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# Regular completions of Cauchy spaces via function algebras

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A regular completion with the universal property is obtained for each member of a certain class of Cauchy spaces by embedding the Cauchy space in a complete function algebra with the continuous convergence structure.

## 1. Preliminaries

A Cauchy space $(X, C)$ is a set $X$ equipped with a Cauchy structure $C$ ; that is, $C$ is a set of filters on $X$ (called "Cauchy filters") subject to the following conditions:

$(C_1)$ the ultrafilter $\dot{x}$ containing $\{x\}$ is in $C$ for all $x$ in $X$ ;

$(C_2)$ $F \in C$ and $G \geq F$ implies $G \in C$ ;

$(C_3)$ if $F, G \in C$ and $F \vee G \neq 0$ (meaning each set in $F$ intersects each set in $G$ ), then $F \cap G \in C$ .

A Cauchy structure $C$ generates a convergence structure on $X$ defined as follows: $F \to x$ in $X$ if $\dot{x} \cap F \in C$ . Keller [5] showed that a set of filters on $X$ satisfying $(C_1)$, $(C_2)$, and $(C_3)$ is precisely the set of Cauchy filters for some uniform convergence structure (see [3]) on $X$ . A definitive study of Cauchy spaces has been made by Ramaley and Wyler ([8], [9]).

A Cauchy space $(X, C)$ is *complete* if each Cauchy filter is convergent. Each convergence space which admits a uniform convergence

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structure (see [5] for a characterization of such spaces) can be regarded as a complete Cauchy space if the Cauchy structure is taken to be the set of all Cauchy filters. A function $f : (X, C) \to (X_1, C_1)$ is *Cauchy-continuous* if $F \in C$ implies $f(F) \in C_1$. The pair $((X_1, C_1), j)$ is a *completion* of a Cauchy space $(X, C)$ if $(X_1, C_1)$ is a complete Cauchy space, and $j : (X, C) \to (X_1, C_1)$ is a dense Cauchy embedding. If, in addition, $F \to x$ in $X$ implies the existence of a filter $G \in C$ such that $jG \to x$ in $X$ and $F \geq \mathrm{cl}_X(jG)$, the completion is said to be *strict*.

Reed [10], showed that each completion of a Cauchy space $(X, C)$ induces a completion of each uniform convergence space $(X, J)$ for which $C$ is the set of $J$-Cauchy filters. A Cauchy space $(X, C)$ is regular if $\mathrm{cl}_X F \in C$ whenever $F \in C$. It is shown in [6] that a strict, regular, Hausdorff completion of a Cauchy space, if it exists, is unique up to Cauchy-homeomorphism.

The Cauchy filters in a Cauchy space $(X, C)$ are partitioned into *Cauchy equivalence classes* by the following equivalence relation: $F \sim G$ if and only if $F \cap G \in C$. Reed [10] used the set $\{[F] : F \in C\}$ of these equivalence classes to construct a family of completions for any Cauchy space. In particular, she showed that every Hausdorff Cauchy space (that is, Hausdorff in its underlying convergence structure) has a strict Hausdorff completion; we will make use of this fact later.

A Cauchy space $(X, C)$ is said to be *pseudo-topological* if a filter $F$ is in $C$ whenever all ultrafilters finer than $F$ are in the same Cauchy equivalence class relative to $C$. A convergence space is *pseudo-topological* if a filter converges to $x$ whenever each refining ultrafilter converges to $x$. Note that a pseudo-topological convergence space can be regarded as a complete pseudo-topological Cauchy space.

Let $X$ be a convergence space, $C(X)$ the $R$-algebra of all continuous real valued functions on $X$. The *continuous convergence structure* $c$ on $X$ is the coarsest relative to which the function $w : C(X) \times X \to R$, defined by $w(f, x) = f(x)$, is continuous. The resulting convergence function space $C_c(X)$ has been studied extensively

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(see, for instance, [1], [2], [4], [11], and [12]). The natural map $i : X \to C_c C_c(X)$ is defined by $i(x) = F_x$, where $F_x(f) = f(x)$, all $f \in C(X)$. Binz [1], has shown that $i(X) = \mathrm{hom} C_c(X)$, the set of all continuous algebra homomorphisms from $C_c(X)$ into $R$ which preserve the multiplicative identity. The space $X$ is said to be *c-embedded* if the function $i$ is an embedding. In order to state a characterization of *c-embedded* spaces, we need a few preliminary definitions.

