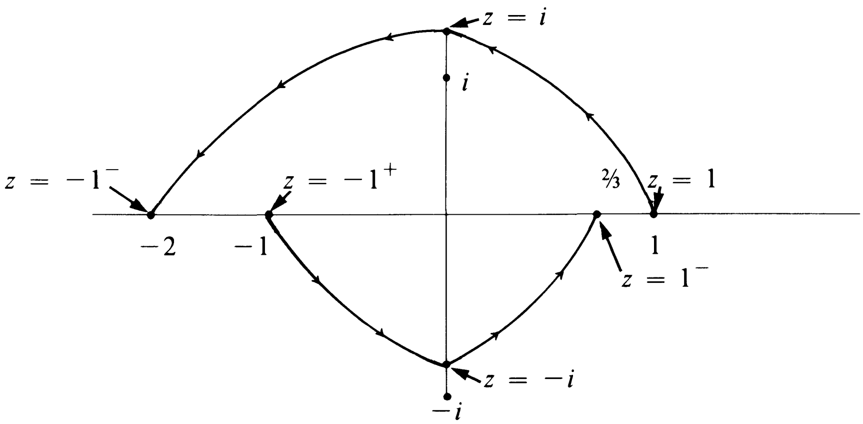
We need to solve the equation

$$z - \lambda - \bar{\lambda}\phi(z) = 0.$$

Taking the complex conjugate of this equation and solving for z, we get

$$\lambda = \frac{z - \bar{z}\phi(z)}{1 - |\phi(z)|^2} = \frac{z - \bar{z}\dfrac{\arg z^2}{4\pi}}{1 - \left(\dfrac{\arg z^2}{4\pi}\right)^2} \equiv \lambda(z).$$

As z ranges around T, counterclockwise beginning at $z = 1$, we see that $\lambda = \lambda(z)$ traces a curve something like the one below:



(The discontinuities occur at $z = \pm 1$.)

Thus the relative spectrum of M_z with respect to M_ϕ is disconnected.

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