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THE POINCARÉ MAP IN MIXED EXTERIOR ALGEBRA

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ABSTRACT. The Poincaré map of mixed exterior algebra generalizes the Hodge star operator and it plays a central rôle in the proofs of many classical identities of linear algebra. The principal purpose of this paper is to derive a new formula for it. This formula is useful in circumstances when the definition is too implicit. Several applications are discussed.

§1. **Introduction.** Let E be an n-dimensional vector space over a field Γ of characteristic 0. Let E^* denote its dual, relative to a scalar product $\langle,\rangle$. Then we may form vector spaces $\Lambda E^*, \Lambda E$ and their tensor product, which we denote by $\Lambda(E^*, E)$. This space has a natural inner product, which is induced from $\langle,\rangle$ and which we denote by the same symbol. Also the exterior algebra structures of ΛE^* and ΛE give rise to an algebraic structure on $\Lambda(E^*, E)$. This is denoted by a dot and is called the mixed exterior algebra. As well, $\Lambda(E^*, E)$ has a "composition" product "$\circ$". The inner product and both algebraic structures restrict to the diagonal subalgebra

$$\Delta(E) = \bigoplus_{p=0}^{n} \Delta_p(E), \qquad \Delta_p(E) = \Lambda^p E^* \otimes \Lambda^p E,$$

and the dot product is commutative there. Since $\Delta_0(E) = \Gamma$ and $\Delta_1(E)$ generate $\Delta(E)$, it is often easy to prove results for the dot product by working with decomposed elements and using linearity. On the other hand, many important results involve the composition algebra. Thus it is essential to know relations between the two products. Several such relations have been developed in [1] and used to give intrinsic proofs of many classical results of linear algebra. Others were announced in [2]. The purpose of this paper is to give a complete proof of one of the formulae of [2].

§2. **Algebraic preliminaries.** The proofs of all the results stated in this section are to be found in [1].

If $L(E)$ denotes the algebra of linear transformations of E, then the map

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$T: \Delta_1(E) \to L(E)$, given by

$$(1) \qquad T(x^* \otimes x)y = \langle x^*, y\rangle x, \qquad x, y \in E, x^* \in E^*,$$

is an isomorphism of composition algebras. $T^{-1}(\iota) = t$ is called the unit tensor. It satisfies

$$(2) \qquad z \circ t = t \circ z = z \qquad z \in \Delta_1(E).$$

We define $t^p \in \Delta_p(E)$, $p \in \mathbb{Z}$, by

$$(3) \quad t^0 = 1, \qquad t^p = \frac{1}{p!}(\underbrace{t \cdot t \cdots t}_{(p \text{ factors})}), \quad p = 1, 2, \ldots, n \text{ and } t^p = 0, \quad \text{otherwise.}$$

For each $u \in \Lambda(E^*, E)$, let $\mu(u)$ denote left multiplication by u in the dot algebra. Thus

$$(4) \qquad \mu(u \cdot v) = \mu(u) \circ \mu(v) \qquad u, v \in \Lambda(E^*, E).$$

The dual of $\mu(u)$ is written $i(u)$:

$$\langle \mu(u)v, w\rangle = \langle v, i(u)w\rangle \qquad u, v, w \in \Lambda(E^*, E)$$

and thus (4) implies

$$(4)' \qquad i(u \cdot v) = i(v) \circ i(u) \qquad u, v \in \Lambda(E^*, E).$$

The identity

$$(5) \qquad i(t^q)t^p = \binom{n-p+q}{q} t^{p-q}$$

holds.

The Poincaré map $D: \Lambda(E^*, E) \to \Lambda(E^*, E)$ is defined by

$$(6) \qquad Du = i(u)t^n, \qquad u \in \Lambda(E^*, E).$$

It is a linear isomorphism and an isometry. It satisfies

$$(7) \qquad D^2u = u, \qquad u \in \Delta(E).$$

From (4)′ and the commutativity of $\Delta(E)$, we have

$$(8) \qquad i(u) \circ D = D \circ \mu(u), \qquad u \in \Delta(E),$$

as an operator identity on $\Delta(E)$. From (5), we also have

$$(9) \qquad Dt^p = t^{n-p}.$$

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The following identity will play a central rôle in the sequel:

$$
\begin{aligned}
i(z)(z_1 \cdots z_p) = {} & \sum_{\mu=1}^{p} z_1 \cdots \langle z, z_\mu \rangle \cdots z_p \\
& - \sum_{1 \le \mu < \nu \le p} (z_\mu \circ z \circ z_\nu + z_\nu \circ z \circ z_\mu) \\
& \times z_1 \cdots \hat{z}_\mu \cdots \hat{z}_\nu \cdots z_p \qquad z, z_1, \ldots, z_p \in \Delta_1(E)
\end{aligned}
\tag{10}
$$