Starting with a space $X$, let $\omega X$ denote the finest completely regular topological space coarser than $X$ on the same underlying set; $\omega X$ is called the completely regular modification of $X$. The fact that $C(X) = C(\omega X)$ is useful and will be used without explicit mention in what follows. We will use the symbol "$\mathrm{cl}_X$" to represent the closure operator for any space $X$. $X$ is said to be *ω-regular* if $\mathrm{cl}_{\omega X} F \to x$ in $X$ whenever $F \to x$ in $X$. The first proposition is established in [12] and, implicitly, in [11].

PROPOSITION 1.1. *A space $X$ is c-embedded iff $X$ is pseudo-topological, Hausdorff, and ω-regular.*

Binz [2], has shown that the space $C_c(X)$ is always $c$-embedded.

Next, let $(X, C)$ be a Cauchy space, and let $C\hat{}(X, C)$ be the $R$-algebra of all real-valued Cauchy-continuous functions on $(X, C)$. Let $C\hat{}$ denote the *continuous Cauchy structure* on $C\hat{}(X, C)$, defined as follows: a filter $\Phi$ on $C\hat{}(X, C)$ is in $C\hat{}$ iff $\Phi(F)$ is $R$-Cauchy (that is, $R$-convergent) whenever $F \in C$. If one defines the notion of product Cauchy structure in the natural way, then there is no difficulty in establishing the following proposition; we omit the details.

PROPOSITION 1.2. *For any Cauchy space $(X, C)$, $C\hat{}$ is the coarsest Cauchy structure on $C\hat{}(X, C)$ relative to which the function $w : C\hat{}(X, C) \times (X, C) \to R$, defined by $w(f, x) = f(x)$, is Cauchy-continuous.*

PROPOSITION 1.3. *For any Cauchy space $(X, C)$, the Cauchy function space $(C\hat{}(X, C), C\hat{})$ is complete.*

Proof. Let $\Phi \in C\hat{}$, and let $f : X \to R$ be defined by

$f(x) = \lim \Phi(\dot{x})$ , all $x \in X$ .

To show that $f \in C^{\wedge}(X)$ , let $\mathcal{F} \in \mathcal{C}$ ; then $\Phi(\mathcal{F}) \to a$ , for some $a \in R$ . Let $W$ be a closed neighborhood of $a$ in $R$ ; then there is $A \in \Phi$ and $F \in \mathcal{F}$ such that $A(F) \subset W$ . If $x \in F$ , then $A(x) \subset W$ , and so $f(x) \in \mathrm{cl}_R A(x) \subset W$ . Thus $f(F) \subset W$ , and $f(\mathcal{F}) \to a$ in $R$ .

To show that $\Phi \to f$ , it must be shown that $\Phi \cap \dot{f} \in \mathcal{C}^{\wedge}$ . If $F \subset X$ , $A \in \Phi$ , and $x \in F$ , then $A(F) \in \Phi(\dot{x})$ , and so $f(F) \subset \mathrm{cl}_R A(F)$ . Thus, if $\mathcal{F} \in \mathcal{C}$ , $\dot{f}(\mathcal{F}) > \mathrm{cl}_F \Phi(\mathcal{F})$ , and hence $\dot{f}(\mathcal{F})$ and $\Phi(\mathcal{F})$ must converge to the same point in $R$ .

A complete Cauchy space is merely a convergence space in which the convergent filters are designated to be Cauchy. Thus the Cauchy structure $\mathcal{C}^{\wedge}$ on the function space is somewhat redundant and will usually be deleted. Also, it is convenient to shorten the notation from $C^{\wedge}(X, \mathcal{C})$ to $C^{\wedge}(X)$ if there is no danger of confusing which Cauchy structure on $X$ is being considered.

The next proposition follows by inspection of the respective definitions.

PROPOSITION 1.4. *If* $(X, \mathcal{C})$ *is a complete Cauchy space, then* $C^{\wedge}(X)$ *coincides with* $C_c(X)$ .

Reed [10], proved that each Hausdorff Cauchy space $(X, \mathcal{C})$ has a strict, Hausdorff completion $\left(\left(X_1, \mathcal{C}_1\right), j\right)$ with the *regular extension property*. The latter term means that each Cauchy-continuous function from $(X, \mathcal{C})$ into a Hausdorff, regular, complete Cauchy space $\left(X_2, \mathcal{C}_2\right)$ has a Cauchy-continuous extension $f_1 : \left(X_1, \mathcal{C}_1\right) \to \left(X_2, \mathcal{C}_2\right)$ . The existence of such a completion (but not the details of the construction) are needed for the following proposition.

PROPOSITION 1.5. *Let* $(X, \mathcal{C})$ *be a Hausdorff Cauchy space, and let* $\left(\left(X_1, \mathcal{C}_1\right), j\right)$ *be a strict, Hausdorff completion with the regular extension property. Then* $C^{\wedge}(X)$ *and* $C^{\wedge}\left(X_1\right)$ *are both isomorphic and homeomorphic.*

Proof. Let $\Psi : C^{\wedge}(X) \to C^{\wedge}\left(X_1\right)$ be defined by $\Psi(f) = f_1$ , where $f_1$

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is the extension from $X$ to $X_1$ whose existence is derived from the regular extension property. The uniqueness of $f_1$ is guaranteed by the Hausdorff property, and so $\Psi$ is well defined. It is a trivial matter to verify that $\Psi$ is an algebra isomorphism.

To show that $\Psi$ is Cauchy continuous, let $\Phi$ be $C^{\wedge}(X)$-Cauchy and let $A \in C_1$. Since $X_1$ is complete, $A$ converges in $X_1$, and by the strictness of the completion, there is $F \in C$ such that $A \geq \mathrm{cl}_{X_1} jF$. It then follows easily that $(\Psi\Phi)(A) \geq \mathrm{cl}_R \Phi(F)$. But $\Phi(F)$ is $R$-convergent, by definition of the Cauchy structure on $C^{\wedge}(X)$, and, since $R$ is regular, $(\Psi\Phi)(A)$ is likewise $R$-convergent. Thus $\Psi\Phi$ is $C^{\wedge}(X_1)$-Cauchy.

To show that $\Psi^{-1}$ is continuous, let $\Lambda$ be a $C^{\wedge}(X_1)$-Cauchy filter, and let $F \in C$. Then $(\Psi^{-1}\Lambda)(F) \geq (jF)$, and it follows that $\Psi^{-1}\Lambda$ is $C^{\wedge}(X)$-Cauchy.

COROLLARY 1.6. *For any Hausdorff Cauchy space* $(X, C)$, *there is a convergence space* $X_1$ *such that* $C^{\wedge}(X)$ *is isomorphic and homeomorphic to* $C_c(X_1)$. *Consequently,* $C^{\wedge}(X)$ *is a Hausdorff,* $\omega$*-regular, pseudo-topological convergence space.*

Proof. These statements follow immediately from Proposition 1.1, the remark following Proposition 1.1, Proposition 1.4, and Proposition 1.5.

It follows from the preceding results that, for any Cauchy space $(X, C)$, $C^{\wedge}C^{\wedge}(X)$ is equal to $C_cC^{\wedge}(X)$ and is equivalent, in the sense of Corollary 1.6, to $C_cC_c(X_1)$, where $(X_1, C_1)$ is the completion of Proposition 1.5. A Cauchy space $(X, C)$ is said to be $c^{\wedge}$-embedded if the function $i : (X, C) \to C^{\wedge}C^{\wedge}(X)$, where $i(x)(f) = f(x)$ for all $f$ in $C^{\wedge}(X)$, is a Cauchy embedding. In order to obtain an internal characterization of $c^{\wedge}$-embedded spaces, we will introduce several new Cauchy space concepts.

For any Cauchy space $(X, C)$, let $\mu X$ be the set $X$ with the weak topology induced by the set of functions $C^{\wedge}(X)$. Since $C^{\wedge}(X) \subset C(X)$, it follows that $\mu X \leq \omega X$, and $\mu X = \omega X$ if $(X, C)$ is complete. $(X, C)$ is said to be *μ-regular* if $\text{cl}_{\mu X} F \in C$ whenever $F \in C$. It is clear that μ-regular implies ω-regular implies regular.