Finally, since any element of $L(E)$ induces an unique derivation of $\wedge E$, we write θ_z for the derivation generated by $T(z)$, $z \in \Delta_1(E)$ (cf. (1)).

§3. **The derivations λ_z and ρ_z.** For each $z \in \Delta_1(E)$, define

$$
\lambda_z = \iota \otimes \theta_z, \qquad \rho_z = \lambda_z^* = \theta_z^* \otimes \iota. \tag{11}
$$

Then both λ_z and ρ_z are derivations of the dot algebra. They both restrict to derivations of $\Delta(E)$.

Throughout the remainder of this paper, we will restrict our consideration to $\Delta(E)$. For $u, v \in \Delta(E)$, then, (4), (4)′ imply

$$
\mu(u) \circ \mu(v) = \mu(v) \circ \mu(u) \quad \text{and} \quad i(u) \circ i(v) = i(v) \circ i(u). \tag{12}
$$

LEMMA.

$$
\lambda_z(z_1 \cdots z_p) = \sum_{\nu=1}^{p} z_1 \cdots (z \circ z_\nu) \cdots z_p \tag{13}
$$

$$
\rho_z(z_1 \cdots z_p) = \sum_{\nu=1} z_1 \cdots (z_\nu \circ z) \cdots z_p \tag{13$'$}
$$

$$
z, z_1 \cdots z_p \in \Delta_1(E).
$$

Proof. It is sufficient to assume that $z = x^* \otimes x$, $z_\nu = x^{*\nu} \otimes x_\nu$, where $x^*, x^{*\nu} \in E^*$, $x, x_\nu \in E$, $\nu = 1, \ldots, p$. Then $z_1 \cdots z_p = (x^{*1} \wedge \cdots \wedge x^{*p}) \otimes (x_1 \wedge \cdots \wedge x_p)$ and so, by (11),

$$
\lambda_z(z_1 \cdots z_p) = (x^{*1} \wedge \cdots \wedge x^{*p}) \otimes \sum_{\nu=1}^{p} (x_1 \wedge \cdots \wedge \theta_z x_\nu \wedge \cdots \wedge x_p).
$$

But $\theta_z x_\nu = T(z) x_\nu = \langle x^*, x_\nu \rangle x$ (cf. (1)). Thus

$$
\lambda_z(z_1 \cdots z_p) = \sum_{\nu=1}^{p} (x^{*1} \otimes x_1) \cdots (\langle x^*, x_\nu \rangle x^{*\nu} \otimes x) \cdots (x^{*p} \otimes x_p).
$$

Since $z \circ z_\nu = \langle x^*, x_\nu \rangle x^{*\nu} \otimes x$, this is (13). Equation (13)′ is proved similarly.

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Next observe that (10) may be written

$$i(z)\mu(z_1)(z_2\cdots z_p)=\langle z, z_1\rangle(z_2\cdots z_p)+\mu(z_1)[i(z)(z_2\cdots z_p)]$$
$$-\sum_{\nu=2}^{p}(z_1\circ z\circ z_\nu+z_\nu\circ z\circ z_1)\cdot z_2\cdots\hat{z}_\nu\cdots z_p.$$

In view of the lemma, the last term is

$$-(\lambda_{z_1\circ z}+\rho_{z\circ z_1})(z_2\cdots z_p)$$

and since $z_2,\ldots,z_p\in\Delta_1(E)$ and p are arbitrary, we have an operator identity on $\Delta(E)$:

(14) $$i(z)\circ\mu(z_1)-\mu(z_1)\circ i(z)=\langle z, z_1\rangle\iota-\lambda_{z_1\circ z}-\rho_{z\circ z_1},\qquad z, z_1\in\Delta_1(E).$$