A Cauchy space $(X, C)$ is said to be *Cauchy-separated* if $X$ is Hausdorff and, whenever $F, G \in C$ and $F \cap G \notin C$, there is $f \in C^{\wedge}(X)$ such that $\lim f(F) \neq \lim f(G)$ in $R$. In other words, $(X, C)$ is Cauchy-separated iff there are enough functions in $C^{\wedge}(X)$ to separate Cauchy equivalence classes. Since distinct points in a Hausdorff Cauchy space correspond to distinct equivalence classes, it follows that $\mu X$ is Hausdorff whenever $(X, C)$ is Cauchy-separated.

PROPOSITION 1.7. *Let $(X, C)$ be a Cauchy space, $A$ a $\mu X$-closed subset of $X$, and $x \in X-A$. Then there is a function $f$ in $C^{\wedge}(X)$ such that $f(x) = 0$ and $f(A) = 1$.*

Proof. Let $J$ be the uniformity on $X$ generated by sets of the form $J(\varepsilon, f) = \{(x, y) : |f(x)-f(y)| < \varepsilon\}$, where $\varepsilon > 0$ and $f \in C^{\wedge}(X, C)$. Note that the $J$-Cauchy filters are precisely those filters $F$ such that $f \in C^{\wedge}(X, C)$ implies that $f(F)$ is Cauchy on $R$. Thus, if $C_J$ is the set of $J$-Cauchy filters, $C \subset C_J$. It also follows from this construction that $C_J$ is an admissible Cauchy structure for $\mu X$.

Let $x$ and $A$ be as stated in the proposition, and let $(X_1, J_1)$ be a complete uniform space containing $(X, J)$. Then there is an $X_1$-closed set $B \subset X_1$ such that $A = X_1 \cap B$. Since $X_1$ is a completely regular topological space, there is $g \in C(X_1)$ such that $g(x) = 0$ and $g(B) = 1$. If $f$ is the restriction of $g$ to $X$, then $f \in C^{\wedge}(X, C_J)$, $f(x) = 0$, and $f(A) = 1$. Since $C \subset C_J$, $C^{\wedge}(X, C_J) \subset C^{\wedge}(X, C)$, and the proof is complete.

PROPOSITION 1.8. *A Cauchy space $(X, C)$ which has an ω-regular, Hausdorff completion is μ-regular and Cauchy-separated.*

Proof. Let $((X_1, C_1), j)$ be a Hausdorff, ω-regular completion of $(X, C)$. We will first show that $(X, C)$ is μ-regular. Let $F \in C$; then $\text{cl}_{\omega X_1}(jF) \in C_1$. If $A \subset X$, then

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$$\mathrm{cl}_{\omega X_1}(jA) = \bigcap\left\{f^{-1}(B) : B \text{ closed subset of } R,\ jA \subset f^{-1}(B),\ f \in C(X_1)\right\},$$

by Proposition 1.7, and

$$\mathrm{cl}_{\mu X}A = \bigcap\{f^{-1}(B) : B \text{ closed subset of } R,\ A \subset f^{-1}(B),\ f \in C^{\wedge}(X)\} .$$

If $f \in C(X_1)$, then the restriction of $f$ to $X$ is in $C^{\wedge}(X)$. Thus $j(\mathrm{cl}_{\mu X}A) \subset \mathrm{cl}_{\omega X_1}(jA)$, and it follows that $\mathrm{cl}_{\mu X}F \geq j^{-1}\left(\mathrm{cl}_{\omega X_1}(jF)\right)$. Thus $\mathrm{cl}_{\mu X}F \in \mathrm{C}$, and $(X, \mathrm{C})$ is $\mu$-regular.

To show that $(X, \mathrm{C})$ is Cauchy separated, let $F, G \in \mathrm{C}$ and $F \cap G \notin \mathrm{C}$. Then $jF$ and $jG$ are in distinct Cauchy equivalence classes in $\mathrm{C}_1$, and so $jF \to a$ in $X_1$, $jG \to b$ in $X_1$, and $a \neq b$. Since $X_1$ is $\omega$-Hausdorff, there is a function $f$ in $C(X_1)$ which separates the points $a$ and $b$. If $g = f \circ j$, then $g \in C^{\wedge}(X)$, and $\lim gF \neq \lim gG$.

## 2. The natural completion

In this section, we will characterize the $c^{\wedge}$-embedded Cauchy spaces both internally and externally; the external characterization leads to what we call the *natural completion*. The natural completion is an $\omega$-regular, Hausdorff, pseudo-topological completion obtained by embedding $(X, \mathrm{C})$ into $C^{\wedge}C^{\wedge}(X)$ under the natural injection $i$.

Let $\hom^{\wedge}C^{\wedge}(X)$ denote the subspace of $C^{\wedge}C^{\wedge}(X)$ consisting of all continuous algebra homomorphisms from $C^{\wedge}(X)$ into $R$ which preserve the multiplicative identity. $\hom^{\wedge}C^{\wedge}(X)$ coincides with $\hom_c C^{\wedge}(X)$, and by Corollary 1.6, $\hom^{\wedge}C^{\wedge}(X)$ is an $\omega$-regular, Hausdorff, pseudo-topological convergence space, since all of these properties are inherited from $C_c C^{\wedge}(X)$. We will assume that $\hom^{\wedge}C^{\wedge}(X)$ is equipped with the Cauchy structure inherited from $C^{\wedge}C^{\wedge}(X)$.

PROPOSITION 2.1. *Let* $(X, \mathrm{C})$ *be a Hausdorff Cauchy space,* $i : (X, \mathrm{C}) \to C^{\wedge}C^{\wedge}(X)$ *the natural function.*

*(a)* $i(X)$ *is a dense subset of* $\hom^{\wedge}C^{\wedge}(X)$.

*(b)* $i$ *is Cauchy-continuous.*

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*(c) If* $(X, \mathcal{C})$ *is Cauchy-separated, then* $i$ *is one-to-one.*

Proof. Let $((X_1, \mathcal{C}_1), j)$ be a strict, Hausdorff completion of $(X, \mathcal{C})$ (see remarks preceding Proposition 1.5). Consider the following commutative diagram:

$$\begin{array}{ccccc} (X, \mathcal{C}) & & \xrightarrow{\quad i \quad} & & C^\wedge C^\wedge(X) \\ {\scriptstyle j}\downarrow & & & & \uparrow{\scriptstyle \Gamma} \\ (X_1, \mathcal{C}_1) & \xrightarrow{i_1} & \hom_c C_c(X_1) & \xrightarrow{\text{id}} & C^\wedge C^\wedge(X_1) \end{array}$$

In this diagram, id is the identity map and $\Gamma$ the homeomorphism and algebra isomorphism derived from the homeomorphism $\Psi : C^\wedge(X) \to C^\wedge(X_1)$ of Proposition 1.5. By the work of Binz [*1*], $i_1$ is always continuous and onto $\hom_c C_c(X_1)$. Thus $i$ is a composition of Cauchy-continuous functions (recall that all the spaces in the above diagram except $(X, \mathcal{C})$ are complete Cauchy spaces, and continuous functions between complete Cauchy spaces are Cauchy continuous). $jX$ is dense in $X_1$, and therefore $i_1 jX$ is dense in $\hom_c C_c(X_1)$. Since $\Gamma$ is an algebra isomorphism, $\Gamma$ carries $\hom_c C_c(X_1)$ onto $\hom^\wedge C^\wedge(X)$, and *(a)* is thus established.

Finally, assume that $(X, \mathcal{C})$ is Cauchy-separated, and let $i(x) = i(y)$. This implies $f(x) = f(y)$, all $f \in C^\wedge(X)$. But $\mu X$ has the weak topology induced by $C^\wedge(X)$, and $x = y$ follows from the fact that $\mu X$ is Hausdorff whenever $(X, \mathcal{C})$ is Cauchy-separated (see the remark preceding Proposition 1.7).