Let $\theta(z)$, $\Gamma(z)$ be the self-dual operators on $\Delta(E)$ defined by

(15) $$\theta(z)=\lambda_z+\rho_z,\qquad \Gamma(z)=\langle t, z\rangle\iota-\theta(z),\qquad z\in\Delta_1(E).$$

Then (14) implies

(16) $$i(t)\circ\mu(z)-\mu(z)\circ i(t)=\Gamma(z)=i(z)\circ\mu(t)-\mu(t)\circ i(z),\qquad z\in\Delta_1(E),$$

where the latter equality follows from the former by dualizing.

§4. **Commutation formulae.** The lemma of §3 implies that

$$\lambda_z(t\cdot z_1\cdots z_p)=(z\circ t)\cdot z_1\cdots z_p+t\cdot\lambda_z(z_1\cdots z_p)$$

and hence yields the operator identity:

(17) $$\lambda_z\circ\mu(t)-\mu(t)\circ\lambda_z=\mu(z),\qquad z\in\Delta_1(E).$$

Similarly

(17)′ $$\rho_z\circ\mu(t)-\mu(t)\circ\rho_z=\mu(z)\qquad z\in\Delta_1(E).$$

In view of (15), these imply

(18) $$\theta(z)\circ\mu(t)-\mu(t)\circ\theta(z)=2\mu(z),\qquad \Gamma(z)\circ\mu(t)-\mu(t)\circ\Gamma(z)=-2\mu(z),\qquad z\in\Delta_1(E),$$

and their duals.

We will obtain higher order commutation formula via the

LEMMA. *Let F be any vector space. Fix $\phi\in L(F)$ and for each $\psi\in L(F)$ consider the ϕ-commutators $\psi_p=\psi\circ\phi^p-\phi^p\circ\psi$, $p=0,1,\ldots$. Then*

(19) $$\psi_p=\sum_{\nu=0}^{p-1}\phi^\nu\circ\psi_1\circ\phi^{p-1-\nu}.$$

Proof. From the definition

$$\psi_{p+1} = \phi \circ \psi_p - \phi \circ \psi_{p-1} \circ \phi + \psi_p \circ \phi.$$

This recursion has an unique solution if ψ_0 and ψ_1 are specified. But it is easy to check that (19) satisfies it (with $\psi_0 = 0$).

COROLLARY 1. *For all* $p \in \mathbb{Z}$,

$$\Gamma(z) \circ \mu(t^p) - \mu(t^p) \circ \Gamma(z) = -2\mu(t^{p-1}) \circ \mu(z), \tag{20}$$

$$i(t^p) \circ \Gamma(z) - \Gamma(z) \circ i(t^p) = -2i(t^{p-1}) \circ i(z), \qquad z \in \Delta_1(E), \tag{20*}$$

Proof. In the lemma, let $\phi = \mu(t)$, $\psi = \Gamma(z)$ and note that $\psi_1 = -2\mu(z)$, by (18). This gives (20) for $p = 1, 2, \ldots$. For $p \le 0$, it follows from (3). (20)* is its dual.

COROLLARY 2. *For all* $p \in \mathbb{Z}$,

$$i(z) \circ \mu(t^p) - \mu(t^p) \circ i(z) = \mu(t^{p-1}) \circ \Gamma(z) - \mu(t^{p-2}) \circ \mu(z), \tag{21}$$

$$\begin{aligned} i(t^p) \circ \mu(z) - \mu(z) \circ i(t^p) = \Gamma(z) \circ i(t^{p-1}) \\ - i(z) \circ i(t^{p-2}), \qquad z \in \Delta_1(E). \end{aligned} \tag{21*}$$

Proof. In the lemma, let $\phi = \mu(t)$, $\psi = i(z)$ and note that $\psi_1 = \Gamma(z)$, by (16). This gives

$$\psi_p = \sum_{\nu=0}^{p-1} \mu(t)^\nu \circ [\mu(t)^{p-1-\nu} \circ \Gamma(z) - 2(p-1-\nu)\mu(t)^{p-2-\nu} \circ \mu(z)],$$

after the use of (20). Simplifying, we find

$$\psi_p = p\mu(t)^{p-1} \circ \Gamma(z) - p(p-1)\mu(t)^{p-2} \circ \mu(z),$$

which, in view of (3), yields (21). (21)* is its dual.