Let $(X, \mathcal{C})$ be a Cauchy space. A collection $\mathcal{B}$ of subsets of $X$ will be called a *Cauchy covering system* if each member of $\mathcal{C}$ contains a member of $\mathcal{B}$. This definition is the obvious extension to Cauchy spaces of the notion of "covering system" defined for convergence spaces in [*4*].

Starting with a Cauchy system $\mathcal{B}$ for $(X, \mathcal{C})$, let $\mathcal{U}$ be the uniformity on the set $C^\wedge(X)$ with basic sets of the form

$$T(B_1, \ldots, B_n, \varepsilon) = \{(f, g) : |f(x)-g(x)| < \varepsilon \text{ for all } x, y \in \cup B_i\} \cup \Delta ,$$

where $\Delta$ is the diagonal in $C^\wedge(X) \times C^\wedge(X)$, $\{B_1, \ldots, B_n\}$ is an arbitrary finite subcollection of $\mathcal{B}$, and $\varepsilon > 0$. Let $\overline{0}$ be the zero

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function in $C^{\wedge}(X)$ , and $N(\overline{0})$ the neighborhood filter at $\overline{0}$ in the uniform topology on $C^{\wedge}(X)$ .

LEMMA 2.2. *In the notation of the preceding paragraph,* $N(\overline{0}) \in C^{\wedge}$ , *the continuous Cauchy structure on* $C^{\wedge}(X)$ .

Proof. Let $H \in C$ ; then $H$ contains a set $B \in B$ , and, for any $\varepsilon > 0$ , $\left(T(B, \varepsilon)(\overline{0})\right)(B) \subset (-\varepsilon, \varepsilon)$ .

THEOREM 2.3. *The following statements about a Cauchy space* $(X, C)$ *are equivalent:*

*(a)* $(X, C)$ *is* $c^{\wedge}$*-embedded;*

*(b)* $\left(\hom^{\wedge}C^{\wedge}(X), i\right)$ *is a completion of* $(X, C)$ *;*

*(c)* $(X, C)$ *has an* $\omega$*-regular, Hausdorff, pseudo-topological completion;*

*(d)* $(X, C)$ *is* $\mu$*-regular, Cauchy-separated, and pseudo-topological.*

Proof. *(a)* implies *(b)*. By Proposition 2.1, $i(X)$ is a dense subset of $\hom^{\wedge}C^{\wedge}(X)$ . $\hom^{\wedge}C^{\wedge}(X)$ is a closed subspace of the complete Cauchy space $C^{\wedge}C^{\wedge}(X)$ , so $\hom^{\wedge}C^{\wedge}(X)$ is also a complete Cauchy space.

*(b)* implies *(c)*. We noted in the second paragraph of Section 2 that $\hom^{\wedge}C^{\wedge}(X)$ is $\omega$-regular, Hausdorff, and pseudo-topological.

*(c)* implies *(d)*. See Proposition 1.8.

*(d)* implies *(a)*. The function $i : (X, C) \to \hom^{\wedge}C^{\wedge}(X)$ is Cauchy-continuous and one-to-one by Proposition 2.1. Thus it remains only to show that if $G$ is a filter on $X$ such that $iG$ is convergent in $\hom^{\wedge}C^{\wedge}(X)$ , then $G \in C$ . Since $(X, C)$ is pseudo-topological, it is sufficient to show:

(1) each ultrafilter finer than $G$ is in $C$ ;

(2) all ultrafilters finer than $G$ are in the same equivalence class.

To establish (1), assume that $F$ is an ultrafilter, $F \geq G$ , and $F \notin C$ . Then $i(F)$ converges in $\hom^{\wedge}C^{\wedge}(X)$ . Since $F \notin C$ , and $(X, C)$ is $\mu$-regular, $F \not\geq \mathrm{cl}_{\mu X}H$ for all $H \in C$ . Thus, for each $H \in C$ , there

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is $H_H$ in $H$ , but not in $F$ , such that $H_H$ is $\mu X$-closed. The set $B = \{H_H : H \in C\}$ is a Cauchy covering system for $(X, C)$ . Let $U$ be the uniformity on the set $C^\wedge(X)$ derived from $B$ , as in the paragraph preceding Lemma 2.2, and let $N(\overline{0})$ be the uniform neighborhood filter at the constant function $\overline{0}$ . By Lemma 2,2, $N(\overline{0}) \in C^\wedge$ . Since $i(F)$ is $\hom^\wedge C^\wedge(X)$-Cauchy, $i(F)(N(\overline{0})) \to 0$ ; which implies that $(N(\overline{0}))(F) \to 0$ in $R$ .