§5. **The main identity.** In this section we will prove the identity (26). It has many applications, some of which will be described in the next section.

For notational simplicity, we introduce the elements of $L(\Delta(E))$ defined by

$$\tau_q^p = \mu(t^{p+q}) \circ i(t^p), \qquad p, q \in \mathbb{Z} \tag{22}$$

and

$$T_q = \sum_p (-1)^p \tau_q^p, \qquad q \in \mathbb{Z}. \tag{23}$$

Observe that $\tau_q^p = 0$, unless $p, p+q \in \{0, 1, \ldots, n\}$. Thus the sum in (23) is finite and

$$\sum_p (-1)^p (\tau_q^p + \tau_q^{p-1}) = 0, \qquad q \in \mathbb{Z}. \tag{24}$$

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Furthermore

$$\tau_q^p(1)=\delta_0^p t^q, \qquad p, q\in\mathbb{Z}, \tag{25}$$

where $\delta_0^0=1$ and $\delta_0^p=0$, for $p\neq 0$. It follows from (23) and (25) that

$$T_q(1)=t^q, \qquad q\in\mathbb{Z}. \tag{25$'$}$$

Now, by (22), if $z\in\Delta_1(E)$, we have $i(z)\circ\tau_q^p=[i(z)\circ\mu(t^{p+q})]\circ i(t^p)$. Thus successive use of (21), (20), (21)* and (20)* yields

$$\begin{aligned} i(z)\circ\tau_q^p &= [\mu(t^{p+q})\circ i(z)+\mu(t^{p+q-1})\circ\Gamma(z)-\mu(t^{p+q-2})\circ\mu(z)]\circ i(t^p)\\ &= \mu(t^{p+q})\circ i(z)\circ i(t^p)+\mu(t^{p+q-1})\circ[i(t^p)\circ\Gamma(z)+2i(t^{p-1})\circ i(z)]\\ &\quad -\mu(t^{p+q-2})\circ\{i(t^p)\circ\mu(z)-[i(t^{p-1})\circ\Gamma(z)+2i(t^{p-2})\circ i(z)]+i(z)\\ &\qquad\qquad \circ\, i(t^{p-2})\}. \end{aligned}$$

In view of (12), (22) this may be written

$$i(z)\circ\tau_q^p=[(\tau_q^p+\tau_q^{p-1})+(\tau_q^{p-1}+\tau_q^{p-2})]\circ i(z)+(\tau_{q-1}^p+\tau_{q-1}^{p-1})\circ\Gamma(z)-\tau_{q-2}^p\circ\mu(z).$$

Summation on p, therefore, yields

$$i(z)\circ T_q=-T_{q-2}\circ\mu(z), \qquad q\in\mathbb{Z}, \quad z\in\Delta_1(E), \tag{26}$$

because of the definition (23) and the identity (24).

This formula is the key to all that follows, as was pointed out to the author by W. H. Greub, to whom the author wishes to express his indebtedness.

From (25), we have $T_q(1)=t^q$ and hence

$$T_q(z)=[T_q\circ\mu(z)](1)=-[i(z)\circ T_{q+2}](1)=-i(z)t^{q+2}, \qquad z\in\Delta_1(E).$$

An inductive argument then leads to

$$T_q(z_1\cdot z_2\cdots z_p)=(-1)^p i(z_1\cdot z_2\cdots z_p)t^{q+2p}, \qquad z_1,\ldots,z_p\in\Delta_1(E).$$

Thus

$$T_q(u)=(-1)^p i(u)t^{q+2p}, \qquad u\in\Delta_p(E).$$

This result has been proved for $p=1,2,\ldots$ but it is clearly true for $p\leq 0$ and, hence, for $p\in\mathbb{Z}$. It may be written

$$i(u)t^{q+2p}=(-1)^p\sum_\nu(-1)^\nu\mu(t^{\nu+q})i(t^\nu)u, \qquad p,q\in\mathbb{Z},\ u\in\Delta_p(E). \tag{27}$$

In particular (cf. (6)), if $q+2p=n$, we have the following formula for the Poincaré map:

$$Du=(-1)^p\sum_\nu(-1)^\nu\mu(t^{n-2p+\nu})i(t^\nu)u, \qquad p\in\mathbb{Z},\ u\in\Delta_p(E). \tag{28}$$

In view of the interesting properties of D such as (7), (8), (9), the equation (28)

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is the source of many other identities. The precise structure of D may be found by using (28) and the orthogonal direct sum decomposition of $\Delta(E)$ announced in [2]. We will discuss this in a later paper; however, some immediate applications will be given in the next section.