Let $W$ be a neighborhood of 0 in $R$ not containing 1 . Then there are sets $B_1, \ldots, B_n$ in $B$ , $\varepsilon > 0$ , and $F \in F$ such that $(T(B_1, \ldots, B_n, \varepsilon))(F) \subset W$ . Since $F$ is an ultrafilter and $B_i \notin F$ for $i = 1, \ldots, n$ , $F$ is not a subset of $B_0 = \cup\{B_i : i = 1, \ldots, n\}$ . Let $x \in F-B_0$ . Since $B_0$ is $\mu X$-closed, it follows from Proposition 1.7 that there is a function $f \in C^\wedge(X)$ such that $f(x) = 1$ and $f(B_0) = 0$ . By this construction, $f \in T(B_1, \ldots, B_n, \varepsilon)$ , but $f(x) \notin W$ for $x \in F$ , a contradiction. This contradiction establishes that $F \in C$ , and assertion (1) is proved.

To establish (2), suppose that $F$ and $H$ are ultrafilters finer than $G$ ; thus $i(F \cap H) \geq i(G)$ . Let $f \in C^\wedge(x)$ . Then $(i(F \cap H))(\dot{f})$ is equal to $f(F) \cap f(H)$ , and so the latter filter is Cauchy in $R$ . Thus $f(F)$ and $f(H)$ converge to the same point in $R$ , and this is true for all $f \in C^\wedge(X)$ . Thus, since $(X, C)$ is Cauchy-separated, $F \cap G \in C$ . This completes the proof of Theorem 2.3.

If $(X, C)$ is a $c^\wedge$-embedded Cauchy space, we will refer to $\hom^\wedge C^\wedge(X)$ as the *natural completion* of $(X, C)$ . As our concluding result in this paper, we will prove that the natural completion of a $c^\wedge$-embedded space has the universal property in the class of all $c^\wedge$-embedded Cauchy spaces.

Let $\phi : (X, C) \to (X_1, C_1)$ be a Cauchy continuous function from one $c^\wedge$-embedded Cauchy space into another. Then it is a routine matter to show that the function $\phi_1 : C^\wedge(X_1) \to C^\wedge(X)$ , defined by $\phi_1(f) = f\phi$ for all $f \in C^\wedge(X_1)$ , is a Cauchy-continuous algebra homomorphism.

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THEOREM 2.4. *If* $(X, C)$ *is a* $c^{\wedge}$*-embedded Cauchy space,* $(X_1, C_1)$ *a complete* $c^{\wedge}$*-embedded Cauchy space, and* $\phi : (X, C) \to (X_1, C_1)$ *is Cauchy-continuous, then* $\phi$ *has a unique Cauchy-continuous extension* $\psi : \text{hom}^{\wedge}C^{\wedge}(X) \to (X_1, C_1)$ .

Proof. Define the natural function $\phi_2 : C^{\wedge}C^{\wedge}(X) \to C^{\wedge}C^{\wedge}(X_1)$ in the manner described in the preceding paragraph. Let $\sigma$ be the restriction of $\phi_2$ to $\text{hom}^{\wedge}C^{\wedge}(X)$ . Since $\phi_2$ is a Cauchy-continuous homomorphism, $\sigma : \text{hom}^{\wedge}C^{\wedge}(X) \to \text{hom}^{\wedge}C^{\wedge}(X_1)$ is Cauchy-continuous. Since $(X_1, C_1)$ is complete, $(X_1, C_1)$ is Cauchy-homeomorphic to $\text{hom}^{\wedge}C^{\wedge}(X_1)$ . Let $\psi = i_1^{-1}\sigma$ , where $i_1$ is the natural embedding from $X_1$ onto $\text{hom}^{\wedge}C^{\wedge}(X_1)$ . It is clear from this construction that $\psi$ is an extension of $\phi$ (in the sense that $\phi = \psi i$ ) and also that this extension is unique.

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