§6. **Some applications.** Define subspaces F_p, G_p of $\Delta_p(E)$ by

$$F_p = \Delta_p(E) \cap \ker i(t), \qquad G_p = \Delta_p(E) \cap \ker \mu(t), \qquad p \in \mathbb{Z}. \tag{29}$$

Then (28) yields

$$Du = (-1)^p \mu(t^{n-2p})u, \qquad u \in F_p, \quad p \in \mathbb{Z}. \tag{30}$$

It follows that $Du = 0$ for $2p > n$ and since D is an isomorphism, we conclude that

$$F_p = 0 \quad \text{for} \quad 2p > n.$$

Also, applying $\mu(t)$ to (30), we have

$$F_p \subset \ker \mu(t^{n-2p+1}), \qquad p \in \mathbb{Z},$$

in view of (8).

Next, we prove the

LEMMA. *D maps F_p isomorphically onto G_{n-p}.*

Proof. If $u \in F_p$, then $\mu(t)Du = Di(t)u = 0$, by (8). Thus $DF_p \subset G_{n-p}$. A similar argument shows that $DG_{n-p} \subset F_p$. The result then follows from (7).

COROLLARY 1:

$$Dv = (-1)^p i(t^{n-2p})v, \qquad v \in G_{n-p}, \quad p \in \mathbb{Z}, \tag{31}$$

Proof. Put $Du = v$ in (30) and apply D to the result.

It follows, as above, that

$$G_{n-p} = 0 \quad \text{for} \quad 2p > n$$

and

$$G_{n-p} \subset \ker i(t^{n-2p+1}), \qquad p \in \mathbb{Z}.$$

From (30), (31) we also conclude that

$$G_{n-p} = \mu(t^{n-2p})F_p, \qquad F_p = i(t^{n-2p})G_{n-p}, \qquad p \in \mathbb{Z}.$$

COROLLARY 2. *If $u \in \Delta_p(E)$ is an eigenvector of D, then $n = 2p$. In this case $F_p = G_p$ are eigenspaces of D corresponding to the eigenvalue $(-1)^p$.*

For a different type of result, put $u = t^p$ in equation (28) and use (5) and (9). The result is

$$t^{n-p} = (-1)^p \sum_{\nu=0}^{p} (-1)^\nu \binom{n-p+\nu}{\nu} t^{n-2p+\nu} \cdot t^{p-\nu}.$$

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Thus we obtain a new proof of the binomial identity:

$$\sum_{\nu=0}^{p} (-1)^\nu \binom{n-p+\nu}{\nu}\binom{n-p}{p-\nu} = (-1)^p.$$

As a final application, let $q=0$ and $u \in F_p$ in (27):

$$i(u)t^{2p} = (-1)^p u, \qquad u \in F_p. \tag{32}$$

To see the meaning of this in component form, choose a basis for E and construct the dual basis for E^*. The components of $u \in F_p$ are then scalars $u^{i_1 \cdots i_p}_{j_1 \cdots j_p}$, $i_1 \cdots j_p = 1, 2, \ldots, n$, skew-symmetric in the i's and j's and such that $u^{ki_2 \cdots i_p}_{kj_2 \cdots j_p} = 0$ (summation convention). The components of t^{2p} are the generalized Kronecker deltas. Then (32) reads

$$u^{i_1 \cdots i_p}_{j_1 \cdots j_p} \delta^{j_1 \cdots j_{2p}}_{i_1 \cdots i_{2p}} = (-1)^p (p!)^2 u^{j_{p+1} \cdots j_{2p}}_{i_{p+1} \cdots i_{2p}}.$$

This results may be proved directly but it does not seem to have been noticed.

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